ESTIMATING THE FRACTAL DIMENSION OF SETS DETERMINED BY NONERGODIC PARAMETERS

A thesis submitted to the faculty of

San Francisco State University

In partial fulfillment of

The Requirements for

The Degree

Master of Arts

In

Mathematics

by

Joseph Paul Squillace

San Francisco, California

May 2014

CERTIFICATION OF APPROVAL

I certify that I have read *Estimating the Fractal Dimension of Sets Determined by Nonergodic Parameters* by Joseph Paul Squillace, and that in my opinion this work meets the criteria for approving a thesis submitted in partial fufillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

Yitwah Cheung
Associate Professor of Mathematics

Eric Hayashi
Professor of Mathematics

Professor of Mathematics

ESTIMATING THE FRACTAL DIMENSION OF SETS DETERMINED BY NONERGODIC PARAMETERS

Joseph Paul Squillace
San Francisco, California
2014

In 1969, Veech introduced two subsets $K_1(\theta)$, $K_0(\theta)$ of $\mathbb{R}/\mathbb{Z}$ which are defined in terms of the continued fraction representation of $\theta \in \mathbb{R}/\mathbb{Z}$. These subsets are known to give information about the dynamics of certain skew products of the unit circle. We show that the Hausdorff dimension of the sets $K_i(\theta)$ can achieve any value between 0 and 1, inclusive.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee   Date

TABLE OF CONTENTS

LIST OF FIGURES

1 INTRODUCTION

Let $\{x : 0 \leq x < 1\}$ be the compact group of real numbers modulo 1. Any irrational element θ in this set has a continued fraction representation

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $(a_i)_{i=1}^{\infty}$ is a sequence of positive integers known as the *terms* or *partial quotients* of the continued fraction. Take $\left(\frac{m_k}{n_k}\right)_{k=1}^{\infty}$ to be the sequence of *principal convergents* of θ; that is,

$$\frac{m_k}{n_k} = [0; a_1, a_2, \ldots, a_k]$$

for each positive integer k. From the theory of continued fractions, we have

$$n_{k+1} = a_{k+1} n_k + n_{k-1} \tag{1}$$

for $k \in \mathbb{N}$, with $n_0 := 1$ and $n_1 = a_1$. Let $(b_k)_{k \in \mathbb{N}}$ be a sequence of even integers satisfying $|b_j| \leq a_{j+1}$ for each $j \in \mathbb{N}$. Given any integer m,

$$\langle m; b_1, b_2, \ldots \rangle_\theta := \left(m\theta + \sum_{j=1}^{\infty} b_j n_j \theta \right) \bmod 1$$

defines a point of $\{x : 0 \leq x < 1\}$. We call such a representation $\langle \cdot \rangle_\theta$ a $\theta-$*expansion.* From now on, we omit the “mod 1” notation. Veech [7] defines the sets

$$K_0(\theta) := \{\langle m; b_1, b_2, \ldots \rangle_\theta : b_j \text{ is eventually even}, \lim_{j\to\infty} b_j n_j \|n_j \theta\| = 0\}$$

and

$$K_1(\theta) := \left\{ \langle m; b_1, b_2, \ldots \rangle_\theta : b_j \text{ is eventually even}, \sum_{j=1}^{\infty} |b_j| \, n_j \, \|n_j \theta\| < \infty \right\}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. Clearly, $K_1(\theta) \subset K_0(\theta)$. In Section 2 we find an upper bound formula for the Hausdorff dimension of $K_0(\theta)$. In Section 4.1 we apply a result from [5] to obtain a lower bound formula for the Hausdorff dimension of $K_1(\theta)$. Using these formulas,

in Section 5 we show that for any $\delta \in [0, 1]$ we can construct a number θ such that the Hausdorff dimension of $K_i(\theta)$ equals δ, where $i = 0, 1$.

1.1 Veech's Examples and a Billiard Interpretation of these Examples

In [7], Veech constructs examples of minimal and not uniquely ergodic dynamical systems as follows (see [6]). Take two copies of the unit circle and mark off segment a J of length $2\pi\alpha$ in the counterclockwise direction on each one with endpoint at 0. Now choose an irrational θ and consider the following dynamical system. Start with a point p in the first circle. Rotate counterclockwise by $2\pi\theta$ repeatedly until the the orbit lands in J; then switch to the corresponding point in the second circle, rotate by $2\pi\theta$ until the first time the point lands in J; switch back to the first circle and so forth. Veech showed there exists irrational α for which this system is minimal and the Lebesgue measure is not uniquely ergodic. We may describe Veech's dynamical system by a flow on a surface arising from a billiard. Consider billiards in the table formed by a $\frac{1}{2} \times 1$ rectangle with a horizontal barrier of length α with one end touching at the midpoint of a vertical side. We can identify the top half of the table as the positive side and the bottom half as the negative side.

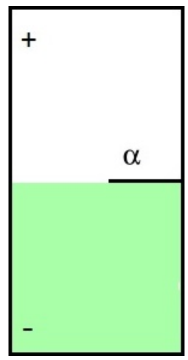

Figure 1: A Billiard table B_α with a barrier of length α.

A standard unfolding of this billiard table is shown in Figure 2. We can view the new figure as having two (identified) barriers of length 2α.

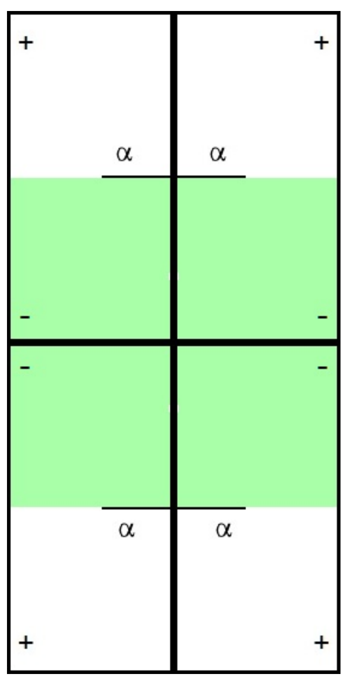

Figure 2: Unfolding of Billiard Table B_α

(See [6]) If we consider the billiard flow in this rectangle with direction θ, there is an unfolding of this billiard such that the restriction of the flow to the map of first return to the horizontal barrier is equivalent to the function described in the examples given by Veech. In [4], Cheung, Hubert, and Masur showed that if we fix the parameter α then the dimension of the set of all θ for which this flow is not ergodic is either 0 or $\frac{1}{2}$. Specifically, Cheung, Hubert, and Masur show that the dimension is $\frac{1}{2}$ if and only if

$$\sum_{k\in\mathbb{N}} \frac{\log\log q_{k+1}}{q_k} < \infty$$

where $\{q_k\}_{k\in\mathbb{N}}$ is the sequence of denominators of the continued fraction representation of α. Our goal is to consider what happens in the case in which we fix θ and allow α to vary.

