Lévy's Constant

MATH 696/697

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Objective

The goal of this project is to compute Lévy's constant, as explicitly as possible to two or more decimal places by using a triple integral formula found by Cheung and Chevallier through a rectilinear domain. The formula derived involves a seven-dimensional integral; however, in this project we have simplified the complex region into three two-dimensional rectangular regions, hence, the triple integral. We focus on Lévy’s constant in higher dimensions due to our interest in the approximation of both rational and irrational vectors of the same length. Our ultimate goal is to provide a benchmark to assist those who wish to evaluate the Lévy's constant in closed form in the future.

Background

Lévy’s constant (also known as the Khinchin-Lévy constant) was first introduced in 1936 by a French mathematician named Paul Lévy. Its role in mathematics is concerned with approximating irrational numbers. Cheung and Chevallier have also discovered that the Lévy’s constant has an interesting relationship with zeta values. Similar to all constants, Lévy’s constant is a continued fraction. Continued fractions are a sum of two numbers—one of which is a fraction whose denominator is another sum of two numbers in which one is also a fraction whose denominator is another sum, etc.—in which the repeated expansion of the denominator may continue infinitely, generating a never-ending staircase.
That is, any continued fraction takes the form, \( a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} \). Lévy’s constant has the form,

\[ C(\alpha) = \lim_{k \to \infty} \ln \frac{q_k}{k} \]

where \( q_k \) is the denominator of the \( k \)th convergent, i.e. best approximate, of \( \alpha \). Lévy’s Constant is not known in dimensions two and higher, thus we are interested in discovering what the constant is.

**Triple Integral**

The area of interest involves a seven-dimensional cylinder-like object; however, we begin by only focusing on the two-dimensional subpart of the seven-dimensional object. The shaded region below (Diagram a) is the area of interest. We divide the region into omega positive and omega negative (see Diagram b). Our focus region is the omega positive region, in which we divide that region into three smaller subsections (Diagram c).

Diagrams a, b, and c illustrates our area of interest.

Note: All diagrams are not drawn to scale. They are merely an illustration for visual purposes.

We are interested in the two-dimensional aspect of those regions. Therefore, the two-dimensional view of those three regions is as follows:
We begin with the expression,

$$\mu(S) = 4\pi \int_{\Omega_+} da_1 da_2 \int_0^1 db \int_{F(a,b)} dc_1 dc_3 \int_{-\infty}^{\sqrt{1-c_1^2}} \frac{dc_2}{(1-a_1 b_c_2 + a_2 (bc_1 - c_3))}.$$  

Integrating the last integral, we result in the following expression:

$$\mu(S) = 2\pi \int_{\Omega_+} da_1 da_2 \int_0^1 db \left( \frac{1}{1-a_1 b} \int_{F(a,b)} \frac{dc_1 dc_3}{\Xi^2} \right),$$  

where $\Xi = (1 - a_1 b) \sqrt{1 - c_1^2} - a_2 (bc_1 - c_3)$.

Applying Green’s Theorem to $\frac{1}{\Xi^2} = \frac{\partial Q}{\partial c_1} - \frac{\partial P}{\partial c_3}$ we result in $Q=0$ and $P = \frac{1}{a_2 \Xi}$. $Q=0$ because it is a constant of the vertical segment; therefore, the derivative of any constant is zero implies $Q=0$. After Green’s Theorem, we achieve the following expression:

$$\mu(S) = \int_{\Omega_+} da_1 da_2 \int_0^1 db \left( \frac{2\pi}{a_2 (1-a_1 b)} \int_{\partial F(a,b)} \frac{dc_1}{\Xi} \right).$$  

