Parity

**Lesson Objective:**
The purpose of this lesson is to show students that parity can be used to correspond between numbers and colors. The students play a familiar hat problem where each student in a line tries to guess at the color of the hat on their head. They discuss strategies for guessing their hat colors and try them out. The idea is for the students to find the optimal strategy. While the lesson only uses two colors, it can be expanded upon to include more to add to the difficulty.

**Mathematical Background:**
Students will only need to understand the difference of even and odd numbers, and multiples of distinct positive integers (i.e. multiples of threes, fours, sevens, etc.).

**Instruction Plan:**
First, please refer to the handouts section, which immediately follows this section. It consists of two pages. The first page details the “game” that is played between the evil villain (sometimes referred to as moderator) and the prisoners. After the demonstration (which will be covered shortly), this page is given to each student. It is recommended that the adults emphasize that there is no “cheating” among the prisoners (i.e. extra sounds, saying the colors in a special way, peeking at their own hat colors, etc.). The second page is a list of successive hints to be given to the students (if necessary), with the last hint being the “code.”

The first part of the lesson is the demonstration. It is best to have at least four adults present for this part, one to play the evil villain, the others to be the prisoners. The moderator can demonstrate with the prisoners as well, once it is clear to the students what the purpose of the game is. The lesson starts with one of the adults being the moderator, or “evil villain.” The moderator explains the scenario and the game (stated in the handout) and introduces his prisoners (played by other adults). The moderator asks for a student volunteer to come up and distribute the hats, so that it appears to not be staged. When the volunteer distributes these hats, they should do so starting at the back of the line, and make sure none of the prisoners see their own hat color, or those of the people behind them. The prisoners play the game. After the first demonstration, it is recommended that the adults do it twice more, each time adding one or two adults to the line of prisoners to make it longer. It is quite alright if the adults make some mistakes so that some prisoners die. The students enjoy this and find it humorous.

After the demonstrations, the students break up into groups to discuss strategies. They spend about 20-30 minutes on this part of the lesson. One of the common strategies they come up with is for the first prisoner to say the color of the prisoner in front of him. Then the second prisoner knows their hat color and says it. The third repeats what the first prisoner did, so the fourth lives. This strategy continues for the rest of the prisoners in line. This guarantees that at least one less than half the prisoners in line will survive. Another strategy is the first prisoner says the color that he sees the most of, and every prisoner says that color also. Lead them with hints if they get stuck. The students should also test out their strategies by getting in lines and trying out the game themselves. They have a lot of fun with this part.
The last 10-15 minutes are devoted to testing out the optimal strategy, whether they have figured it out or not (this part is up to the adults, but they seem to really enjoy trying it out). They will find that it isn’t quite as easy as it seems to keep track of all the hats that are said.

Handouts:
Math Circle – Hats!

The Evil Witch has captured some goblins! She is holding them prisoner, but decides to give them a chance to live. The Witch tells the goblins the following:

“In a few minutes, you will line up, and I will put hats on you. The hats will be red or blue. You will be able to see the hat colors of the goblins in front of you, but you won't be able to see your own hat or the hats of the goblins behind you. Starting at the back of the line, I will ask you to say your hat color. You are allowed to say the single word “red” or “blue” just once, and otherwise, you are not allowed to communicate with each other at all.

If you answer incorrectly, you will die! If you answer correctly, I will reluctantly let you live.

You can take a few minutes to discuss a strategy before I line you up. After that, you may not communicate with each other except to say your hat color.”

What do you think is the best strategy? Can all of the goblins live? Can we guarantee that at least some of them live?

We encourage you to try out your strategy! Line up and do it!
Hints

Here are the hints to give the students if they ask for them or we want to give them when it seems appropriate.

1. The first goblin to speak gives *extremely useful information*, even though they only say one word! They don't just say a color, but are giving a code to the other goblins!

2. The first goblin is the **only** one using the code. This goblin might die, but the rest can all live!

3. Each goblin needs to listen carefully to everyone who speaks before, in order to figure out their hat color. The last goblin to speak doesn't know his hat color until he has heard all the other goblins before him!

4. When the first goblin speaks, the code gives information about the number of red and blue hats he sees. He doesn't just say a color, but tells everyone else how many red and blue hats there are!

5. Here's the code: “red” means “I see an **odd** number of red hats!” “Blue” means “I see an **even** number of red hats!” This is why the first goblin may die, but the rest can live. Try out the code!
Reflection:
This lesson went very well with the students at Burton. We had seven adults present for the
demonstration. The students were interested immediately because of the demonstration and really wanted
to know the trick. They went right into discussing strategies, and were eager to test them out by playing
the game themselves. The only part of the lesson that went a little iffy was leading them towards the
optimal strategy. It took quite a bit of prodding for them to reach the idea of having an even or odd
number of red hats and how that was conveyed by just the two words. Once the students realized the code,
they were eager to try it out themselves. They did it twice. Once was just the students, about seven of
them in a line. The second time included some adults, and there were eleven in the line. On this second
attempt, nine of them got the right hat color! Not only that, but the two that missed were next to each
other in line. We realized that the two mistakes cancelled each other out and allowed everyone else
thereafter to still get their colors right. A good follow up lesson for this might be to do the same lesson,
but to start with three hat colors, then four hat colors, and try to generalize it to as many hat colors as
there are prisoners in line. The follow up lesson hasn’t been tested yet, but we might do it for Math Circle.

This lesson went less well at Lowell. I forgot the cups (we used red and blue party cups as the hats) at
home, so I had to buy new ones. The two types of cups were different colors, but NOT different sizes. We
found that it was important to have the exact same type of cups, but different in color, because the
prisoners often need to hold the cups in place on their heads without looking at them. When the cups are
different sizes, they can easily tell which cups are on their heads. We fixed this situation during the lesson
by using a marker to color some of the red cups a different color.

After we fixed this, the rest of the lesson went pretty well. We only had two adults, but we had about ten
students, split into three groups. They spent about twenty minutes discussing strategies, and then Steven
and I pulled them all together for a group discussion. Through some hints, they got to the concept of even
and odd numbers, and we filled in the gap by clarifying how this related to the two colors. They spent the
last ten minutes testing out the optimal strategy, but I felt they didn’t try as hard as the students at
Burton did. Some of the students answered quickly as if they were guessing right away.

The final school we ran this lesson at was Mission High School. The lesson also went really well here.
There were about 15 students and five or six adults. We did the demonstration twice, then the students
discussed the strategies in about four groups. At Lowell and Burton, the students didn’t come up with the
optimal strategy themselves, but came up with some strategies that were close and help many of the
students in line survive. However, at Mission there were two groups of students that found the optimal
strategy within about 10-15 minutes of discussion. It was astounding how fast they found it. They tested
out the strategy afterwards, and not only had fun, but did well in keeping track of the hat colors called
even with how noisy the room was overall. A third group got extremely close to the optimal strategy at the
end of the lesson, but were just shy of it. Overall, it was a great lesson.

References: