Slowly divergent geodesics in moduli space

Yitwah Cheung

August 13, 2005

Abstract

Slowly divergent Teichmüller geodesics in the moduli space of Riemann surfaces of genus $g \geqslant 2$ are constructed via cyclic branched covers of the torus. Nonergodic examples (i.e. geodesics whose defining quadratic differential has nonergodic vertical foliation) diverging to infinity at sublinear rates are constructed using a Diophantine condition. Examples with an arbitrarily slow prescribed rate of divergence are also exhibited.

1 Introduction

Let $\mathcal{M}_g$ denote the moduli space of closed Riemann surfaces of genus $g \geqslant 2$, endowed with the Teichmüller metric τ. A geodesic in $\mathcal{M}_g$ is determined by a pair (X_0, q) where X_0 is a Riemann surface and q is a holomorphic quadratic differential on X_0. The differential q defines a flat metric with isolated singularities on X_0 together with a pair of transverse measured foliations defined by $q > 0$ (the horizontal) and $q < 0$ (the vertical). By a theorem of Masur [Ma92] the vertical foliation of q is uniquely ergodic if X_t accumulates in $\mathcal{M}_g$ as $t \to \infty$. Therefore, a *nonergodic* geodesic, by which we mean a geodesic determined by a pair (X_0, q) such that the vertical foliation of q is not uniquely ergodic, must eventually leave every compact set. A geodesic with this latter property is said to be *divergent*. The original motivation of this study is to answer a question of C. McMullen regarding the existence of *slowly divergent* nonergodic geodesics:

$$\lim_{t\to\infty} \frac{\tau(X_0, X_t)}{t} = 0. \tag{1}$$

The examples are realized using branched covers of the torus satisfying a Diophantine condition. Let (X, q) be the g-cyclic branched cover of $T = (\mathbb{C}/\mathbb{Z}[i], dz^2)$ obtained by cutting along an embedded linear arc γ. (See §2 for a precise definition.) Each $\theta \in S^1$ determines a Teichmüller geodesic X_t^θ in $\mathcal{M}_g$ starting at $X_0^\theta = X$. A direction θ is also said to be *nonergodic, divergent, slowly divergent, etc.* if the corresponding geodesic X_t^θ has the same property. For a slowly divergent direction it makes sense to consider the *sublinear rate*:

$$r_+(\theta) := \limsup_{t\to\infty} \frac{\log \tau(X_t^\theta, X_0^\theta)}{\log t} \tag{2}$$

A pair $(x_0, y_0) \in \mathbb{R}^2$ is said to satisfy a Diophantine condition if there are constants $c_0 > 0$ and $d_0 > 0$ such that for all pairs of integers $(m, n) \in \mathbb{Z}^2 \setminus \{0\}$

$$\inf_{l \in \mathbb{Z}} |mx_0 + ny_0 + l| > \frac{c_0}{\max(|m|, |n|)^{d_0}}. \tag{3}$$

Let $x_0 + iy_0 \in \mathbb{C}$ be the affine holonomy $\int_\gamma dz$ of γ.

Theorem 1. *If (x_0, y_0) satisfies a Diophantine condition with exponent d_0 then for every $e_0 > \max(d_0, 2)$ there is a Hausdorff dimension $1/2$ set of slowly divergent nonergodic directions θ with sublinear rate $r_+(\theta) \leqslant 1 - 1/e_0$.*

It should be emphasized that Theorem 1 does not give any examples of nonergodic directions with $r_+(\theta) \leqslant 1/2$. In fact, after this paper had been accepted, it was shown that if $r_+(\theta) \leqslant 1/2$ then θ is uniquely ergodic. See [CE].

As a complement to Theorem 1, we also prove

Theorem 2. *If $(x_0, y_0) \notin \mathbb{Q}^2$ then there are directions which are divergent with an arbitrarily slow prescribed rate, i.e. given any function $R(t)$ with $R(t) \to \infty$ as $t \to \infty$ there exists a divergent direction θ such that $\tau(X_t^\theta, X_0^\theta) \leqslant R(t)$ for all sufficiently large t.*

In $\mathcal{M}_1$, the asymptotic behavior of a geodesic is determined by the arithmetic properties of its endpoint in $\mathbb{R} \cup \{\infty\}$. For example, (1) holds iff the endpoint is a Roth number: for any $\varepsilon > 0$, there exists $c_0 > 0$ such that for any $p, q \in \mathbb{Z}$, $|\alpha - p/q| > c_0/|q|^{2+\varepsilon}$. The question asked by C. McMullen was inspired by a recent result of Marmi-Moussa-Yoccoz concerning interval exchange maps, which give another a source of Teichmüller geodesics in $\mathcal{M}_g$. In [MMY], they construct examples of uniquely ergodic interval exchange maps based on a certain "Roth type" condition, which is apparently stronger than (1). Theorem 1 shows that the condition (1) alone is not sufficient to ensure unique ergodicity.

Acknowledgments. The author would like to thank the referees for numerous suggestions that significantly improved the exposition of the paper. In addition thanks must go to Howard Masur, Curt McMullen and Barak Weiss for many enlightening discussions on the topic of slow divergence. Last, but not least, the author is indebted to his wife Ying Xu for her constant and unwavering support.

2 Cyclic branched covers along a slit

The g-cyclic branched cover of T along γ is defined as follows. Endow the complement of γ with the metric defined by shortest path and let T' be its metric completion. T' is a compact surface with a single boundary component and is known as a *slit torus*, i.e. T slit along γ. Let $\gamma_\pm$ denote the two lifts of γ under the natural projection $T' \to T$ which maps $\partial T'$ onto γ. For convenience, we assume γ is defined on the unit interval. Let X be the quotient space of $T' \times \mathbb{Z}/g\mathbb{Z}$ obtained by identifying $(\gamma_-(t), n)$ with $(\gamma_+(t), n+1)$ for all $t \in [0, 1]$ and $n \in \mathbb{Z}/g\mathbb{Z}$. The map $\pi : X \to T$ induced by projection onto the first factor is a branched cover of degree g, holomorphic with respect to a unique complex structure on X. The pair $(X, \pi^* dz^2)$ is called the g-cyclic cover of $T = (\mathbb{C}/\mathbb{Z}[i], dz^2)$ along γ.

Note that X is a closed Riemann surface of genus g. The map π is branched at two points corresponding to the zeros of the quadratic differential $\pi^* dz^2$. Each branch point lies over an endpoint of γ.

2.1 Teichmüller geodesics and saddle connections

The Teichmüller geodesic X_t^θ will be described explicitly. Let $g_t^\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which contracts distances by a factor of $e^{t/2}$ in the θ direction while expanding by $e^{t/2}$ in the direction perpendicular to θ. There is an atlas of charts $\{U_\alpha, \varphi_\alpha\}$ covering X away from the branch points such that $d\varphi_\alpha = \pi^* dz$. The complex structure of X is uniquely determined by this atlas. It is easy to check that $\{U_\alpha, g_t^\theta \circ \varphi_\alpha\}$ defines a new atlas of charts uniquely determining a new complex structure on X. (Here, we used the standard identification $\mathbb{C} = \mathbb{R}^2$.) The space X with this new complex structure is the Riemann surface X_t^θ referred to in the introduction. The family X_t^θ defines a unit speed geodesic in $\mathcal{M}_g$ with respect to the Teichmüller metric τ. It carries a quadratic differential q_t^θ which is the square the holomorphic 1-form determined by the new charts.

A *saddle connection* is a geodesic segment which joins a pair of branch points without passing through one in its interior. Associated to an *oriented* saddle connection α in X is a complex number $\int_\alpha \pi^* dz$ which we identify with the corresponding vector in $\mathbb{R}^2$. The collection of vectors associated to saddle connections in X will be denoted by V.

Let $W = \pm(x_0, y_0) + \mathbb{Z}^2$ and $Z = \{(p, q) \in \mathbb{Z}^2 : \gcd(p, q) = 1\}$ and note that

$$V = W \cup Z.$$

Indeed, a saddle connection in X projects to a path in T whose lift to $\mathbb{R}^2$ lies in $W \cup Z$. Conversely, (3) implies $\{x_0, y_0, 1\}$ is independent over $\mathbb{Q}$ so that the slope of any vector in $W \cup Z$ is irrational. Hence, any vector in $W \cup Z$ can be represented by a geodesic arc in T which joins the endpoints of γ without passing through either one. The lift of this arc to X is a saddle connection.

Note that the set of vectors associated with saddle connections in (X_t^θ, q_t^θ) is simply given by $g_t^\theta V$.

For any discrete subset $S \subset \mathbb{R}^2$, let $\ell(S)$ denote the length of the shortest vector in S. To control distances in $\mathcal{M}_g$, we need the following result which is proved in slightly greater generality in [Ma93].

Proposition 2.1. *There is a constant $C = C(g)$ such that for all $t \in \mathbb{R}$*

$$\tau(X_t^\theta, X_0^\theta) \leqslant -\log \ell(g_t^\theta V)^2 + C. \tag{4}$$

Analysing rates of divergence is reduced to studying the function $\ell(g_t^\theta V)$, which is carried out in §3.

Remark 2.2. The square in (4) does not appear in [Ma93] due to a different normalisation of the Teichmüller metric. In our case, the sectional curvature along Teichmüller disks is -1, instead of -4.

2.2 Summable cross products condition

The surface X carries a flat metric induced by $\pi^* dz$ so that it makes sense to talk about parallel lines, area measure, etc. For any $\theta \in S^1$, let F_θ denote the foliation of X by lines parallel to θ. The foliation F_θ is *ergodic* (with respect to area measure) if X cannot be written as a disjoint union of two invariant sets of positive measure. (An invariant set is one that can be written as a union of leaves.) By definition, θ is a nonergodic direction iff F_θ is not ergodic.

The next lemma will be useful for finding nonergodic directions.

Lemma 2.3. *Let $\pi' : X' \to T$ be the g-cyclic branched cover of T along another arc γ' with the same endpoints as γ. Then π' is biholomorphically equivalent to π if and only if $\gamma - \gamma'$ represents the trivial element in $H_1(T, \mathbb{Z}/g\mathbb{Z})$.*

Proof. Let $U \subset X$ be the set of points lying over the complement of γ in T and let $U' \subset X'$ be defined similarly. We shall identify a dense subset of U with a dense subset of U' as follows. Fix a base point $z_0 \notin \gamma \cup \gamma'$ and let $\mathcal{U}$ be the set of paths in X starting at z_0 which are transverse to $\pi^{-1}\gamma$. For any $\alpha \in \mathcal{U}$, the intersection number $i_\gamma(\alpha) \in \mathbb{Z}/g\mathbb{Z}$ is the number of times α crosses γ positively. (This notion depends on a choice of orientation for T, which we assume has been fixed.) The map $\alpha \mapsto (\alpha(1), i_\gamma(\alpha))$ (where $\alpha(1)$ denotes the terminal point of α) induces a bijection between U and $\mathcal{U}/\sim$ where

$$\alpha \sim \alpha' \qquad \text{iff} \qquad \alpha(1) = \alpha'(1) \ \text{ and } \ i_\gamma(\alpha) = i_\gamma(\alpha').$$

Similarly, U' may be identified with classes of paths transverse to γ' using the above with γ replaced by γ'.

Note $i_\gamma(\alpha - \alpha') = i_{\gamma'}(\alpha - \alpha')$ iff the homology intersection of the cycles $\gamma - \gamma'$ and $\alpha - \alpha'$ vanishes. Therefore, if γ is homologous to γ', there exists a bijection of $U \cap \pi^{-1}\pi'(U')$ with $U' \cap \pi'^{-1}\pi(U)$ which extends uniquely to a biholomorphic equivalence between π and π'. Conversely, if γ is not homologous to γ', then there is a closed curve β disjoint from γ' such that $i_\gamma(\beta) \neq 0$. Since its lift is closed in X' but not in X, π' cannot be equivalent to π. ∎

In the sequel, the cross product of two vectors in $\mathbb{R}^2$ is defined to be a scalar

$$\langle a, b \rangle \times \langle c, d \rangle := ad - bc.$$

Lemma 2.4. *Let $(w_j)_{j \geqslant 0}$ be a sequence of vectors of vectors in $\pm\langle x_0, y_0 \rangle + g\mathbb{Z}^2$ whose Euclidean lengths form an increasing sequence and suppose that*

$$\sum_{j=0}^{\infty} |w_j \times w_{j+1}| < \infty. \tag{5}$$

Then $|w_j|^{-1} w_j$ converges to a nonergodic direction in X.

Proof. Note that the direction of the vector $\langle x_0, y_0\rangle$ associated to γ is nonergodic because $\pi^{-1}\gamma$ partitions X into g invariant sets of equal area. Similarly, there is associated to each w_j a g-partition of some branched cover of T that is biholomorphically equivalent to π, by Lemma 2.3. Since a biholomorphism preserves partitions by invariant sets, it follows that the direction of each w_j is also nonergodic. Now observe that the symmetric difference of the g-partitions of X associated to a consecutive pair of vectors in the sequence is a union parallelograms whose area is bounded above by a constant times the absolute cross product. An elementary argument[1] shows $\lim |w_j|^{-1}w_j$ exists while (5) implies the sequence of g-partitions converge measure-theoretically to a g-partition invariant in the limit direction. ☐

Remark 2.5. Lemma 2.4 is due to Masur-Smillie in genus 2 and is the precursor to a general criterion for nonergodicity developed in [MS]. Their original motivation was to give a geometric interpretation, in the context of rational billiards, of certain $\mathbb{Z}/2$ skew-products studied by Veech in [Ve].

3 Analysis of the shortest vector function

The main result of this section is Proposition 3.6, which is used to control rates. Its hypotheses are motivated by Lemmas 3.3, 3.4 and 3.5 while its conclusion is motivated by Lemmas 3.1 and 3.2.

Lemma 3.1. *Let* $\lambda(t) = -\log|g_t^\theta v|^2$ *where* $\theta \in S^1$ *and* $v \in \mathbb{R}^2$ *and assume* $|v \times \theta| \neq 0$ *and* $|v \cdot \theta| \neq 0$. *Then the unique maximum* (T, M) *of* $\lambda(t)$ *satisfies*

$$T = \log \frac{|v||v'|}{|v \times v'|} + O(1) \tag{6}$$

$$M = \log \frac{|v'|}{|v||v \times v'|} + O(1) \tag{7}$$

provided $\max(\angle v\theta,\ \angle v'\theta,\ \angle vv') \leqslant \pi/4$ *and* $|v' \times \theta||v|/|v \times v'| \leqslant 1/2$.

Proof. From

$$|g_t^\theta v|^2 = |v \cdot \theta|^2 e^{-t} + |v \times \theta|^2 e^t \tag{8}$$

we see that (T, M) is given by

$$T = \log|v \cdot \theta| - \log|v \times \theta|, \tag{9}$$

$$M = -\log|v \cdot \theta| - \log|v \times \theta| - \log 2. \tag{10}$$

Rewrite (9) by eliminating $|v \cdot \theta|$ in favor of $|v|$ to get

$$e^{2T} = \frac{|v \cdot \theta|^2}{|v \times \theta|^2} = \frac{|v|^2}{|v \times \theta|^2} - 1. \tag{11}$$

[1] For more details see the proof of Lemma 1.1 in [Ch].