After substituting $c_1 = \frac{2\tau}{1+\tau^2}$ and $dc_1 = \frac{2(1-\tau^2)}{(1+\tau^2)^2} d\tau$, and using partial fractions, the resulting term is

$$\mu(S) = \int_{\Omega_+} da_1 da_2 \int_0^1 db \frac{4\pi}{(1-a_1 b)^2 + a_2^2 \tau^2} \int_{\partial F(a,b)} \frac{(-c_3) d\tau}{1-a_1 b a_2 c_3 - 2a_2 b \tau - (1-a_1 b - a_2 c_3) \tau^2}.$$  

These areas are denoted as the $F(a,b)$ region, where its area is $1-a_1 b$. 
Finally, we achieve the following triple integral that is to be computed by Octave:

$$
\mu(S) = \int_{\Omega^+} da_1 da_2 \int_0^1 db \sum_v \frac{(-1)^v 2\pi c_3}{(1-a_1 b)^2 + a_2^2 b^2} \ln \left( \frac{(\phi_+ - 2a_2 b \tau_+ - \phi_{-\tau}^2)^2}{(\sqrt{D} + a_2 b + \phi_{-\tau_+})^2} \right).
$$

**Octave**

In order to solve the triple integral, $\mu(S)$, in Octave, we created ten functions. Alpha, beta, delta, kappa, and tau are the variable functions.

$$
\alpha = \frac{a_2 b}{1-a_1 b-a_2 c_3}, \beta = \frac{1-a_1 b+a_2 c_3}{1-a_1 b-a_2 c_3}, \delta = \alpha^2 + \beta, \kappa(a) = \frac{a_1}{2} - a_2 \sqrt{|a|^2 - \frac{1}{4}}, \text{ and}
$$

$$
\tau = \begin{cases} 
\frac{1}{c_1} - \sqrt{\frac{1}{c_1^2} - 1} & c_1 > 0 \\
0 & c_1 = 0 \\
\frac{1}{c_1} + \sqrt{\frac{1}{c_1^2} - 1} & c_1 < 0
\end{cases}
$$

We created one function for the movement along the horizontal edges called cor3, and one function for the movement along the vertical edges called cor1. We set up the functions using switch cases. Both functions’ inputs are the coordinates from $F(a, b)$, corners $(\nu, v)$, and the region of interest, but with the difference in the coordinates. For example, cor3’s input is $b$, which the y coordinate of $F(a, b)$; therefore, we move along the horizontal edges as travel to and from the same y coordinate. The two functions return a value.

We created a function, fp, for practical purposes for Octave to evaluate. Function fp is the paired terms with the same $c_3$ that are joined by the horizontal segment on $\partial F(a, b)$.

$$
fp = \frac{2\pi c_3 / \sqrt{D}}{(1-a_1 b)^2 + a_2^2 b^2} \ln \left\{ \left( \frac{\phi_+ - 2a_2 b \tau_+ - \phi_{-\tau}^2}{\sqrt{\phi_+ - 2a_2 b \tau_+ - \phi_{-\tau}^2}} \right)^2 \right\}
$$

The following table provided by Dr. Cheung, illustrates fp.
c₃ illustrates the \( b \) coordinates, and \( c_1 \) illustrates the \( a \) coordinates. The function \( f_p \) returns values that are passed to the \( s_e r_a \) function.

Finally, we created a function to perform all of the calculations, and called it \( s_e r_a \). The \( s_e r_a \) function takes inputs \( \lambda (\lambda) \) — the area we integrate over, \( r \)—polar values, and \( b \)—the region of interest. In order to calculate the region, we converted our triple integral into polar coordinates. Let \( a_1 = r \cos \theta \), and \( a_2 = r \sin \theta \). Given the \( \Omega^\pm \) boundaries, we have computed the following polar coordinates:

\[
\Omega^+_I: |a - 1|^2 = 1
\]
\[
(a_1 - 1)^2 + a_2^2 = 1
\]
\[
a_1^2 - 2a_1 + 1 + a_2^2 = 1
\]
\[
a_1^2 + a_2^2 = 2a_1
\]
\[
r^2 = 2r \cos \theta
\]
\[
r = 2 \cos \theta
\]
\[
\therefore \theta^* = \cos^{-1} \frac{r}{2}
\]
\[
\Omega^+_I: |a - \xi|^2 = 1
\]
\[
(a_1 - \frac{1}{2})^2 - (a_2 - \frac{\sqrt{3}}{2})^2 = 1
\]