Moreover, the sets $K_1(\theta)$ and $K_0(\theta)$ are known to give information (see [3]) about the dynamics of Veech's examples described above.

2 UPPER BOUND ON HAUSDORFF DIMENSION

For $A \subset \mathbb{R}^n$, we denote by $\mathrm{Hdim}A$ the *Hausdorff dimension* of A (see [4]). In Section 2.1 we give an upper bound for $\mathrm{Hdim}K_0(\theta)$ by applying what Cheung [2] has defined as a *self-similar covering* to subsets which we call $K_0^i(\theta)$ of $K_0(\theta)$. In particular, the self-similar covering will give an upper bound for $\mathrm{Hdim}K_0^i(\theta)$, and we show that this upper bound is also an upper bound for $\mathrm{Hdim}K_0(\theta)$.

The following definition is motivated by the fact that (i) Veech showed in [7] that if θ has bounded partial quotients (i.e., $\sup_{j\in\mathbb{N}} a_j < \infty$) then $\mathrm{Hdim}K_0(\theta) = 0$, and

(ii) Lothar Narins conjectured that $\mathrm{Hdim} K_1(\theta) = 1$ when $a_j \to \infty$.

Definition. Let $M \in \mathbb{N}$. An irrational θ with unbounded partial quotients is *divergent relative to* $[1, M]$ if the sequence of partial quotients formed by the terms that are greater than M diverges to ∞.

If θ is divergent relative to $[1, M]$, define

$$\begin{aligned} \kappa^\theta &:= \{k_i : a_{k_i+1} > M\} \\ k_0 &:= \min \kappa^\theta \end{aligned}$$

Theorem 1. Let θ be divergent relative to $[1, M]$, and let $(k_i)_{i=0}^\infty$ enumerate the values of k satisfying $a_{k+1} > M$. Then

$$\mathrm{Hdim} K_0(\theta) \le \limsup_{i\to\infty} \frac{\log a_{k_i+1}}{\log n_{k_{i+1}} - \log n_{k_i}} \tag{2}$$

2.1 Reduction

We reduce computing an upper bound of $\mathrm{Hdim} K_0(\theta)$ directly by introducing new sets $K_0^i(\theta)$ for nonnegative integers i. We will show that this suffices for finding an upper bound of $\mathrm{Hdim} K_0(\theta)$.

Definition. Suppose θ is divergent relative to $[1, M]$.

$$K_0^i(\theta) := \left\{ \langle m; b_1, b_2, \ldots \rangle_\theta \in K_0(\theta) : j > k_i \Rightarrow |b_j|\, n_j \, \|n_j \theta\| < \frac{1}{4M} \right\}$$

The sets $K_0^i(\theta)$ are nonempty for all sufficiently large i since, by definition of $K_0(\theta)$,

$$\lim_{j\to\infty} |b_j|\, n_j \, \|n_j\theta\| \quad = \quad 0$$

Further, the next three lemmas along with the fact that we can find an upper bound for $\mathrm{Hdim}K_0^i(\theta)$ that does not depend on i will furnish an upper bound for $\mathrm{Hdim}K_0(\theta)$.

Lemma 1.

$$K_0(\theta) \subset \bigcup_{i=0}^{\infty} K_0^i(\theta)$$

Proof. If $y \in K_0(\theta)$, then there is a sequence $m, b_1, b_2, \ldots$ such that $y = \langle m; b_1, b_2, \ldots \rangle_\theta$, where $b_j n_j \|n_j\theta\|$
$\to 0$. Hence, there is an i for which $j > k_i$ implies $b_j n_j \|n_j\theta\| < \frac{1}{4M}$. Therefore $y \in K_0^i(\theta)$.

∎

Lemma 2. *(Countable Stability)*

$$\mathrm{Hdim} \bigcup_{i=0}^{\infty} K_0^i(\theta) = \sup\left\{\mathrm{Hdim}K_0^i(\theta) : i \geq 0\right\}$$

Proof. For any set $F \subset \mathbb{R}^n$ and $s \geq 0$, denote by $\mathcal{H}^s(F)$ the *s-dimensional Hausdorff measure* of F (see [5]). $\mathrm{Hdim}K_0^i(\theta) \leq \mathrm{Hdim} \bigcup_i K_0^i(\theta)$ for each i, so $\sup_i \mathrm{Hdim}K_0^i(\theta) \leq$

Hdim $\bigcup_i K_0^i(\theta)$. Conversely, let $s = \sup_i \text{Hdim} K_0^i(\theta)$. It suffices to show $\mathcal{H}^{s+\varepsilon} \bigcup_i K_0^i(\theta) = 0$ when $\varepsilon > 0$. Let $\delta > 0$. For each i we can cover $K_0^i(\theta)$ by intervals A_{i_j} such that the sum of their radii by the power $s+\varepsilon$ is less than $2^{-i}\delta$. The union of all intervals A_{i_j}, over i and j, covers $\bigcup_i K_0^i(\theta)$ and the sum of their radii raised by the power $s+\varepsilon$ is less than δ. Therefore, $\mathcal{H}^{s+\varepsilon} \bigcup_i K_0^i(\theta) = 0$.

■

Lemma 3. *There exists an* i_0 *such that if* $j \geq k_{i_0}$, $j \notin \kappa^\theta$ *and* $|b_j| n_j \|n_j\theta\| < \frac{1}{4M}$, *then* $b_j = 0$.

Proof. Take i_0 large enough such that $k_{i_0} \geq \min k^\theta$. Since $j \notin \kappa^\theta$, $a_{j+1} \leq M$. Hence

$$\begin{aligned} \frac{1}{4M} &> |b_j| n_j \|n_j\theta\| \\ &> \frac{|b_j| n_j}{2n_{j+1}} \\ &> \frac{|b_j|}{4a_{j+1}} \end{aligned}$$

Hence, $a_{j+1} > M|b_j|$. Since $a_{j+1} < M$, $b_j = 0$.

■

A consequence of Lemma 3 is $x \in K_0^{i_0}(\theta)$ has a $\theta-$expansion $\langle m; b_1, b_2, \ldots \rangle$ such that for all $j > k_{i_0}$ either $a_{j+1} > M$ or $b_j = 0$.

2.2 Specification of Self-Similar Cover

Given $|b_i| \le a_{i+1}$ and an even b satisfying $|b| \le a_{k+1}$, define by

$$I\left(m; b_1, b_2, \ldots, b_{k-1}, b\right)$$

the *interval* of length $\frac{8}{n_k}$ centered at

$$m\theta + \sum_{j=1}^{k-1} b_j \left\|n_j\theta\right\| + b\left\|n_k\theta\right\|$$

Lemma 4. $x \in I\left(m; b_1, \ldots b_k\right)$ for any $x \in K_0\left(\theta\right)$.

Proof. Let $x = \langle m; b_1, \ldots\rangle_\theta \in K_0\left(\theta\right)$. Since θ is between $\frac{m_k}{n_k}$ and $\frac{m_{k+1}}{n_{k+1}}$, it follows that $\left|\frac{m_k}{n_k} - \theta\right| \le \left|\frac{m_k}{n_k} - \frac{m_{k+1}}{n_{k+1}}\right|$. Multiplying by n_k gives $|m_k - n_k\theta| \le \left|m_k - n_k\frac{m_{k+1}}{n_{k+1}}\right|$. Therefore,

$$\begin{aligned} \left\|n_k\theta\right\| &\le |m_k - n_k\theta| \\ &\le \left|m_k - n_k\frac{m_{k+1}}{n_{k+1}}\right| \\ &= \frac{1}{n_{k+1}} \end{aligned}$$

We also note that since $n_{k+2} > 2n_k$ we have $\frac{1}{n_{k+1}} > \frac{2^i}{n_{k+(2i+1)}}$ and $\frac{1}{n_{k+2}} > \frac{2^i}{n_{k+(2i+2)}}$. Therefore, $\sum_{i=0}^{\infty} \frac{1}{n_{k+(2i+1)}}$
$< \sum_{i=0}^{\infty} \frac{1}{2^i n_{k+1}}$, which is a geometric series that simplifies to $\frac{2}{n_{k+1}}$. Similarly, $\sum_{i=1}^{\infty} \frac{1}{n_{k+(2i)}} < \frac{2}{n_{k+2}}$.

Without loss of generality, suppose $m = 0$. Then

$$\begin{aligned}
\left| x - \sum_{i=1}^{k} b_i \left\| n_i \theta \right\| \right| &= \left| \sum_{j=k+1}^{\infty} b_j \left\| n_j \theta \right\| \right| \\
&\leq \sum_{j=k+1}^{\infty} \left| b_j \right| \left\| n_j \theta \right\| \\
&\leq \sum_{j=k+1}^{\infty} \frac{\left| b_j \right|}{n_{j+1}} \\
&\leq \sum_{j=k+1}^{\infty} \frac{a_{j+1}}{n_{j+1}} \\
&\leq \sum_{j=k+1}^{\infty} \frac{1}{n_j} \\
&= \sum_{i=0}^{\infty} \frac{1}{n_{k+(2i+1)}} + \sum_{i=1}^{\infty} \frac{1}{n_{k+2i}} \\
&< \frac{2}{n_{k+1}} + \frac{2}{n_{k+2}} \\
&\leq \frac{4}{n_{k+1}}
\end{aligned}$$

Therefore, $x \in I\,(m; b_1, \ldots b_k)$.

■

Definition. (Section 3 of [2]) Let X be a metric space and J a countable set. Given $\sigma \subset J \times J$ and $\alpha \in J$ we let $\sigma(\alpha)$ denote the set of all $\alpha' \in J$ such that $(\alpha, \alpha') \in \sigma$. We say a sequence $(\alpha_j)_{j \in \mathbb{N}}$ of elements in J is *σ-admissible* if $\alpha_{j+1} \in \sigma(\alpha_j)$ for all $j \in \mathbb{N}$; and we let J^{σ} denote the set of all σ-admissible sequences in J. By a *self-similar covering* of X we mean a triple $(\mathcal{B}, J, \sigma)$ where $\mathcal{B}$ is a collection of bounded subsets of X, J a countable index set for $\mathcal{B}$, and $\sigma \subset J \times J$ such that there is a map $\mathcal{E} : K_0(\theta) \to J^{\sigma}$ that assigns to each $x \in X$ a σ-admissible sequence $\left(\alpha_j^x\right)_{j \in \mathbb{N}}$ such that for all $x \in X$

(i) $\bigcap_{j=1}^{\infty} B\left(\alpha_j^x\right) = \{x\}$, and

(ii) diam $B\left(\alpha_j^x\right) \to 0$ as $j \to \infty$, where $B(\alpha)$ denotes the element of $\mathcal{B}$ indexed by α.

2.3 A Self-similar covering of $K_0^{i_0}(\theta)$

We have access to a self-similar covering of $K_0^{i_0}(\theta)$. Define

$$J := \left\{ (m; b_1, b_2, \ldots, b_{k-1}) : m \in \mathbb{Z},\ k \in \kappa^\theta,\ b_j \in \mathbb{Z}\,(1 \le j \le k),|b_j|\le a_{j+1}, \right.$$

$$\left. \text{if and } b_j = 0 \text{ if both } k_{i_0} < j < k, j \notin \kappa^\theta \right\}$$

$$\mathcal{B} := \{I(\beta) : \beta \in J\}$$

and define $\sigma \subset J \times J$ such that for each $\alpha_{k_i} = (m; b_1, b_2, \ldots, b_{k_i - 1}) \in J$ we have

$$\sigma(\alpha_{k_i}) = \left\{ \left(m; b_1, b_2, \ldots, b_{k_{i+1}-2}, b\right) : |b| \le a_{k_{i+1}} \right\}$$

[1]Let J^σ denote the set of all σ-admissible sequences in J. By Lemma 4 we can define

$$\begin{aligned} \mathcal{E} : K_0(\theta) &\to J^\sigma \\ x &\mapsto \left(\alpha_j^x\right)_{j=0}^{\infty} \end{aligned}$$

where $x \in I(\alpha_j) = I(m; b_1, \ldots, b_{j-1}) \in J_0$ for each $\alpha_j \in \left(\alpha_j^x\right)$. Suppose $x \in K_0^{i_0}(\theta)$. Our triple satisfies (i) of the definition of a self-similar covering; apply Lemma 4 to

[1]As mentioned immediately after Theorem 3.1 of [2], we can take elements of $\mathcal{B}$ to be subsets of the ambient space X.

show $x \in I(\beta_j)$ for all $\beta_j \in J_0$. Since $\lim_{j\to\infty} \operatorname{diam} I\left(\alpha_j^x\right) = 0$, $\bigcap_{j=0}^{\infty} I\left(\alpha_j^x\right) = \{x\}$. Thus, (ii) is also satisfied.