Since $\angle v\theta \leqslant \pi/4$ iff $T \geqslant 0$, (11) implies $\sqrt{2}|v \times \theta| < |v|$ so that

$$\frac{|v|}{\sqrt{2}|v \times \theta|} \leqslant e^T \leqslant \frac{|v|}{|v \times \theta|}. \tag{12}$$

Using the triangle inequality, the second hypothesis and the fact that the sine function is increasing and nonnegative on $[0, \pi/2]$ we have

$$\frac{|v \times v'|}{|v||v'|} - \frac{|v' \times \theta|}{|v'|} \leqslant \frac{|v \times \theta|}{|v|} \leqslant \frac{|v \times v'|}{|v||v'|} + \frac{|v' \times \theta|}{|v'|}.$$

The first hypothesis now implies

$$(1 - 1/2)\frac{|v \times v'|}{|v||v'|} \leqslant \frac{|v \times \theta|}{|v|} \leqslant (1 + 1/2)\frac{|v \times v'|}{|v||v'|} \tag{13}$$

which together with (12) gives (6). Finally, combining (9) and (10) we have $e^{-M} = 2|v \times \theta|^2 e^T$, so that (7) follows from (12) and (13). □

Lemma 3.2. *Let θ and v be as in Lemma 3.1 and assume v' is another vector with $|v' \times \theta| \neq 0$ and $|v' \cdot \theta| \neq 0$. If $|v| < |v'|$, $|v' \times \theta| \leqslant |v \times \theta|/\sqrt{2}$ and $|v' \cdot \theta| \geqslant \sqrt{2}|v \cdot \theta|$ then there exists a unique $t > 0$ such that $\lambda(t) = -\log |g_t^\theta v'|^2$. Moreover, t and $m := \lambda(t)$ satisfy the following estimates.*

$$t = \log \frac{|v'|^2}{|v \times v'|} + O(1) \tag{14}$$

$$m = \log \frac{1}{|v \times v'|} + O(1) \tag{15}$$

Proof. From (8) we see the unique solution to $|g_t^\theta v| = |g_t^\theta v'|$ is determined by

$$e^{2t} = \frac{|v' \cdot \theta|^2 - |v \cdot \theta|^2}{|v \times \theta|^2 - |v' \times \theta|^2}. \tag{16}$$

The second and third hypotheses imply t is well-defined and that $e^{2t} > 1$ iff $|v| < |v'|$. Hence, the first hypothesis implies $t > 0$.

Note that the second and third hypotheses hold after v and v' are replaced by the vectors $g_t^\theta v$ and $g_t^\theta v'$. Since these vectors have the same length, an elementary calculation shows the sine of the angle ϕ between them is at least 1/3. (Note this is the sine of the angle between $\langle \sqrt{2}, 1\rangle$ and $\langle 1, \sqrt{2}\rangle$.) Since g_t^θ preserves cross products, we have $|v \times v'| = e^{-m} \sin \phi$, which implies (15).

To get (14) we first consider the unique maximum time T' of the function $t \to -\log |g_t^\theta v'|^2$. The analog of (11) for T' is

$$e^{2T'} = \frac{|v' \cdot \theta|^2}{|v' \times \theta|^2} = \frac{|v'|^2}{|v' \times \theta|^2} - 1. \tag{17}$$

Note that $t \leqslant T'$ for the second hypothesis together with (16) and (17) implies

$$e^{2t} \leqslant \frac{|v' \cdot \theta|^2 - |v \times \theta|^2}{|v' \times \theta|^2} \leqslant e^{2T'}.$$

Now using the definition of m, the analog of (8) for v', (16) and (17) we have

$$\begin{aligned} e^{-m} &= |v' \cdot \theta|^2 e^{-t} + |v' \times \theta|^2 e^t \\ &= |v'|^2 e^{-t} \left(\frac{|v' \cdot \theta|^2 + |v' \times \theta|^2 e^{2t}}{|v' \cdot \theta|^2 + |v' \times \theta|^2} \right) \\ &= |v'|^2 e^{-t} \left(1 + \frac{e^{2t} - 1}{e^{2T'} + 1} \right) \end{aligned}$$

so that $|v'|^2 \leqslant e^{t-m} \leqslant 2|v'|^2$ since $0 < t \leqslant T$. (14) now follows from (15). $\square$

Lemma 3.3. *Let V be a discrete subset of $\mathbb{R}^2$ such that $\mathbb{R}v \cap V = \{\pm v\}$ for all $v \in V$ and assume $V \neq \emptyset$. Then $\ell(g_t^\theta V) = |g_t^\theta v|$ for some $t \in \mathbb{R}$ if*

$$2|v||v \times \theta| \leqslant \min\{|v \times u| : |u| \leqslant \sqrt{2}|v|, u \in V, u \neq \pm v\}. \tag{18}$$

In fact, the condition (18) implies $\ell(g_t^\theta V) = |g_t^\theta v|$ for some open interval of t near the unique (possibly infinite) time when $|g_t^\theta v|$ is minimized.

