ESTIMATING THE FRACTAL DIMENSION OF SETS DETERMINED BY NONERGODIC PARAMETERS

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I certify that I have read *Estimating the Fractal Dimension of Sets Determined by Nonergodic Parameters* by Joseph Paul Squillace, and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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In 1969, Veech introduced two subsets $K_1(\theta), K_0(\theta)$ of $\mathbb{R}/\mathbb{Z}$ which are defined in terms of the continued fraction representation of $\theta \in \mathbb{R}/\mathbb{Z}$. These subsets are known to give information about the dynamics of certain skew products of the unit circle. We show that the Hausdorff dimension of the sets $K_i(\theta)$ can achieve any value between 0 and 1, inclusive.
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1 INTRODUCTION

Let \( \{x : 0 \leq x < 1\} \) be the compact group of real numbers modulo 1. Any irrational element \( \theta \) in this set has a continued fraction representation

\[
\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}
\]

where \((a_i)_{i=1}^\infty\) is a sequence of positive integers known as the terms or partial quotients of the continued fraction. Take \( \left(\frac{m_k}{n_k}\right)_{k=1}^\infty \) to be the sequence of principal convergents of \( \theta \); that is,

\[
\frac{m_k}{n_k} = [0; a_1, a_2, \ldots, a_k]
\]

for each positive integer \( k \). From the theory of continued fractions, we have

\[
n_{k+1} = a_{k+1}n_k + n_{k-1}
\]

for \( k \in \mathbb{N} \), with \( n_0 := 1 \) and \( n_1 = a_1 \). Let \((b_k)_{k \in \mathbb{N}}\) be a sequence of even integers satisfying \( |b_j| \leq a_{j+1} \) for each \( j \in \mathbb{N} \). Given any integer \( m \),

\[
\langle m; b_1, b_2, \ldots \rangle_\theta := \left( m\theta + \sum_{j=1}^{\infty} b_jn_j\theta \right) \mod 1
\]

defines a point of \( \{x : 0 \leq x < 1\} \). We call such a representation \( \langle \cdot \rangle_\theta \) a \( \theta \)-expansion.

From now on, we omit the “mod 1” notation. Veech [7] defines the sets

\[
K_0(\theta) := \{ \langle m; b_1, b_2, \ldots \rangle_\theta : b_j \text{ is eventually even, } \lim_{j \to \infty} b_jn_j \|n_j\theta\| = 0 \}
\]

and
\[ K_1(\theta) := \left\{ (m; b_1, b_2, \ldots)_\theta : b_j \text{ is eventually even}, \sum_{j=1}^{\infty} |b_j| n_j \| n_j \theta \| < \infty \right\} \]

where \( \| \cdot \| \) denotes the distance to the nearest integer. Clearly, \( K_1(\theta) \subset K_0(\theta) \). In Section 2 we find an upper bound formula for the Hausdorff dimension of \( K_0(\theta) \).
In Section 4.1 we apply a result from [5] to obtain a lower bound formula for the Hausdorff dimension of \( K_1(\theta) \). Using these formulas, in Section 5 we show that for any \( \delta \in [0, 1] \) we can construct a number \( \theta \) such that the Hausdorff dimension of \( K_i(\theta) \) equals \( \delta \), where \( i = 0, 1 \).

1.1 Veech’s Examples and a Billiard Interpretation of these Examples

In [7], Veech constructs examples of minimal and not uniquely ergodic dynamical systems as follows (see [6]). Take two copies of the unit circle and mark off segment a \( J \) of length \( 2\pi \alpha \) in the counterclockwise direction on each one with endpoint at 0. Now choose an irrational \( \theta \) and consider the following dynamical system. Start with a point \( p \) in the first circle. Rotate counterclockwise by \( 2\pi \theta \) repeatedly until the the orbit lands in \( J \); then switch to the corresponding point in the second circle, rotate by \( 2\pi \theta \) until the first time the point lands in \( J \); switch back to the first circle and so forth. Veech showed there exists irrational \( \alpha \) for which this system is minimal and the Lebesgue measure is not uniquely ergodic. We may describe Veech’s dynamical system by a flow on a surface arising from a billiard. Consider billiards in the table formed by a \( \frac{1}{2} \times 1 \) rectangle with a horizontal barrier of length \( \alpha \) with one end touching at the midpoint of a vertical side. We can identify the top half of the table as the positive side and the bottom half as the negative side.
Figure 1: A Billiard table $B_\alpha$ with a barrier of length $\alpha$.

A standard unfolding of this billiard table is shown in Figure 2. We can view the new figure as having two (identified) barriers of length $2\alpha$.

Figure 2: Unfolding of Billiard Table $B_\alpha$
(See [6]) If we consider the billiard flow in this rectangle with direction $\theta$, there is an unfolding of this billiard such that the restriction of the flow to the map of first return to the horizontal barrier is equivalent to the function described in the examples given by Veech. In [4], Cheung, Hubert, and Masur showed that if we fix the parameter $\alpha$ then the dimension of the set of all $\theta$ for which this flow is not ergodic is either 0 or $\frac{1}{2}$. Specifically, Cheung, Hubert, and Masur show that the dimension is $\frac{1}{2}$ if and only if
\[
\sum_{k \in \mathbb{N}} \log \log \frac{q_{k+1}}{q_k} < \infty
\]
where $\{q_k\}_{k \in \mathbb{N}}$ is the sequence of denominators of the continued fraction representation of $\alpha$. Our goal is to consider what happens in the case in which we fix $\theta$ and allow $\alpha$ to vary.

Moreover, the sets $K_1(\theta)$ and $K_0(\theta)$ are known to give information (see [3]) about the dynamics of Veech's examples described above.

2 UPPER BOUND ON HAUSDORFF DIMENSION

For $A \subset \mathbb{R}^n$, we denote by $\text{Hdim} A$ the Hausdorff dimension of $A$ (see [4]). In Section 2.1 we give an upper bound for $\text{Hdim} K_0(\theta)$ by applying what Cheung [2] has defined as a self-similar covering to subsets which we call $K_0^i(\theta)$ of $K_0(\theta)$. In particular, the self-similar covering will give an upper bound for $\text{Hdim} K_0^i(\theta)$, and we show that this upper bound is also an upper bound for $\text{Hdim} K_0(\theta)$.

The following definition is motivated by the fact that (i) Veech showed in [7] that if $\theta$ has bounded partial quotients (i.e., $\sup_{j \in \mathbb{N}} a_j < \infty$) then $\text{Hdim} K_0(\theta) = 0$, and
(ii) Lothar Narins conjectured that $\text{Hdim} K_1(\theta) = 1$ when $a_j \to \infty$.

**Definition.** Let $M \in \mathbb{N}$. An irrational $\theta$ with unbounded partial quotients is *divergent relative to* $[1, M]$ if the sequence of partial quotients formed by the terms that are greater than $M$ diverges to $\infty$.

If $\theta$ is divergent relative to $[1, M]$, define

$$
\kappa^\theta := \{k_i : a_{k_i + 1} > M\}
$$

$$
k_0 := \min \kappa^\theta
$$

**Theorem 1.** Let $\theta$ be divergent relative to $[1, M]$, and let $(k_i)_{i=0}^{\infty}$ enumerate the values of $k$ satisfying $a_{k+1} > M$. Then

$$
\text{Hdim} K_0(\theta) \leq \limsup_{i \to \infty} \frac{\log a_{k_i + 1}}{\log n_{k_i + 1} - \log n_{k_i}}
$$

(2)

### 2.1 Reduction

We reduce computing an upper bound of $\text{Hdim} K_0(\theta)$ directly by introducing new sets $K_i^0(\theta)$ for nonnegative integers $i$. We will show that this suffices for finding an upper bound of $\text{Hdim} K_0(\theta)$.

**Definition.** Suppose $\theta$ is divergent relative to $[1, M]$.

$$
K_i^0(\theta) := \left\{ \langle m; b_1, b_2, \ldots \rangle_\theta \in K_0(\theta) : j > k_i \Rightarrow |b_j| n_j \|n_j \theta\| < \frac{1}{4M} \right\}
$$
The sets $K^i_0(\theta)$ are nonempty for all sufficiently large $i$ since, by definition of $K_0(\theta)$,

$$\lim_{j \to \infty} |b_j| n_j \| n_j \theta \| = 0$$

Further, the next three lemmas along with the fact that we can find an upper bound for $\text{Hdim} K^i_0(\theta)$ that does not depend on $i$ will furnish an upper bound for $\text{Hdim} K^0_0(\theta)$.

Lemma 1.

$$K^i_0(\theta) \subset \bigcup_{i=0}^{\infty} K^i_0(\theta)$$

Proof. If $y \in K^i_0(\theta)$, then there is a sequence $m, b_1, b_2, \ldots$ such that $y = \langle m; b_1, b_2, \ldots \rangle_\theta$, where $b_j n_j \| n_j \theta \| \to 0$. Hence, there is an $i$ for which $j > k_i$ implies $b_j n_j \| n_j \theta \| < \frac{1}{4M}$. Therefore $y \in K^i_0(\theta)$.

Lemma 2. (Countable Stability)

$$\text{Hdim} \bigcup_{i=0}^{\infty} K^i_0(\theta) = \sup \{ \text{Hdim} K^i_0(\theta) : i \geq 0 \}$$

Proof. For any set $F \subset \mathbb{R}^n$ and $s \geq 0$, denote by $\mathcal{H}^s(F)$ the $s$-dimensional Hausdorff measure of $F$ (see [5]). $\text{Hdim} K^i_0(\theta) \leq \text{Hdim} \bigcup_i K^i_0(\theta)$ for each $i$, so $\sup_i \text{Hdim} K^i_0(\theta) \leq$
Hdim $\bigcup_i K_0^i (\theta)$. Conversely, let $s = \sup_i \text{Hdim} K_0^i (\theta)$. It suffices to show
\[ \mathcal{H}^{s+\epsilon} \bigcup_i K_0^i (\theta) = 0 \] when $\epsilon > 0$. Let $\delta > 0$. For each $i$ we can cover $K_0^i (\theta)$ by intervals
$A_{ij}$ such that the sum of their radii by the power $s + \epsilon$ is less than $2^{-i}\delta$. The union of all intervals $A_{ij}$, over $i$ and $j$, covers $\bigcup_i K_0^i (\theta)$ and the sum of their radii raised by the power $s + \epsilon$ is less than $\delta$. Therefore, $\mathcal{H}^{s+\epsilon} \bigcup_i K_0^i (\theta) = 0$.

\[ \square \]

**Lemma 3.** There exists an $i_0$ such that if $j \geq k_{i_0}$, $j \not\in \kappa^\theta$ and $|b_j| n_j \parallel n_j \theta\parallel < \frac{1}{4M}$, then $b_j = 0$.

**Proof.** Take $i_0$ large enough such that $k_{i_0} \geq \min \kappa^\theta$. Since $j \not\in \kappa^\theta$, $a_{j+1} \leq M$. Hence
\[
\frac{1}{4M} > |b_j| n_j \parallel n_j \theta\parallel > \frac{|b_j| n_j}{2n_{j+1}} > \frac{|b_j|}{4a_{j+1}}
\]
Hence, $a_{j+1} > M |b_j|$. Since $a_{j+1} < M$, $b_j = 0$.

\[ \square \]

A consequence of Lemma 3 is $x \in K_0^{i_0} (\theta)$ has a $\theta$–expansion $\langle m; b_1, b_2, \ldots \rangle$ such that for all $j > k_{i_0}$ either $a_{j+1} > M$ or $b_j = 0$. 
2.2 Specification of Self-Similar Cover

Given $|b_i| \leq a_{i+1}$ and an even $b$ satisfying $|b| \leq a_{k+1}$, define by

$$I(m; b_1, b_2, \ldots, b_{k-1}, b)$$

the interval of length $\frac{8}{n_k}$ centered at

$$m \theta + \sum_{j=1}^{k-1} b_j \|n_j \theta\| + b \|n_k \theta\|$$

Lemma 4. $x \in I(m; b_1, \ldots, b_k)$ for any $x \in K_0(\theta)$.

Proof. Let $x = (m; b_1, \ldots) \in K_0(\theta)$. Since $\theta$ is between $\frac{m_k}{n_k}$ and $\frac{m_{k+1}}{n_{k+1}}$, it follows that $\left| \frac{m_k}{n_k} - \theta \right| \leq \left| \frac{m_k}{n_k} - \frac{m_{k+1}}{n_{k+1}} \right|$. Multiplying by $n_k$ gives $|m_k - n_k \theta| \leq |m_k - n_k \frac{m_{k+1}}{n_{k+1}}|$. Therefore,

$$\|n_k \theta\| \leq |m_k - n_k \theta| \leq \left| m_k - n_k \frac{m_{k+1}}{n_{k+1}} \right| = \frac{1}{n_{k+1}}$$

We also note that since $n_{k+2} > 2n_k$ we have $\frac{1}{n_{k+1}} > \frac{2^i}{n_{k+(2i+1)}}$ and $\frac{1}{n_{k+2}} > \frac{2^i}{n_{k+(2i+2)}}$. Therefore, $\sum_{i=0}^{\infty} \frac{1}{n_{k+(2i+1)}} < \sum_{i=0}^{\infty} \frac{1}{2^{n_k+1}}$, which is a geometric series that simplifies to $\frac{2}{n_{k+1}}$. Similarly, $\sum_{i=1}^{\infty} \frac{1}{n_{k+(2i)}} < \frac{2}{n_{k+2}}$. 
Without loss of generality, suppose \( m = 0 \). Then
\[
| x - \sum_{i=1}^{k} b_i \| n_i \theta \| | = \left| \sum_{j=k+1}^{\infty} b_j \| n_j \theta \| \right|
\leq \sum_{j=k+1}^{\infty} | b_j | \| n_j \theta \|
\leq \sum_{j=k+1}^{\infty} \frac{| b_j |}{n_j+1}
\leq \sum_{j=k+1}^{\infty} \frac{a_{j+1}}{n_j+1}
\leq \sum_{j=k+1}^{\infty} \frac{1}{n_j}
= \sum_{i=0}^{\infty} \frac{1}{n_{k+2i+1}} + \sum_{i=1}^{\infty} \frac{1}{n_{k+2i}}
< \frac{2}{n_{k+1}} + \frac{2}{n_{k+2}}
\leq \frac{4}{n_{k+1}}
\]

Therefore, \( x \in I (m; b_1, \ldots b_k) \).

**Definition.** (Section 3 of [2]) Let \( X \) be a metric space and \( J \) a countable set. Given \( \sigma \subset J \times J \) and \( \alpha \in J \) we let \( \sigma (\alpha) \) denote the set of all \( \alpha' \in J \) such that \( (\alpha, \alpha') \in \sigma \). We say a sequence \( (\alpha_j)_{j \in \mathbb{N}} \) of elements in \( J \) is \( \sigma \)-admissible if \( \alpha_{j+1} \in \sigma (\alpha_j) \) for all \( j \in \mathbb{N} \); and we let \( J^\sigma \) denote the set of all \( \sigma \)-admissible sequences in \( J \). By a self-similar covering of \( X \) we mean a triple \( (\mathcal{B}, J, \sigma) \) where \( \mathcal{B} \) is a collection of bounded subsets of \( X \), \( J \) a countable index set for \( \mathcal{B} \), and \( \sigma \subset J \times J \) such that there is a map \( E : K_0 (\theta) \to J^\sigma \) that assigns to each \( x \in X \) a \( \sigma \)-admissible sequence \( (\alpha_j^x)_{j \in \mathbb{N}} \) such that for all \( x \in X \)
(i) $\cap_{j=1}^{\infty} B(\alpha_j^x) = \{x\}$, and
(ii) $\text{diam } B(\alpha_j^x) \to 0$ as $j \to \infty$, where $B(\alpha)$ denotes the element of $B$ indexed by $\alpha$.

2.3 A Self-similar covering of $K_{i0}^0(\theta)$

We have access to a self-similar covering of $K_{i0}^0(\theta)$. Define

$$J := \left\{ (m; b_1, b_2, \ldots, b_{k-1}) : m \in \mathbb{Z}, k \in \kappa^\theta, b_j \in \mathbb{Z} (1 \leq j \leq k), |b_j| \leq a_{j+1}, \right. \left. \text{if and } b_j = 0 \text{ if both } k_0 < j < k, j \notin \kappa^\theta \right\}$$

$$B := \{ I(\beta) : \beta \in J \}$$

and define $\sigma \subset J \times J$ such that for each $\alpha_{k_i} = (m; b_1, b_2, \ldots, b_{k_i-1}) \in J$ we have

$$\sigma(\alpha_{k_i}) = \{ (m; b_1, b_2, \ldots, b_{k_{i+1}-2}, b) : |b| \leq a_{k_{i+1}} \}$$

Let $J^\sigma$ denote the set of all $\sigma$-admissible sequences in $J$. By Lemma 4 we can define

$$\mathcal{E} : K_0(\theta) \to J^\sigma$$

$$x \mapsto (\alpha_j^x)_{j=0}^{\infty}$$

where $x \in I(\alpha_j) = I(m; b_1, \ldots, b_{j-1}) \in J_0$ for each $\alpha_j \in (\alpha_j^x)$. Suppose $x \in K_{i0}^0(\theta)$. Our triple satisfies (i) of the definition of a self-similar covering; apply Lemma 4 to

\footnote{As mentioned immediately after Theorem 3.1 of [2], we can take elements of $B$ to be subsets of the ambient space $X$.}
show \( x \in I(\beta_j) \) for all \( \beta_j \in J_0 \). Since \( \lim_{j \to \infty} \text{diam} \ I(\alpha_j^x) = 0, \cap_{j=0}^{\infty} I(\alpha_j^x) = \{x\} \). Thus, \((ii)\) is also satisfied.

Define

\[
E_i := \bigcup I(m; b_1, \ldots, b_{k_i-1})
\]

such that the union is over all finite sequences \( m, b_1, \ldots, b_{k_i-1} \), where \( k_i \in \kappa^{\theta} \) and \( |b_j| \leq a_{j+1} \) for each \( j \), and define

\[
E := \bigcap_{i=0}^{\infty} E_i
\]

2.4 Calculation

In this section we give an upper bound on \( \text{Hdim}K_0^{\omega}(\theta) \). The following lemma is a direct consequence of the definition of \( \sigma \).

**Lemma 5.** Let \( \alpha \in J \) and \( k = |\alpha| \).

\[
\#\sigma(\alpha) \leq a_{k+1}
\]

**Proof.** Let \( \alpha \) be a sequence in \( J \) of length \( k_i \). Then \( \alpha = (m; b_1, \ldots, b_{k_i-1}) \) where both \( j > k_{i_0} \) and \( j \notin \kappa^{\theta} \) imply \( b_j = 0 \). Since \( |\alpha| = k \), \( \alpha \) corresponds to an interval belonging to \( E_i \). Hence, the centers determined by the children of the intervals comprising \( E_i \) are determined by all even integers \( b_k \) satisfying \( b_k \leq a_{k_i+1} \).
For \( s \geq 0 \), since the value of \(|I(\beta)|\) does not depend on the choice of \( \beta \in \sigma(\alpha) \), a direct consequence of equation (3) is

\[
\sum_{\beta \in \sigma(\alpha)} \left( \frac{|I(\beta)|}{|I(\alpha)|} \right)^s = \#\sigma(\alpha) \left( \frac{|I(\beta)|}{|I(\alpha)|} \right)^s
\]

(4)

Further, for \( \beta \in \sigma(\alpha) \)

\[
\#\sigma(\alpha) \left( \frac{|I(\beta)|}{|I(\alpha)|} \right)^s \leq 1 \iff \#\sigma(\alpha) \left( \frac{n_{k_i}}{n_{k_{i+1}}} \right)^s \leq 1
\]

The critical value of \( s \) is \( s = \frac{\log \#\sigma(\alpha)}{\log n_{k_{i+1}} - \log n_{k_i}} \). Let \( \varepsilon > 0 \), and let \( s' = s + \varepsilon \). Then

\[
\sum_{\beta \in \sigma(\alpha)} \left( \frac{|I(\beta)|}{|I(\alpha)|} \right)^{s'} \leq 1 \iff \left( \frac{n_{k_i}}{n_{k_{i+1}}} \right)^{s'} \#\sigma(\alpha) \leq 1 \quad \text{(by (4))}
\]

\[
\iff \left( \frac{n_{k_i}}{n_{k_{i+1}}} \right)^{s} \leq 1
\]

Theorem 5.3 of [1] implies \( H\text{dim}K_0(\theta) \leq s' \). Since \( \varepsilon \) can be arbitrarily small, we have \( H\text{dim}K_0(\theta) \leq s \). Hence

\[
H\text{dim}K_0(\theta) \leq \limsup_{i \to \infty} \frac{\log \#\sigma(\alpha)}{\log n_{k_{i+1}} - \log n_{k_i}} = \limsup_{i \to \infty} \frac{\log a_{k_{i+1}}}{\log n_{k_{i+1}} - \log n_{k_i}} \quad \text{(by (3))}
\]

\[\blacksquare\]
3 A NONTRIVIAL EXAMPLE OF $\theta$ SATISFYING $\text{Hdim} K_0(\theta) = 0$

In [7], Veech showed that the Hausdorff dimension of $K_i(\theta)$ vanishes when $\theta$ has bounded partial quotients. Using our upper bound formula (2), we give an example for which $\sup_j a_j = \infty$ and $\text{Hdim} K_0(\theta) = 0$. Let $0 < \delta \leq 1$ be given and fix $M = \lceil \frac{1}{2 \delta} \rceil$. We specify the continued fraction representation of $\theta$ recursively. We choose $a_1, a_2, \ldots, a_{k_0} = M$, where $k_0$ is the smallest index for which $n^\delta_{k_0} > \max \left\{ 2M, \frac{1}{2 \delta} \right\}$. Note that there exists an integer in between $n^\delta_{k_0}$ and $2^\delta n^\delta_{k_0}$ since $n^\delta_{k_0} (2^\delta - 1) > 1$.

Choose $a_{k_0+1} \in \mathbb{Z}$ such that $n^\delta_{k_0} < a_{k_0+1} < 2n^\delta_{k_0}$. Recursively, given $k_i$, define

$$k_{i+1} := k_i + 1 + \left\lceil \frac{\delta i \log n^\delta_{k_i}}{\log M} \right\rceil \quad (5)$$

set $a_{k_i + 2} = a_{k_i + 3} = \cdots = a_{k_{i+1}} = M$, and choose $a_{k_{i+1}}$ such that $n^\delta_{k_i} < a_{k_{i+1}} < 2n^\delta_{k_i}$. This completes the recursive definition of the sequence $(a_{k_i})_{i \geq 0}$. Moreover, a direct consequence of the definition of (5) is

$$M^{k_{i+1} - k_i - 1} \geq n^\delta_{k_i} \quad (6)$$

Claim. Under the above construction, $\theta$ is divergent relative to $[1, M]$.

Proof. By construction of $(a_{k_i+1})_{i \geq 0}$, we have, for each $i \geq 0$, $a_{k_i+1} > M$ and $n^\delta_{k_{i+1}} < 2n^\delta_{k_i}$. Therefore $a_{k_{i+1}}$ diverges to $\infty$.