Define

$$E_i := \bigcup I\left(m; b_1, \ldots, b_{k_i-1}\right)$$

such that the union is over all finite sequences $m, b_1, \ldots, b_{k_i-1}$, where $k_i \in \kappa^\theta$ and $|b_j| \leq a_{j+1}$ for each j, and define

$$E := \bigcap_{i=0}^{\infty} E_i$$

2.4 Calculation

In this section we give an upper bound on $\mathrm{Hdim} K_0^{i_0}(\theta)$. The following lemma is a direct consequence of the definition of σ.

Lemma 5. Let $\alpha \in J$ and $k = |\alpha|$.

$$\#\sigma(\alpha) \leq a_{k+1} \tag{3}$$

Proof. Let α be a sequence in J of length k_i. Then $\alpha = (m; b_1, \ldots, b_{k_i-1})$ where both $j > k_{i_0}$ and $j \notin \kappa^\theta$ imply $b_j = 0$. Since $|\alpha| = k$, α corresponds to an interval belonging to E_i. Hence, the centers determined by the children of the intervals comprising E_i are determined by all even integers b_k satisfying $b_k \leq a_{k_i+1}$.

■

For $s \geq 0$, since the value of $|I(\beta)|$ does not depend on the choice of $\beta \in \sigma(\alpha)$, a direct consequence of equation (3) is

$$\sum_{\beta \in \sigma(\alpha)} \left(\frac{|I(\beta)|}{|I(\alpha)|} \right)^s = \#\sigma(\alpha) \left(\frac{|I(\beta)|}{|I(\alpha)|} \right)^s \tag{4}$$

Further, for $\beta \in \sigma(\alpha)$

$$\#\sigma(\alpha) \left(\frac{|I(\beta)|}{|I(\alpha)|} \right)^s \leq 1 \quad \Longleftrightarrow \quad \#\sigma(\alpha) \left(\frac{n_{k_i}}{n_{k_{i+1}}} \right)^s \leq 1$$

The critical value of s is $s = \frac{\log \#\sigma(\alpha)}{\log n_{k_{i+1}} - \log n_{k_i}}$. Let $\varepsilon > 0$, and let $s' = s + \varepsilon$. Then

$$\begin{aligned} \sum_{\beta \in \sigma(\alpha)} \left(\frac{|I(\beta)|}{|I(\alpha)|} \right)^{s'} \leq 1 \quad & \Longleftrightarrow \quad \left(\frac{n_{k_i}}{n_{k_{i+1}}} \right)^{s'} \#\sigma(\alpha) \leq 1 \qquad \text{(by (4))} \\ & \Longleftrightarrow \quad \left(\frac{n_{k_i}}{n_{k_{i+1}}} \right)^{\varepsilon} \leq 1 \end{aligned}$$

Theorem 5.3 of [1] implies $\mathrm{Hdim} K_0(\theta) \leq s'$. Since ε can be arbitrarily small, we have $\mathrm{Hdim} K_0(\theta) \leq s$. Hence

$$\begin{aligned} \mathrm{Hdim} K_0(\theta) & \leq \limsup_{i \to \infty} \frac{\log \#\sigma(\alpha)}{\log n_{k_i+1} - \log n_{k_i}} \\ & = \limsup_{i \to \infty} \frac{\log a_{k_i+1}}{\log n_{k_{i+1}} - \log n_{k_i}} \qquad \text{(by (3))} \end{aligned}$$

■

3 A NONTRIVIAL EXAMPLE OF θ SATISFYING Hdim$K_0(\theta) = 0$

In [7], Veech showed that the Hausdorff dimension of $K_i(\theta)$ vanishes when θ has bounded partial quotients. Using our upper bound formula (2), we give an example for which $\sup_j a_j = \infty$ and $\mathrm{Hdim}K_0(\theta) = 0$. Let $0 < \delta \leq 1$ be given and fix $M = \left\lceil 2^{\frac{1}{\delta}} \right\rceil$. We specify the continued fraction representation of θ recursively. We choose $a_1, a_2, \ldots, a_{k_0} = M$, where k_0 is the smallest index for which $n_{k_0}^{\delta} > \max\left\{2M, \frac{1}{2^{\delta}-1}\right\}$. Note that there exists an integer in between $n_{k_0}^{\delta}$ and $2^{\delta} n_{k_0}^{\delta}$ since $n_{k_0}^{\delta}\left(2^{\delta}-1\right) > 1$. Choose $a_{k_0+1} \in \mathbb{Z}$ such that $n_{k_0}^{\delta} < a_{k_0+1} < 2n_{k_0}^{\delta}$. Recursively, given k_i, define

$$k_{i+1} \coloneqq k_i + 1 + \left\lceil \frac{\delta i \log n_{k_i}}{\log M} \right\rceil \tag{5}$$

set $a_{k_i+2} = a_{k_i+3} = \cdots = a_{k_{i+1}} = M$, and choose a_{k_i+1} such that $n_{k_i}^{\delta} < a_{k_i+1} < 2n_{k_i}^{\delta}$. This completes the recursive definition of the sequence $(a_{k_i})_{i \geq 0}$. Moreover, a direct consequence of the definition of (5) is

$$M^{k_{i+1}-k_i-1} \geq n_{k_i}^{\delta i} \tag{6}$$

Claim. Under the above construction, θ is divergent relative to $[1, M]$.

Proof. By construction of $(a_{k_i+1})_{i \geq 0}$, we have, for each $i \geq 0$, $a_{k_i+1} > M$ and $n_{k_i}^{\delta} < a_{k_i+1} < 2n_{k_i}^{\delta}$. Therefore a_{k_i+1} diverges to ∞.

■

Theorem 2. *Let θ be constructed as above. Then* $\mathrm{Hdim}K_0(\theta) = 0$.

Proof.

$$\begin{aligned}
n_{k_{i+1}} &> a_{k_{i+1}} n_{k_{i+1}-1} \\
&= M n_{k_{i+1}-1} \\
&> M a_{k_{i+1}-1} n_{k_{i+1}-2} \\
&= M^2 n_{k_{i+1}-2} \\
&\vdots \\
&= M^{k_{i+1}-k_i-1} n_{k_i} \\
&> n_{k_i}^{\delta i} n_{k_i} \qquad \text{(by (6))} \\
&= n_{k_i}^{\delta i+1}
\end{aligned}$$

Therefore, $\log n_{k_{i+1}} > (\delta i + 1) \log n_{k_i}$, and this implies

$$\lim_{i\to\infty} \frac{\log n_{k_{i+1}}}{\log n_{k_i}} = \infty \tag{7}$$

Since θ is half-divergent,

$$\begin{aligned}
\text{Hdim} K_0(\theta) &\leq \limsup_{i\to\infty} \frac{\log a_{k_i+1}}{\log n_{k_{i+1}} - \log n_{k_i}} \qquad \text{(by (2))} \\
&\leq \limsup_{i\to\infty} \frac{\log 2 n_{k_i}^{\delta}}{\log n_{k_{i+1}} - \log n_{k_i}} \\
&\leq \limsup_{i\to\infty} \frac{\delta \log n_{k_i} + \log 2}{\log n_{k_{i+1}} - \log n_{k_i}} \\
&= \limsup_{i\to\infty} \frac{\delta}{\frac{\log n_{k_{i+1}}}{\log n_{k_i}} - 1} \\
&= 0 \qquad \text{(by (7))}
\end{aligned}$$

■

4 EXAMPLE OF θ FOR WHICH Hdim$K_1(\theta) = 1$

4.1 Falconer's Lower Bound Formula

(See [4]) Let $[0,1] = F_0 \supset F_1 \supset F_2 \supset \ldots$ be a decreasing sequence of sets, with each F_k a union of a finite number of disjoint closed intervals (called kth *level basic intervals*), which each interval of F_k containing at least two intervals of F_{k+1}, and the maximum length of kth level intervals tending to 0 as $k \to \infty$. Then the set

$$F = \bigcap_{k=0}^{\infty} F_k$$

is a totally disconnected subset of $[0,1]$. The condition needed to apply Falconer's lower bound formula is

($*$) *Each* $(j-1)$*th level interval contains at least* $m_j \geq 2$ j*th level intervals* $(j = 1, 2, \ldots)$ *which are separated by gaps of at least* γ_j*, where* $0 < \gamma_{j+1} < \gamma_j$ *for each* j*.*

Falconer's lower bound formula is

$$\text{H}dimF \geq \liminf_{k\to\infty} \frac{\log(m_0 m_1 \cdots m_j)}{-\log(m_{j+1}\gamma_{j+1})} \tag{8}$$

4.2 Lower Bound for Hdim$K_1(\theta)$

Fix $\delta \in (0,1)$ and $\varepsilon \in (0,\delta)$. For the remainder of Section 4, let θ be an element for which there exists a k_0 sufficiently large so that $n_{k_0}^{\delta-\varepsilon} \geq 3$, $n_{k_0}^{\varepsilon} \geq 9$ and for all $k \geq k_0$, $n_k^{\delta} < a_{k+1} < 2n_k^{\delta}$, b_k is even and $|b_k| < \lfloor n_k^{\delta-\varepsilon} \rfloor$. Our strategy is to construct an infinite family of Cantor sets contained in $K_1(\theta)$, each of which satisfies the conditions needed to apply Falconer's lower bound formula, allowing us to give lower bounds of $\mathrm{Hdim} K_1(\theta)$ arbitrarily close to 1.

Given $|b_i| \le a_{i+1}$, define by

$$L(m; b_1, b_2, \ldots, b_{k-1})$$

or L_{k-1} the *interval* of length $\|n_{k-1}\theta\|$ concentric with $I(m; b_1, \ldots, b_{k-1})$. We require that for each k the length of the gaps between the intervals L_k is constant. Hence, we have the following lemma.

Lemma 6. *The gaps between consecutive children intervals of L_k are of length $\|n_k\theta\|$.*

Proof. The distance between the centers of adjacent intervals of L_{k+1} is $2\|n_k\theta\|$. Between these centers is the gap between them as well as two half intervals of L_{k+1}. Hence, the size of the gap is $\|n_k\theta\|$.

■

The following lemma gives a sufficient condition for $L_{k+1} \subset L_k$.

Lemma 7. *Suppose we are given $m, b_1, \ldots, b_{k-1}, b$. If $|b| \le \frac{a_{k+1}}{8}$, then*

$$L(m; b_1, \ldots, b_{k-1}, b) \subset L(m; b_1, \ldots, b_{k-1})$$

Proof. We may suppose $a_{k+1} \ge 9$ since the claim is vacuously true otherwise. Let $y \in L(m; b_1, \ldots, b_{k-1}, b)$, and let x be the center of the corresponding parent interval

$L(m; b_1, \ldots, b_{k-1})$. Let us call x' the center of the interval $L(m; b_1, \ldots, b_{k-1}, b)$. Then

$$\begin{aligned} |x-y| &\leq |x'-x| + |x'-y| \\ &\leq |b| \, \|n_k\theta\| + \frac{\|n_k\theta\|}{2} \\ &\leq \frac{\frac{a_{k+1}}{8} + \frac{1}{2}}{n_{k+1}} \\ &< \frac{\left(\frac{a_{k+1}}{4} - \frac{1}{2}\right) + \frac{1}{2}}{n_{k+1}} \qquad \text{(since } a_{k+1} > 4) \\ &= \frac{a_{k+1}}{4n_{k+1}} \\ &< \frac{a_{k+1}}{4a_{k+1}n_k} \\ &= \frac{1}{4n_k} \\ &< \frac{\|n_{k-1}\theta\|}{2} \end{aligned}$$

■

The following result was anticipated by Lothar Narins.

Theorem 3. *The number θ, as constructed in 4.2, satisfies the following:*

$$HdimK_1(\theta) = 1$$

Proof. Define

$$F_j := \bigcup L(m; b_1 \ldots, b_{k_0+j-1})$$

where the union is over all finite sequences such that $m = b_1 = \cdots = b_{k_0-1} = 0$. If $k \geq k_0$, then

$$\begin{aligned}|b_k| &\leq \left\lfloor n_k^{\delta-\varepsilon} \right\rfloor \\ &\leq n_k^{\delta-\varepsilon} \\ &= \frac{n_k^{\delta}}{n_k^{\varepsilon}} \\ &< \frac{a_{k+1}}{8}\end{aligned}$$

so that, by Lemma 7,

$$F_{j+1} \subset F_j$$

for $j \geq 0$.