Proof. If $v \times \theta = 0$ then $|g_t^\theta v| = e^{-t/2}|v| < |g_t^\theta u|$ for any $u \in V$ with $|u| > |v|$ since g_t^θ shrinks Euclidean lengths by a factor of at most $e^{t/2}$. Since V is discrete, there are only finitely many $u \in V$ with $|u| \leqslant |v|$. For each such $u \neq \pm v$ we have $|g_t^\theta u| > |g_t^\theta v|$ for some t. Therefore, $\ell(g_t^\theta V) = |g_t^\theta v|$ for all large enough t.

If $v \times \theta \neq 0$ let $\varepsilon = |g_T^\theta v|$ where T is the unique time when $|g_t^\theta v|$ is minimized. Note that the angle between $g_T^\theta v$ and the line $\mathbb{R}\theta$ is $\pi/4$ so that

$$|v \times \theta| e^{T/2} = \varepsilon/\sqrt{2} = |v \cdot \theta| e^{-T/2}.$$

Suppose $u \in V$ is a vector with $|g_T^\theta u| \leqslant |g_T^\theta v|$. Since g_T^θ stretches Euclidean lengths by a factor of at most $e^{|T|/2}$, we have $|u| \leqslant \varepsilon e^{|T|/2} \leqslant \sqrt{2}|v|$ by the above equations. Observing that g_T^θ preserves cross products, we have

$$|v \times u| = |g_T^\theta u \times g_T^\theta v| \leqslant \varepsilon^2 = 2|v \cdot \theta||v \times \theta| < 2|v||v \times \theta|$$

as $v \times \theta \neq 0$. Now (18) implies $u = \pm v$ so that $\ell(g_T^\theta V) = |g_T^\theta v|$. This proves the first part of the lemma while the second part follows by discreteness of V. $\square$

Let $\tilde{W} \subset W$ consist of those vectors w for which there is some vector $v \in Z$ satisfying $|v| \leqslant \sqrt{2}|w|$ and $|w \times v| \leqslant 1/2\sqrt{2}$.

Lemma 3.4. *For any $w \in \tilde{W}$ there exists a unique $v \in Z$ up to sign such that $|w \times v| = \min\{|w \times u| : |u| \leqslant \sqrt{2}|w|, u \in V, u \neq \pm w\}$.*

Proof. Since $W \cap Z = \emptyset$, the hypothesis implies the minimum exists. In fact, it must be realized by some vector in Z, for if $w' \in W$ then $w' = w + du$ for some $u \in Z$ and positive integer d so that $|w \times w'| = |w \times du| \geqslant |w \times u|$. Now let v be the vector associated to w and consider the capped rectangle R defined by the inequalities $|u| \leqslant \sqrt{2}|w|$ and $|w \times u| \leqslant |w \times v|$. It is enough to show $R \cap Z = \{\pm v\}$. Suppose there exists $u \in R \cap Z$ such that $u \neq \pm v$. Note that the area of R is $< 4\sqrt{2}|w \times v| \leqslant 2$ while the area of the parallelogram P with vertices at $\pm u$ and $\pm v$ is exactly 2. This is absurd since $P \subset R$. Therefore, $R \cap Z = \{\pm v\}$. $\square$

For any $w \in \tilde{W}$ define

$$I(w) = \{\theta \in S^1 : |w \times \theta| < |w \times v|/2|w|, w \cdot \theta > 0\}$$

where v is the vector given by the Lemma 3.4.