\[
a_1^2 + a_2^2 - a_1 - \sqrt{3}a_2 = 0
\]

\[
r^2 = a_1 + \sqrt{3}a_2 = r\cos\theta + \sqrt{3}r\sin\theta
\]

\[
\frac{r}{2} = \frac{1}{2} \cos\theta + \frac{\sqrt{3}}{2} \sin\theta
\]

\[
r = 2 \cos \left(\theta - \frac{\pi}{3}\right)
\]

\[
\therefore \theta^* = \frac{\pi}{3} + \cos^{-1} \frac{r}{2}
\]

\[
\Omega_{III}: |a - \xi|^2 = 1
\]

\[
(a_1 + \frac{1}{2})^2 - (a_2 + \frac{\sqrt{3}}{2})^2 = 1
\]

\[
a_1^2 + a_2^2 = -a_1 - \sqrt{3}a_2
\]

\[
r = -\cos\theta - \sqrt{3}\sin\theta
\]

\[
r = 2\cos \left(\frac{4\pi}{3} - \theta\right)
\]

\[
r = 2 \cos \left(\theta - \frac{\pi}{3}\right)
\]

\[
\therefore \theta^* = \frac{4\pi}{3} - \cos^{-1} \frac{r}{2}
\]

The upper and lower bounds for the polar coordinates are as follows:

\[
\frac{\pi}{3} \leq \cos^{-1} \frac{r}{2} \leq \frac{\pi}{3} + \cos^{-1} \frac{r}{2} \leq \frac{4\pi}{3} - \cos^{-1} \frac{r}{2} \leq \pi
\]

The difference in the polar regions are computed by \(r(\theta_2(r) - \theta_1(r))\).

\[
\Omega_I^* : \frac{\pi}{3} + \cos^{-1} \frac{r}{2} - \cos^{-1} \frac{r}{2} = \frac{\pi}{3}
\]
The final $\theta$ that is inputted into $a_1$ and $a_2$ is computed by

$$\theta = \theta^* + \lambda \ast (\text{difference in polar regions}).$$

The sera function also utilizes the switch case function to compute the function $fp$ based on the given region. Within the sera function, $kappa (\kappa)$ is converted into a complex number. The sera function returns a value that is to be computed in the triple integral.

In an updated code, Cheung and Chevallier have discovered that the sera function could be further simplified to only depend on $tau (\tau)$ instead of $kappa (\kappa)$, thus making the function much simpler. The function continues to return a value that is to be computed in the triple integral. However, we are able to achieve a value that is a better approximate with the reformed function.

Results

Using the functions, $sf=(x, y, z)$, $sera(x, y, z, \omega \text{ region that we are interested in—either 1,2,or 3 })$, and $\text{triplequad}(sf, 0, 1, 0, 1, 0, 1, \text{tolerance level})$. After running the functions, we summed over the three regions using tolerance levels of one, two, or three decimal places of accuracy, and resulted in $3.44609, 3.44567, 3.44775$. The average of these three results is $3.4465$. Using Borel-Cantelli’s statement, $C = dC^*$ for $d=2$, $C = 2C^* = 2 \frac{\zeta(2) \zeta(3)}{\mu(S)}$, for $\zeta(2) = 1.2020569031 \ldots$ and $\zeta(3) = 1.649340668 \ldots$ with $\frac{1}{2} \mu(S) = 3.4465$, we yield $C \approx 0.5738$. However, Cheung and Chevallier recently discovered that with further simplifications and by altering the sera and $tau$ functions, $\mu(S) = 3.49277 \ldots$ Therefore, we conclude that Lévy’s constant in dimension two is $1.13525 \ldots$