$\blacksquare$

Theorem 2. Let $\theta$ be constructed as above. Then $\text{Hdim} K_0(\theta) = 0$. 
Proof.

\[ n_{ki+1} > a_{ki+1} n_{ki+1} - 1 \]
\[ = Mn_{ki+1} - 1 \]
\[ > Ma_{ki+1} - 1 n_{ki+1} - 2 \]
\[ = M^2 n_{ki+1} - 2 \]
\[ \vdots \]
\[ = M^{ki+1 - ki - 1} n_{ki} \]
\[ > n^{\delta_i} n_{ki} \text{ (by (6))} \]
\[ = n^{\delta_i+1} \]

Therefore, \( \log n_{ki+1} > (\delta + 1) \log n_{ki} \), and this implies

\[
\lim_{i \to \infty} \frac{\log n_{ki+1}}{\log n_{ki}} = \infty \tag{7}
\]

Since \( \theta \) is half-divergent,

\[
\text{Hdim}_0(\theta) \leq \limsup_{i \to \infty} \frac{\log a_{ki+1}}{\log n_{ki+1} - \log n_{ki}} \tag{by (2)}
\]
\[
\leq \limsup_{i \to \infty} \frac{\log 2 n^{\delta_i} n_{ki}}{\log n_{ki+1} - \log n_{ki}}
\]
\[
\leq \limsup_{i \to \infty} \frac{\delta \log n_{ki} + \log 2}{\log n_{ki+1} - \log n_{ki}}
\]
\[
= \limsup_{i \to \infty} \frac{\log n_{ki+1} - \log n_{ki}}{\log n_{ki}} = 1
\]
\[
= 0 \text{ (by (7))}
\]

\[ \square \]
4  EXAMPLE OF $\theta$ FOR WHICH $\text{Hdim}K_1(\theta) = 1$

4.1  Falconer’s Lower Bound Formula

(See [4]) Let $[0, 1] = F_0 \supset F_1 \supset F_2 \supset \ldots$ be a decreasing sequence of sets, with each $F_k$ a union of a finite number of disjoint closed intervals (called $k$th level basic intervals), which each interval of $F_k$ containing at least two intervals of $F_{k+1}$, and the maximum length of $k$th level intervals tending to 0 as $k \to \infty$. Then the set

$$F = \bigcap_{k=0}^{\infty} F_k$$

is a totally disconnected subset of $[0, 1]$. The condition needed to apply Falconer’s lower bound formula is

$$(*)  \text{Each } (j-1)\text{th level interval contains at least } m_j \geq 2 \ j \text{th level intervals } (j = 1, 2, \ldots) \text{ which are separated by gaps of at least } \gamma_j, \text{ where } 0 < \gamma_{j+1} < \gamma_j \text{ for each } j.$$

Falconer’s lower bound formula is

$$\text{Hdim} F \geq \liminf_{k \to \infty} \frac{\log (m_0 m_1 \cdots m_j)}{-\log (m_{j+1} \gamma_{j+1})}$$

(8)

4.2  Lower Bound for $\text{Hdim}K_1(\theta)$

Fix $\delta \in (0, 1)$ and $\varepsilon \in (0, \delta)$. For the remainder of Section 4, let $\theta$ be an element for which there exists a $k_0$ sufficiently large so that $n_{k_0}^{\delta - \varepsilon} \geq 3$, $n_{k_0}^{\varepsilon} \geq 9$ and for all $k \geq k_0$, $n_k^\delta < a_{k+1} < 2n_k^\delta$, $b_k$ is even and $|b_k| < \lfloor n_k^{\delta - \varepsilon} \rfloor$. Our strategy is to
construct an infinite family of Cantor sets contained in $K_1(\theta)$, each of which satisfies the conditions needed to apply Falconer's lower bound formula, allowing us to give lower bounds of $\text{Hdim} K_1(\theta)$ arbitrarily close to 1.

Given $|b_i| \leq a_{i+1}$, define by

$$L(m; b_1, b_2, \ldots, b_{k-1})$$

or $L_{k-1}$ the interval of length $\|n_{k-1}\theta\|$ concentric with $I(m; b_1, \ldots, b_{k-1})$. We require that for each $k$ the length of the gaps between the intervals $L_k$ is constant. Hence, we have the following lemma.

**Lemma 6.** The gaps between consecutive child intervals of $L_k$ are of length $\|n_k\theta\|$.

**Proof.** The distance between the centers of adjacent intervals of $L_{k+1}$ is $2\|n_k\theta\|$.

Between these centers is the gap between them as well as two half intervals of $L_{k+1}$. Hence, the size of the gap is $\|n_k\theta\|$.

The following lemma gives a sufficient condition for $L_{k+1} \subset L_k$.

**Lemma 7.** Suppose we are given $m, b_1, \ldots, b_{k-1}, b$. If $|b| \leq \frac{a_{k+1}}{8}$, then

$$L(m; b_1, \ldots, b_{k-1}, b) \subset L(m; b_1, \ldots, b_{k-1})$$

**Proof.** We may suppose $a_{k+1} \geq 9$ since the claim is vacuously true otherwise. Let $y \in L(m; b_1, \ldots, b_{k-1}, b)$, and let $x$ be the center of the corresponding parent interval
Let us call $x'$ the center of the interval $L(m; b_1, \ldots, b_{k-1}, b)$. Then

$$|x - y| \leq |x' - x| + |x' - y|$$

$$\leq |b||n_k\theta| + \frac{\|n_k\theta\|}{2}$$

$$\leq \frac{a_{k+1}}{8} + \frac{1}{2}$$

$$\leq \frac{1}{2n_{k+1}}$$

$$< \left( \frac{a_{k+1}}{4} - \frac{1}{2} \right) + \frac{1}{2}$$ (since $a_{k+1} > 4$)

$$= \frac{a_{k+1}}{4n_{k+1}}$$

$$< \frac{a_{k+1}}{4a_{k+1}n_k}$$

$$= \frac{1}{4n_k}$$

$$< \frac{\|n_{k-1}\theta\|}{2}$$

The following result was anticipated by Lothar Narins.

