NON-EUCLIDEAN GEOMETRY: A HISTORY AND INTRODUCTION

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INTRODUCTION
High school students are first exposed to geometry starting with Euclid's classic postulates:
1. It is possible to draw a straight line from any one point to another point.
2. It is possible to create a finite straight line continuously on a straight line.
3. It is possible to describe a circle of any center and distance.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the exterior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Postulate #5, the so-called "parallel postulate" has always been a sticking point for mathematicians. Historically, mathematicians encountering Euclid's beautiful work wonder why #5 is a postulate instead of a proven theorem. There have been many, many attempts to prove #5; there have been many, many failures. One such failure was that of Jewish priest and mathematician, Giraldo Saccheri. His failure sparked ideas that led to what is now known as "Non-Euclidean geometry", a branch of geometry that discards #5 and finds out where the geometries lead them. In particular, two Non-Euclidean branches will be discussed: that of Nikolai Lobachevsky and Bernhard Riemann.

LESSON OVERVIEW
The idea is to illustrate why Non-Euclidean geometry opened up rich avenues in mathematics only after the parallel postulate was rejected and re-examined, and to give students a brief, non-confusing idea of how Non-Euclidean geometry works. A series of PowerPoint slides and a worksheet (illustrating Saccheri's problem) will give the students both the basics of Non-Euclidean geometry and the history behind it. These slides give both the background, definitions and the information for the student to understand the material.

LEARNING OBJECTIVES
There are two main objectives: first, to introduce the concept of non-Euclidean geometries to high school geometry students who have examined Euclidean geometry at length, including some basic worksheets so they can study the concept for themselves. Second, to introduce students to the rich history of mathematics and mathematical ideas. To accomplish this, the lesson will include:
A 35 to 60 minute PowerPoint slide show that illustrates the history and concepts of non-Euclidean geometry;
A worksheet on Saccheri quadrilaterals, spherical and hyperbolic geometries. The time required will be 30-45 minutes total.

REFLECTION
This will be presented during the 2nd semester, when students have a stronger grasp of Euclidean geometry and an introduction to the non-Euclidean "taxicab" metric.

REFERENCES


The MacTutor History of Mathematics Archive, Giovanni Giraldo Saccheri http://www-history.mcs.st-andrews.ac.uk/Biographies/Saccheri.html


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THE FIVE POSTULATES
1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance
4. That all right angles are equal to one another.
5. That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

THE FIFTH POSTULATE
Although this postulate does not talk directly about parallel lines, it states that lines where the sum of interior angles equals 180° will never meet if drawn indefinitely.

More mathematicians were bothered by the fact this was a "postulation" and left open. Many mathematicians tried to prove the parallel postulate, and they all failed.

There is a very difficult proof that if you cannot prove the parallel postulate, however, once mathematicians started asking themselves "is this really true?" from some striking results happened.

SACHERCI'S QUADRILATERAL
The sides are called "legs", and are congruent. The base is perpendicular to the legs. The top is called the "summit."

Saccheri noted that: if we could prove the Five Postulates, then Saccheri's Quadrilateral would be right angles.

MORE THAN ONE PARALLEL TO THE LINE?

The Lobachevsky Postulate:
"The sum of angles of a Saccheri quadrilateral are acute."
Lobachevsky Theorem #1:
"The sum of a Saccheri quadrilateral is larger than the base.

Work sheet question #3: If Theorem #1 is true, what happens to the triangle and the segment thereon? Explain why."

In mathematics, we want to know WHY things are rather than HOW things are. Here are some final WHys (and a Why?) to consider:

Why did Euclid's fifth postulate stay unchallenged until Lobachevsky? Even Lobachevsky tried to prove its truth until he realized that it may not be the case.

As it turns out, the universe itself is NOT flat. We don't know exactly what kind of geometry (yet), but we do know it isn't Euclidean. What is the geometry of the universe?

Nonetheless, Euclidean geometry worked, and worked well, for centuries. Why?

FURTHER THOUGHTS
EUCLID (325 B.C. - 265 B.C.E.)
- Taught geometry in Alexandria, Egypt.
- Put together his teacher's theorems, wrote many of his own, and published the Elements, easily the most studied text in all of mathematics and second only to the Bible as the best selling book of all time.
- Not much else is known about his life other than his work on The Elements.

NICOLAI LOBACHEVSKY (1793-1856)
- The history of attempted proofs of the parallel postulate sparked Lobachevsky's interest in the question. He was the first to follow through on the question "what if it is not true?"
- He published the first work ever on non-Euclidean geometry in 1829.
- Janos Bolyai (1802-1860), a Hungarian mathematician, independently came to some conclusions Lobachevsky did about the potential of non-Euclidean geometry.
- Unfortunately, his legacy as a mathematician came after his death. He died in near-poverty, his genius not yet recognized by his contemporaries.

BERNHARD RIEMANN (1826-1866)
- Riemann developed a geometry similar to spherical geometry that postulated no parallels to any line.
- He is best known for "Riemann manifolds", a type of geometric space that can extend beyond three dimensions and has curvature.
- It takes an advanced degree in mathematics to understand most of Riemann's theorems.

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