Define

$$F(k_0, \varepsilon) := \bigcap_{j=0}^{\infty} F_j$$

Lemma 8. $F(k_0, \varepsilon) \subset K_1(\theta)$

Proof. If $y \in F(k_0, \varepsilon)$, then we have a sequence of the form $b_1, b_2, \ldots$ such that $y = \langle m; b_1, b_2, \ldots, b_{k_0}, b_{k_0+1}, \ldots \rangle_\theta$, where $b_1 = \cdots = b_{k_0-1} = 0$ and, by construction, each b_k is even. We show that y satisfies the constraints imposed on elements of $K_1(\theta)$.

$$\begin{aligned}\sum_{j=1}^{\infty} |b_j|\, n_j \|n_j\theta\| &\leq \sum_{j=1}^{\infty} \frac{n_j^{\delta-\varepsilon+1}}{n_{j+1}} \\ &\leq \sum_{j=1}^{\infty} \frac{n_j^{\delta-\varepsilon+1}}{n_j^{1+\delta}} \\ &\leq \sum_{j=1}^{\infty} \frac{1}{n_j^{\varepsilon}}\end{aligned}$$

Using the Ratio Test on the latter series, we have

$$\begin{aligned}\lim_{j\to\infty}\frac{n_j^{\varepsilon}}{n_{j+1}^{\varepsilon}} &\leq \lim_{j\to\infty}\left(\frac{n_j}{a_{j+1}n_j}\right)^{\varepsilon} \\ &\leq \lim_{j\to\infty}\frac{1}{a_{j+1}^{\varepsilon}}=0 \\ &= 0\end{aligned}$$

Therefore, $\sum_{j=1}^{\infty}|b_j|\,n_j\,\|n_j\theta\|<\infty$. Hence $y\in K_1(\theta)$.

■

For convenience, define $k_j := k_0 + j$. We show that $F(k_0,\varepsilon)$ satisfies $(*)$. Denote by m_j the number of children intervals L_{k_j} of F so that m_j counts the number of even integers b satisfying $|b| < \left\lfloor n_{k_j}^{\delta-\varepsilon}\right\rfloor$, so that

$$m_j = \left\lfloor n_{k_j}^{\delta-\varepsilon}\right\rfloor$$

Denote by γ_j the length of the gaps between children intervals of L_{k_j} so that, by Lemma 6,

$$\gamma_j = \left\|n_{k_j}\theta\right\|$$

From basic continued fraction theory,

$$\frac{1}{2n_{k+1}} < \|n_k\theta\| < \frac{1}{n_{k+1}} \tag{9}$$

Since we have $\frac{1}{n_{k_j+2}} < \frac{1}{2n_{k_j+1}}$ for each j, we have $0 < \gamma_{k_j+1} < \gamma_{k_j}$ for each j. By (9), $\lim_{j\to\infty}\gamma_j = 0$. It is the case that each interval $L\left(m; b_1,\ldots,b_{k_j-1}\right)$ contains at least 3 intervals of L_{k_j} and the gaps γ_j decrease monotonically to 0 as $j\to\infty$. Hence, the conditions for Falconer's lower bound formula (inequality 4.7 of [4]) are satisfied.

To simplify our calculation of the lower bound of $\mathrm{Hdim} F(k_0, \varepsilon)$, we use the fact that $k \geq k_0$ implies

$$\begin{aligned} n_{k+1} &< (a_{k+1}+1)\, n_k \\ &< 3n_k^{1+\delta} \end{aligned}$$

Taking log of both sides of the expression $n_{k+1} < 3n_k^{1+\delta}$ gives

$$\log n_{k+1} < \left(1 + \delta + \frac{\log 3}{\log n_{k_0}}\right) \log n_k \tag{10}$$

Further,

$$\begin{aligned} m_{j+1} &= \left\lfloor n_{k_{j+1}}^{\delta-\varepsilon} \right\rfloor \\ &> \frac{n_{k_{j+1}}^{\delta\text{-}\varepsilon}}{2} \end{aligned}$$

and

$$\begin{aligned} \gamma_{j+1} &= \left\| n_{k_{j+1}} \theta \right\| \\ &> \frac{1}{2n_{k_{j+1}+1}} \qquad \text{(by (9))} \\ &> \frac{1}{4a_{k_{j+1}+1} n_{k_{j+1}}} \qquad (\text{since } n_{k+1} < 2a_{k+1} n_k) \\ &> \frac{1}{8n_{k_{j+1}}^{1+\delta}} \end{aligned}$$

imply

$$m_{j+1}\gamma_{j+1} > \frac{1}{16n_{k_{j+1}}^{1+\varepsilon}} \tag{11}$$

Using Falconer's lower bound (8) gives

$$\begin{aligned}
\mathrm{Hdim}F(k_0,\varepsilon) &\geq \liminf_{j\to\infty}\frac{\log(m_0\cdots m_j)}{-\log(m_{j+1}\gamma_{j+1})}\\
&\geq \liminf_{j\to\infty}\frac{(\delta-\varepsilon)\left(\log n_{k_0}+\log n_{k_1}+\cdots+\log n_{k_j}\right)}{(1+\varepsilon)\log n_{k_{j+1}}}\\
&\geq \liminf_{j\to\infty}\frac{\delta-\varepsilon}{1+\varepsilon}\left(\frac{1}{1+\delta+\frac{\log 3}{\log n_{k_0}}}+\cdots+\frac{1}{\left(1+\delta+\frac{\log 3}{\log n_{k_0}}\right)^{j+1}}\right)\\
&= \liminf_{j\to\infty}\frac{\delta-\varepsilon}{1+\varepsilon}\left(\frac{1}{1+\delta+\frac{\log 3}{\log n_{k_0}}}\right)\left(\frac{1-\left(\frac{1}{1+\delta+\frac{\log 3}{\log n_{k_0}}}\right)^{j+1}}{1-\frac{1}{1+\delta+\frac{\log 3}{\log n_{k_0}}}}\right)\\
&= \frac{\delta-\varepsilon}{(1+\varepsilon)\left(\delta+\frac{\log 3}{\log n_{k_0}}\right)}
\end{aligned}$$

$\mathrm{Hdim}F(k_0,\varepsilon)\to 1$ as $\varepsilon\to 0, k_0\to\infty$. Therefore, $\mathrm{Hdim}K_1(\theta)=1$.