**Theorem 3.** The number $\theta$, as constructed in 4.2, satisfies the following:

$$Hdim K_1(\theta) = 1$$

**Proof.** Define

$$F_j := \bigcup L(m; b_1 \ldots, b_{k_0+j-1})$$

where the union is over all finite sequences such that $m = b_1 = \cdots = b_{k_0-1} = 0$. If $k \geq k_0$, then
\[ |b_k| \leq \left\lfloor n_k^{\delta - \varepsilon} \right\rfloor \leq n_k^{\delta - \varepsilon} = \frac{n_k^{\delta}}{n_k} < \frac{a_{k+1}}{8} \]

so that, by Lemma 7,

\[ F_{j+1} \subset F_j \]

for \( j \geq 0 \).

Define

\[ F(k_0, \varepsilon) := \bigcap_{j=0}^{\infty} F_j \]

Lemma 8. \( F(k_0, \varepsilon) \subset K_1(\theta) \)

Proof. If \( y \in F(k_0, \varepsilon) \), then we have a sequence of the form \( b_1, b_2, \ldots \) such that \( y = \langle m; b_1, b_2, \ldots, b_{k_0}, b_{k_0+1}, \ldots \rangle_\theta \), where \( b_1 = \cdots = b_{k_0-1} = 0 \) and, by construction, each \( b_k \) is even. We show that \( y \) satisfies the constraints imposed on elements of \( K_1(\theta) \).

\[
\sum_{j=1}^{\infty} |b_j| n_j \|n_j \theta\| \leq \sum_{j=1}^{\infty} \frac{n_j^{\delta - \varepsilon + 1}}{n_{j+1}} \leq \sum_{j=1}^{\infty} \frac{n_j^{\delta - \varepsilon + 1}}{n_j^{1+\delta}} \leq \sum_{j=1}^{\infty} \frac{1}{n_j^{\delta}}
\]
Using the Ratio Test on the latter series, we have

\[
\lim_{j \to \infty} \frac{n_j^\varepsilon}{n_{j+1}^\varepsilon} \leq \lim_{j \to \infty} \left( \frac{n_j}{a_{j+1}n_j} \right) \varepsilon \\
\leq \lim_{j \to \infty} \frac{1}{a_{j+1}^\varepsilon} = 0 \\
= 0
\]

Therefore, \(\sum_{j=1}^{\infty} |b_j| n_j \|n_j \theta\| < \infty\). Hence \(y \in K_1(\theta)\).

For convenience, define \(k_j := k_0 + j\). We show that \(F(k_0, \varepsilon)\) satisfies (*). Denote by \(m_j\) the number of children intervals \(L_{k_j}\) of \(F\) so that \(m_j\) counts the number of even integers \(b\) satisfying \(|b| < \left\lfloor n_{k_j}^{\delta - \varepsilon} \right\rfloor\), so that

\[m_j = \left\lfloor n_{k_j}^{\delta - \varepsilon} \right\rfloor\]

Denote by \(\gamma_j\) the length of the gaps between children intervals of \(L_{k_j}\) so that, by Lemma 6,

\[\gamma_j = \|n_{k_j} \theta\|\]

From basic continued fraction theory,

\[\frac{1}{2n_{k+1}} < \|n_k \theta\| < \frac{1}{n_{k+1}} \quad (9)\]

Since we have \(\frac{1}{n_{k+1}^2} < \frac{1}{2n_{k+1}^2}\) for each \(j\), we have \(0 < \gamma_{k+1} < \gamma_k\) for each \(j\). By (9), \(\lim_{j \to \infty} \gamma_j = 0\). It is the case that each interval \(L(m; b_1, \ldots, b_{k-1})\) contains at least 3 intervals of \(L_{k_j}\) and the gaps \(\gamma_j\) decrease monotonically to 0 as \(j \to \infty\). Hence, the conditions for Falconer’s lower bound formula (inequality 4.7 of [4]) are satisfied.
To simplify our calculation of the lower bound of $\text{Hdim} F (k_0, \varepsilon)$, we use the fact that $k \geq k_0$ implies

$$n_{k+1} < (a_{k+1} + 1) n_k \leq 3 n_k^{1+\delta}$$

Taking log of both sides of the expression $n_{k+1} < 3 n_k^{1+\delta}$ gives

$$\log n_{k+1} < \left(1 + \delta + \frac{\log 3}{\log n_k} \right) \log n_k \quad (10)$$

Further,

$$m_{j+1} = \left\lfloor \frac{n_{k_{j+1}}^{\delta - \varepsilon}}{n_{k_{j+1}}} \right\rfloor > \frac{n_{k_{j+1}}^{\delta - \varepsilon}}{2}$$

and

$$\gamma_{j+1} = \|n_{k_{j+1}} \theta\| > \frac{1}{2n_{k_{j+1}}+1} \quad (\text{by (9)})$$

$$> \frac{1}{4a_{k_{j+1}}+1n_{k_{j+1}}} \quad (\text{since } n_{k+1} < 2a_{k+1}n_k)$$

$$> \frac{1}{8n_{k_{j+1}}^{1+\delta}}$$

imply

$$m_{j+1} \gamma_{j+1} > \frac{1}{16 n_{k_{j+1}}^{1+\varepsilon}} \quad (11)$$
Using Falconer’s lower bound (8) gives

\[
\text{Hdim} F (k_0, \varepsilon) \geq \liminf_{j \to \infty} \frac{\log (m_0 \cdots m_j)}{\log (m_{j+1} \gamma_{j+1})} - \log (m_{j+1} \gamma_{j+1}) \\
\geq \liminf_{j \to \infty} \frac{(\delta - \varepsilon)(\log n_{k_0} + \log n_{k_1} + \cdots + \log n_{k_j})}{(1 + \varepsilon) \log n_{k_{j+1}}} \\
\geq \liminf_{j \to \infty} \frac{\delta - \varepsilon}{1 + \varepsilon} \left( \frac{1}{1 + \delta + \frac{\log 3}{\log n_{k_0}}} + \cdots + \frac{1}{1 + \delta + \frac{\log 3}{\log n_{k_{j+1}}}} \right) \\
= \liminf_{j \to \infty} \frac{\delta - \varepsilon}{1 + \varepsilon} \left( \frac{1}{1 + \delta + \frac{\log 3}{\log n_{k_0}}} \right) \left( \frac{1 - \frac{1}{1 + \delta + \frac{\log 3}{\log n_{k_0}}}}{1 - \frac{1}{1 + \delta + \frac{\log 3}{\log n_{k_0}}}} \right) \\
= \frac{\delta - \varepsilon}{(1 + \varepsilon) \left( \delta + \frac{\log 3}{\log n_{k_0}} \right)}
\]

