Extensions on the Pythagorean Theorem

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A Formula for Distance in Three Dimensions

After some experimenting, students should be able to formulate a distance formula in three dimensions. The formula is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$ 

This is possible to introduce the notation $\Delta x = x_2 - x_1$, and similarly for the other variables. This turns the two dimensional distance formula into

$$d^2 = \Delta x^2 + \Delta y^2,$$

making the relationship to the Pythagorean theorem extremely clear. This also yields the three-dimensional formula

$$d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2.$$

It should not surprise students at this point to learn that in higher dimensional space with coordinates $u_1, \ldots, u_n$, the distance formula is $d^2 = \sum (\Delta u_i)^2$. It is wise to switch to $u$ as the coordinate variable here, lest the students confuse the coordinate variables $x_i$ with the points $(x_i, y_i, \ldots)$ etc.

Connection to Relativity

At this point, the instructor may end the lesson or introduce a connection to higher mathematics. A metric is a collection of numbers which tell us how to add coordinates to get a measure of distance. For example, the metric of the plane is $d^2 = x^2 + y^2$, where the $d$ stands for distance. The metric of three-dimensional space is

$$d^2 = x^2 + y^2 + z^2.$$

We frequently hear scientists talk about "space-time." What is spacetime, exactly? We will investigate by learning the metric of spacetime. If we think of distance as a measure of how easy it is to get from one point to another, we can figure it out. It’s very easy for you to get up and cross the room; but it is comparatively difficult for you to travel to New York. We say that New York is more distant than the other side of the classroom. But now consider - would it be easier to get to New York in a year, or in a minute? To get to New York in a year, all you have to do is start walking - but if you want to be in New York in a minute, you’ll have to use some kind of amazing rocket technology. By the same logic as before, New York (one minute from now) is more distant than New York (one year from now).

This demonstrates how time is much different than the other “dimensions” - length, width, height. In our everyday experience, the larger the separation in space we are from something, the more distant it is. But with time, it seems the opposite - the larger the separation (for example, a year), the closer it is. Since distance with time seems to work opposite our normal understanding of distance in space, it has a different role in the metric of spacetime: $d^2 = x^2 + y^2 + z^2 - t^2$.

This allows the “space-time distance” (called the interval) to be 0, or even negative! Think of some examples of events (points in spacetime are called events) with a positive, negative, and 0 interval from the here and now.

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