■

5 $\mathbf{Hdim}K_i(\theta) = \delta$

Theorem 4. *For any $\delta \in [0,1]$ there is a θ satisfying $HdimK_i(\theta) = \delta$.*

The cases $\delta = 0$ and $\delta = 1$ are handled by Theorem 2 and Theorem 3, respectively. Let us choose $\delta \in (0,1)$. Our strategy is to construct a half-divergent θ in terms of δ in a particular way which gives

$$\delta \le \mathrm{Hdim}K_1(\theta) \le \mathrm{Hdim}K_0(\theta) \le \delta$$

Proof. Suppose $M \in \mathbb{N}_{\ge 3}$. We construct a θ divergent relative to $[1, M]$. In what follows we construct integers a_k in terms of the indices k_i by taking

$$\begin{cases} 3 \le a_{k+1} \le M & \text{if } k \ne k_i, \\ n_k^\delta < a_{k+1} < 2n_k^\delta & \text{if } k = k_i \end{cases} \tag{12}$$

Choose $a_1, a_2, \ldots, a_{k_0} \in [3, M]$, where k_0 is large enough so that $n_{k_0}^\delta > M$. With this choice of k_0, $a_{k+1} > M$ when $k = k_i$; therefore, $a_{k+1} \in [3, M]$ if and only if $k \ne k_i$. Suppose we are given k_i and that $a_1, a_2, \ldots, a_{k_i}, n_1, n_2, \ldots, n_{k_i+1}$ are defined according to (12). Choose k_{i+1} to be the smallest $k \ge k_i + 1$ such that

$$\begin{cases} n_k < n_{k_i}^2, \text{ and set } a_{k+1} \in [3, M] & \text{if } k < k_{i+1}, \\ n_k \ge n_{k_i}^2, \text{ and set } a_{k+1} \in \left(n_k^\delta, 2n_k^\delta\right) & \text{if } k = k_{i+1} \end{cases}$$

By this recursive definition, $n_{k_{i+1}} \geq n_{k_i}^2$ and $n_{k_{i+1}-1} < n_{k_i}^2$. If $k_{i+1} = k_i + 1$, then

$$\begin{aligned} n_{k_{i+1}} &= a_{k_i+1} n_{k_i} + n_{k_i-1} \\ &< 2a_{k_i+1} n_{k_i} \\ &< 4n_{k_i}^{1+\delta} \\ &< (M+1)\, n_{k_i}^2 \end{aligned}$$

If $k_{i+1} > k_i + 1$, then k_{i+1} is not of the form $k_j + 1$ for any $j > i$; for if $j > i$, then $k_j + 1 > k_j \geq k_{i+1}$. Therefore, in this case, we have $a_{k_{i+1}} \in [3, M]$, so

$$\begin{aligned} n_{k_{i+1}} &= a_{k_{i+1}} n_{k_{i+1}-1} + n_{k_{i+1}-2} \\ &< \left(a_{k_{i+1}} + 1\right) n_{k_{i+1}-1} \\ &< (M+1)\, n_{k_i}^2 \end{aligned}$$

Therefore,

$$n_{k_i}^2 \leq n_{k_{i+1}} < (M+1)\, n_{k_i}^2 \tag{13}$$

and

$$n_{k_i}^{\delta} < a_{k_i+1} < 2n_{k_i}^{\delta}$$

Further, it will be used in the upper and lower bound calculations that (13) implies

$$\lim_{j\to\infty} \frac{\log n_{k_j}}{\log n_{k_{j+1}}} = \frac{1}{2} \tag{14}$$

5.1 Upper Bound

As construction in Section 5, θ is divergent relative to $[1, M]$ since the subsequence $(a_{k_i+1})_{i\geq 0}$ of terms of $(a_k)_{k\geq 1}$ which are larger than M also satisfy $n_{k_i}^{\delta} < a_{k_i+1} < 2n_{k_i}^{\delta}$. Therefore, $\lim_{i\to\infty} a_{k_i+1} = \infty$. Hence,

$$\begin{aligned}
\mathrm{Hdim} K_0(\theta) &\leq \limsup_{j\to\infty} \frac{\log a_{k_j+1}}{\log n_{k_{j+1}} - \log n_{k_j}} && \text{(by (2))} \\
&\leq \limsup_{j\to\infty} \frac{\log n_{k_j}^{\delta} + \log 2}{\log n_{k_{j+1}} - \log n_{k_j}} \\
&= \limsup_{j\to\infty} \frac{\delta \frac{\log n_{k_j}}{\log n_{k_{j+1}}}}{1 - \frac{\log n_{k_j}}{\log n_{k_{j+1}}}} \\
&= \delta && \text{(by (14))}
\end{aligned}$$

Therefore, $\mathrm{Hdim} K_0(\theta) \leq \delta$.

5.2 Lower Bound

Let $\varepsilon \in (0, 1)$. Choose $k_0' \geq k_0$ sufficiently large such that $k_i \geq k_0'$ implies $a_{k_i+1}^{\varepsilon} < \frac{a_{k_{i+1}}}{8}$. Define

$$F_j := \bigcup L\left(m; b_1, b_2, \ldots b_{k_0'-1}, \ldots, b_{k_j-1}\right)$$

to be the union over all finite sequences of even terms $b_1, \ldots b_{k_j-1}$ such that if $k = k_i \geq k_0'$ then b_k is even and satisfies $|b_k| < \lfloor a_{k+1}^{\varepsilon} \rfloor$, otherwise $b_k = 0$. WLOG, let $m = 0$. Define

$$F_\varepsilon = F\left(k_0', \varepsilon\right) := \bigcap_{j=0}^{\infty} F_j$$

Lemma 7 implies $F_{j+1} \subset F_j$ for $j \in \mathbb{N}$ since if $k_i \geq k_0'$, then

$$\begin{aligned} |b_{k_i}| &< \left\lfloor a_{k_i+1}^{\varepsilon} \right\rfloor \\ &\leq a_{k_i+1}^{\varepsilon} \\ &< \frac{a_{k_{i+1}}}{8} \end{aligned}$$