\text{Hdim} F (k_0, \varepsilon) \to 1 \text{ as } \varepsilon \to 0, k_0 \to \infty. \text{ Therefore, } \text{Hdim} K_1 (\theta) = 1.
5 \ Hdim K_i (\theta) = \delta

**Theorem 4.** For any $\delta \in [0, 1]$ there is a $\theta$ satisfying $Hdim K_i (\theta) = \delta$.

The cases $\delta = 0$ and $\delta = 1$ are handled by Theorem 2 and Theorem 3, respectively. Let us choose $\delta \in (0, 1)$. Our strategy is to construct a half-divergent $\theta$ in terms of $\delta$ in a particular way which gives

$$\delta \leq Hdim K_1 (\theta) \leq Hdim K_0 (\theta) \leq \delta$$

**Proof.** Suppose $M \in \mathbb{N}_{\geq 3}$. We construct a $\theta$ divergent relative to $[1, M]$. In what follows we construct integers $a_k$ in terms of the indices $k_i$ by taking

$$\begin{cases} 
3 \leq a_{k+1} \leq M & \text{if } k \neq k_i, \\
 n_k^{\delta} < a_{k+1} < 2n_k^{\delta} & \text{if } k = k_i
\end{cases}$$

Choose $a_1, a_2, \ldots, a_k \in [3, M]$, where $k_0$ is large enough so that $n_{k_0}^{\delta} > M$. With this choice of $k_0$, $a_{k+1} > M$ when $k = k_i$; therefore, $a_{k+1} \in [3, M]$ if and only if $k \neq k_i$. Suppose we are given $k_i$ and that $a_1, a_2, \ldots, a_{k_i}, n_1, n_2, \ldots, n_{k_i+1}$ are defined according to (12). Choose $k_{i+1}$ to be the smallest $k \geq k_i + 1$ such that

$$\begin{cases} 
n_k < n_{k_i}^2, \text{ and set } a_{k+1} \in [3, M] & \text{if } k < k_{i+1}, \\
n_k \geq n_{k_i}^2, \text{ and set } a_{k+1} \in (n_k^\delta, 2n_k^\delta) & \text{if } k = k_{i+1}
\end{cases}$$
By this recursive definition, \( n_{k_{i+1}} \geq n_{k_i}^2 \) and \( n_{k_{i+1}} - 1 < n_{k_i}^2 \). If \( k_{i+1} = k_i + 1 \), then

\[
\begin{align*}
n_{k_{i+1}} & = a_{k_{i+1}} n_{k_i} + n_{k_i - 1} \\
& < 2a_{k_{i+1}} n_{k_i} \\
& < 4n_{k_i}^{1+\delta} \\
& < (M + 1) n_{k_i}^2
\end{align*}
\]

If \( k_{i+1} > k_i + 1 \), then \( k_{i+1} \) is not of the form \( k_j + 1 \) for any \( j > i \); for if \( j > i \), then \( k_j + 1 > k_j \geq k_{i+1} \). Therefore, in this case, we have \( a_{k_{i+1}} \in [3, M] \), so

\[
\begin{align*}
n_{k_{i+1}} & = a_{k_{i+1}} n_{k_i + 1} - 1 + n_{k_{i+1} - 2} \\
& < (a_{k_{i+1}} + 1) n_{k_{i+1} - 1} \\
& < (M + 1) n_{k_i}^2
\end{align*}
\]

Therefore,

\[
n_{k_i}^2 \leq n_{k_{i+1}} < (M + 1) n_{k_i}^2 \quad (13)
\]

and

\[
n_{k_i}^\delta < a_{k_{i+1}} < 2n_{k_i}^\delta
\]

Further, it will be used in the upper and lower bound calculations that (13) implies

\[
\lim_{j \to \infty} \frac{\log n_{k_j}}{\log n_{k_{j+1}}} = \frac{1}{2} \quad (14)
\]
5.1 Upper Bound

As construction in Section 5, \( \theta \) is divergent relative to \([1, M]\) since the subsequence \((a_{k_i+1})_{i \geq 0}\) of terms of \((a_k)_{k \geq 1}\) which are larger than \(M\) also satisfy \(n_{k_i}^\delta < a_{k_i+1} < 2n_{k_i}^\delta\).
Therefore, \(\lim_{i \to \infty} a_{k_i+1} = \infty\). Hence,

\[
\text{Hdim} K_0 (\theta) \leq \limsup_{j \to \infty} \frac{\log a_{k_j+1}}{\log n_{k_j+1} - \log n_k} \quad \text{(by (2))}
\]

\[
\leq \limsup_{j \to \infty} \frac{\log n_{k_j}^\delta + \log 2}{\log n_{k_j+1} - \log n_k}
\]

\[
= \limsup_{j \to \infty} \frac{\delta \log n_{k_j}}{\log n_{k_j+1}}
\]

\[
= \delta \quad \text{(by (14))}
\]

Therefore, \(\text{Hdim} K_0 (\theta) \leq \delta\).