Lemma 9. $F_\varepsilon \subset K_1(\theta)$

Proof. Let $y \in F_\varepsilon$. Then we are given a sequence of the form $b_1, b_2, \ldots$ such that $y = \left\langle m; b_1, b_2, \ldots, b_{k_0'}, b_{k_0'+1}, \ldots \right\rangle_\theta$, $|b_k|$ is even and satisfies $|b_k| < \left\lfloor a_{k+1}^{\varepsilon} \right\rfloor$ if $k = k_i > k_0'$ and $b_k = 0$ otherwise. Further,

$$\begin{aligned} \sum_{j=1}^{\infty} |b_j|\, n_j \, \|n_j\theta\| &\leq \sum_{j=1}^{\infty} \frac{a_{j+1}^{\varepsilon} n_j}{n_{j+1}} \\ &\leq \sum_{j=1}^{\infty} \frac{a_{j+1}^{\varepsilon} n_j}{a_{j+1} n_j} \\ &= \sum_{j=1}^{\infty} \frac{1}{a_{j+1}^{1-\varepsilon}} \\ &\leq \sum_{j=1}^{\infty} \frac{1}{n_j^{\delta(1-\varepsilon)}} \end{aligned}$$

■

Proof. Using the Ratio Test on the latter series, we have

$$\begin{aligned} \lim_{j\to\infty} \frac{n_j^{(1-\varepsilon)\delta}}{n_{j+1}^{(1-\varepsilon)\delta}} &\leq \lim_{j\to\infty} \frac{n_j^{(1-\varepsilon)\delta}}{n_{j+1}^{(1-\varepsilon)\delta}} \\ &\leq \lim_{j\to\infty} \frac{1}{a_{j+1}^{(1-\varepsilon)\delta}} \\ &= 0 \end{aligned}$$

Therefore, $\sum_{j=1}^{\infty} |b_j|\, n_j \|n_j\theta\| < \infty$. Hence $y \in K_1(\theta)$.

■

Define

$$M_j := \left\lfloor a_{k_j+1}^{\varepsilon} \right\rfloor$$

so that M_j is a lower bound on the number of intervals in F_{j+1} contained in intervals of F_j.

Define

$$\Gamma_j := \left\| n_{k_j}\theta \right\|$$

so that, by Lemma 6, Γ_j is a lower bound on the gaps between the intervals of F_{j+1}. Since we have $\frac{1}{n_{k_{j+2}}} < \frac{1}{2n_{k_{j+1}}}$ for each j, we have $0 < \Gamma_{j+1} < \Gamma_j$ for each j. Since $\frac{1}{2n_{k_j+1}} < \Gamma_j < \frac{1}{n_{k_j+1}}$ for each j, $\lim_{j\to\infty} \Gamma_j = 0$. It is the case that each interval $L\left(m; b_1, \ldots, b_{k_j-1}\right)$ contains at least 3 intervals of $F_{k_{j+1}}$ and the gaps Γ_j decrease monotonically to 0 as $j \to \infty$. The conditions for Falconer's lower bound formula are satisfied.

To simplify our calculation of the lower bound of HdimF_ε, we use the fact that

$$\begin{aligned}\Gamma_{j+1} &= \left\|n_{k_{j+1}}\theta\right\| \\ &> \frac{1}{2n_{k_{j+1}+1}} && \text{(by (9))} \\ &> \frac{1}{2\left(2a_{k_{j+1}+1}n_{k_{j+1}}\right)} && (n_{k+1} < 2a_{k+1}n_k) \\ &> \frac{1}{8n_{k_{j+1}}^{1+\delta}}\end{aligned}$$

and

$$\begin{aligned}M_{j+1} &= \left[a_{k_{j+1}+1}^{\varepsilon}\right] \\ &\geq \frac{a_{k_{j+1}+1}^{\varepsilon}}{2} \\ &> \frac{n_{k_{j+1}}^{\delta\varepsilon}}{2}\end{aligned}$$

imply

$$M_{j+1}\Gamma_{j+1} > \frac{1}{16n_{k_{j+1}}^{1+\delta-\delta\varepsilon}} \tag{15}$$

Using Falconer's lower bound gives

$$\begin{aligned}\text{Hdim}F &\geq \liminf_{j\to\infty}\frac{\log\left(M_0\cdots M_j\right)}{-\log\left(M_{j+1}\Gamma_{j+1}\right)} \\ &\geq \liminf_{j\to\infty}\frac{\varepsilon\left(\log a_{k_0+1}+\log a_{k_1+1}+\cdots+\log a_{k_j+1}\right)}{(1+\delta-\delta\varepsilon)\log n_{k_{j+1}}} && \text{(by (15))} \\ &\geq \frac{\delta\varepsilon}{1+\delta-\delta\varepsilon}\liminf_{j\to\infty}\left(\frac{\log n_{k_0}+\log n_{k_1}+\cdots+\log n_{k_j}}{\log n_{k_{j+1}}}\right) \\ &\geq \frac{\delta\varepsilon}{1+\delta-\delta\varepsilon}\liminf_{j\to\infty}\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{j+1}}\right) && \text{(by (14))} \\ &= \frac{\delta\varepsilon}{1+\delta-\delta\varepsilon}\end{aligned}$$

$\mathrm{Hdim} F(k_0', \varepsilon) \to \delta$ as $\varepsilon \to 1$. Therefore, $\mathrm{Hdim} K_1(\theta) \geq \delta$.

■

References

[1] Y. Cheung. *Hausdorff Dimension of the Set of Points on Divergent Trajectories of a Homogeneous Flow on a Product Space*. Ergodic Theory and Dynamical Systems, **27** (2007), 65-85.

[2] Y. Cheung. *Hausdorff Dimension of the Set of Singular Pairs*. Annals of Mathematics, **173** (2011), 127-167.

[3] Y. Cheung, A. Eskin. *Slow Divergence and Unique Ergodicity*. Fields Institute Communications, **51** (2007), 213-222.

[4] Y. Cheung, P. Hubert, and H. Masur. *Dichotomy for the Hausdorff Dimension of the Set of Nonergodic Directions.* Inventiones, **183** (2001), 337-383.

[5] K. Falconer. Fractal Geometry: Mathematical Foundations and Applications. Chichester: John Wiley and Sons, 1990. Print.

[6] H. Masur and S. Tabachnikov. *Rational Billiards and Flat Surfaces*. Handbook of Dynamical Systems, Vol. **1A** (2002), 1015-1089.

[7] W. Veech. *Strict Ergodicity in Zero Dimensional Dynamical Systems and the Kronecker-Weyl Theorem Modulo 2*. Transactions of the American Mathematical Society, **140** (1969), 1-33.