5.2 Lower Bound

Let \(\varepsilon \in (0, 1)\). Choose \(k_0' \geq k_0\) sufficiently large such that \(k_i \geq k_0'\) implies \(a_{k_i+1}^\varepsilon < \frac{a_{k_i+1}}{8}\). Define

\[F_j := \bigcup L (m; b_1, b_2, \ldots b_{k_0'-1}, \ldots, b_{k_j-1})\]

to be the union over all finite sequences of even terms \(b_1, \ldots b_{k_j-1}\) such that if \(k = k_i \geq k_0'\) then \(b_k\) is even and satisfies \(|b_k| < |a_{k_i+1}^\varepsilon|\), otherwise \(b_k = 0\). WLOG, let \(m = 0\). Define
\[ F_\varepsilon = F(k'_0, \varepsilon) := \bigcap_{j=0}^{\infty} F_j \]

Lemma 7 implies \( F_{j+1} \subset F_j \) for \( j \in \mathbb{N} \) since if \( k_i \geq k'_0 \), then

\[
|b_{k_i}| < \left\lfloor a_{k_i+1}^\varepsilon \right\rfloor \\
\leq a_{k_i+1}^\varepsilon \\
< \frac{a_{k+1}}{8}
\]

Lemma 9. \( F_\varepsilon \subset K_1(\theta) \)

Proof. Let \( y \in F_\varepsilon \). Then we are given a sequence of the form \( b_1, b_2, \ldots \) such that

\[
y = \langle m; b_1, b_2, \ldots, b_{k'_0}, b_{k'_0+1}, \ldots \rangle, |b_k| \text{ is even and satisfies } |b_k| < \left\lfloor a_{k+1}^\varepsilon \right\rfloor \text{ if } k = k_i > k'_0 \text{ and } b_k = 0 \text{ otherwise.}
\]

Further,

\[
\sum_{j=1}^{\infty} |b_j| \|n_j\| \|n_j\theta\| \leq \sum_{j=1}^{\infty} \frac{a_{j+1}^\varepsilon n_j}{n_{j+1}} \\
\leq \sum_{j=1}^{\infty} \frac{a_{j+1}^\varepsilon n_j}{a_{j+1} n_j} \\
= \sum_{j=1}^{\infty} \frac{1}{a_{j+1}^{1-\varepsilon}} \\
\leq \sum_{j=1}^{\infty} \frac{1}{n_j^{\delta(1-\varepsilon)}}
\]

\[ \blacksquare \]
Proof. Using the Ratio Test on the latter series, we have

\[
\lim_{j \to \infty} \frac{n_j^{(1-\varepsilon)\delta}}{n_{j+1}^{(1-\varepsilon)\delta}} \leq \lim_{j \to \infty} \frac{n_j^{(1-\varepsilon)\delta}}{n_{j+1}^{(1-\varepsilon)\delta}} \\
\leq \lim_{j \to \infty} \frac{1}{a_{j+1}^{(1-\varepsilon)\delta}} \\
= 0
\]

Therefore, \( \sum_{j=1}^{\infty} |b_j| n_j \|n_j \theta\| < \infty \). Hence \( y \in K_1 (\theta) \).

Define

\[ M_j := \left\lfloor a_{k_j+1}^\varepsilon \right\rfloor \]

so that \( M_j \) is a lower bound on the number of intervals in \( F_j+1 \) contained in intervals of \( F_j \).

Define

\[ \Gamma_j := \|n_k \theta\| \]

so that, by Lemma 6, \( \Gamma_j \) is a lower bound on the gaps between the intervals of \( F_j+1 \).

Since we have \( \frac{1}{n_{k_j+2}} < \frac{1}{2n_{k_j+1}} \) for each \( j \), we have \( 0 < \Gamma_{j+1} < \Gamma_j \) for each \( j \). Since \( \frac{1}{2n_{k_j+1}} < \Gamma_j < \frac{1}{n_{k_j+1}} \) for each \( j \), \( \lim_{j \to \infty} \Gamma_j = 0 \). It is the case that each interval \( L (m; b_1, \ldots, b_{k_j-1}) \) contains at least 3 intervals of \( F_{k_j+1} \) and the gaps \( \Gamma_j \) decrease monotonically to 0 as \( j \to \infty \). The conditions for Falconer’s lower bound formula are satisfied.
To simplify our calculation of the lower bound of $\text{Hdim} F_\varepsilon$, we use the fact that

\[
\Gamma_{j+1} = \| n_{k_{j+1}} \theta \| \\
\geq \frac{1}{2n_{k_{j+1}}+1} \quad \text{(by (9))} \\
\geq \frac{1}{2 \left( 2a_{k_{j+1}}n_{k_{j+1}} \right)} \quad (n_{k+1} < 2a_{k+1}n_k) \\
\geq \frac{1}{8n_{k_{j+1}}^{1+\delta}}
\]

and

\[
M_{j+1} = \left[ a_{k_{j+1}}^{\delta} + 1 \right] \\
\geq a_{k_{j+1}}^{\delta} \\
\geq \frac{2}{n_{k_{j+1}}^{1+\delta}} \\
\geq \frac{1}{2n_{k_{j+1}}^{1+\delta}}
\]

imply

\[
M_{j+1} \Gamma_{j+1} > \frac{1}{16n_{k_{j+1}}^{1+\delta-\delta \varepsilon}} \quad \text{(15)}
\]

Using Falconer’s lower bound gives

\[
\text{Hdim} F \geq \liminf_{j \to \infty} \frac{\log (M_0 \cdots M_j)}{-\log (M_{j+1} \Gamma_{j+1})} \\
\geq \liminf_{j \to \infty} \frac{\varepsilon \left( \log a_{k_{j+1}} + \log a_{k_{j+1}} + \cdots + \log a_{k_{j+1}} \right)}{(1 + \delta - \delta \varepsilon) \log n_{k_{j+1}}} \quad \text{(by (15))} \\
\geq \frac{\delta \varepsilon}{1 + \delta - \delta \varepsilon} \liminf_{j \to \infty} \left( \frac{\log n_{k_{j+1}} + \log n_{k_{j+1}} + \cdots + \log n_{k_{j+1}}}{\log n_{k_{j+1}}} \right) \\
\geq \frac{\delta \varepsilon}{1 + \delta - \delta \varepsilon} \liminf_{j \to \infty} \left( \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{j+1}} \right) \quad \text{(by (14))} \\
= \frac{\delta \varepsilon}{1 + \delta - \delta \varepsilon}
\]
Hdim $F(k'_0, \varepsilon) \rightarrow \delta$ as $\varepsilon \rightarrow 1$. Therefore, $Hdim K_1(\theta) \geq \delta$. 

■
References