Lemma 3.5. *If $w \in \tilde{W}$ and $v' \in Z$ satisfy $w \cdot v' > 0$ and $|w \times v'| \leqslant 1/2\sqrt{2}$, then $w' = w + gv \in \tilde{W}$ and if $\varepsilon = |w||w \times v'|/|v'||w \times v| \leqslant 1/5$ then $\overline{I(w')} \subset I(w)$.*

Proof. Since $|w' \times v'| = |w \times v'| \leqslant 1/2\sqrt{2}$ while $w \cdot v' > 0$ implies $|w'| > |v'|$, we have $w \in \tilde{W}$, easily. The angle between w and w' is at most $2\varepsilon|I(w)|$ (where $|\cdot|$ denotes Lebesgue measure induced by arc length) because $|w'| > g|v|$ implies

$$\sin \angle ww' = \frac{|w \times w'|}{|w||w'|} < \frac{|w \times v'|}{|w||v'|} = \frac{\varepsilon |w \times v|}{|w|^2} = 2\varepsilon \sin \frac{|I(w)|}{2}$$

and $\sin^{-1}(2\varepsilon x) \leqslant 2\varepsilon \sin^{-1} x$, $0 \leqslant x \leqslant 1$. Similarly, $|I(w')| < g^{-1}\varepsilon|I(w)|$ because

$$\frac{|w' \times v'|}{|w'|^2} < \frac{|w \times v'|}{g|w'||v'|} = \frac{\varepsilon|w|}{g|w'|}\frac{|w \times v|}{|w|^2}$$

and $|w| < |w'|$. Hence, $\overline{I(w')} \subset I(w)$ provided $(2 + 1/2g)\varepsilon \leqslant 1/2$, which follows from $g \geqslant 1$ and $\varepsilon \leqslant 1/5$. □

Proposition 3.6. *If $(w_j)_{j \geqslant 0}$ is a sequence in W satisfying*

(i) for all j, $w_{j+1} = w_j + gv'$ for some $v' \in Z$ with $|v'| > |w_j|$ and $w_j \cdot v' > 0$,

(ii) $\limsup |w_j \times w_{j+1}| < g/2\sqrt{2}$, and

(iii) $\limsup |w_j||w_j \times w_{j+1}|/|w_{j+1}||w_j \times w_{j-1}| < 1/(5g+5)$

then there exists a piecewise linear function $\Lambda(t)$ satisfying

$$\limsup_{t \to \infty} |-\log \ell(g_t^\theta V)^2 - \Lambda(t)| < \infty$$

(with $\theta = \lim |w_j|^{-1} w_j$) and whose critical points are given by

$$(T_j, M_j) = \left(\log \frac{|w_j||w_{j+1}|}{|w_j \times w_{j+1}|}, \log \frac{|w_{j+1}|/|w_j|}{|w_j \times w_{j+1}|}\right), \tag{19}$$

$$(t_{j+1}, m_{j+1}) = \left(\log \frac{|w_{j+1}|^2}{|w_j \times w_{j+1}|}, \log \frac{1}{|w_j \times w_{j+1}|}\right). \tag{20}$$

where $j > j_1$ for some $j_1 > 0$.

Proof. First, verify $w_j \in \tilde{W}$ for j large enough. Indeed, by (i) $w_{j+1} = w_j + gv'$ for some $v' \in Z$ with $w_j \cdot v' > 0$. Thus, we easily have $|v'| < \sqrt{2}|w_{j+1}|$ and (ii) implies $|w_{j+1} \times v'| = g^{-1}|w_j \times w_{j+1}| \leqslant 1/2\sqrt{2}$. Since v' is the vector associated to w_{j+1}, we note here that for any $\theta \in I(w_{j+1})$ and j large enough

$$|w_{j+1} \times \theta| < \frac{|w_{j+1} \times v'|}{2|w_{j+1}|} = \frac{|w_j \times w_{j+1}|}{2g|w_{j+1}|}. \tag{21}$$

Next, by Lemma 3.5, whose hypothesis $\varepsilon \leqslant 1/5$ is implied by (iii) and $|w_{j+1}| \leqslant (g+1)|v'|$ (from (i)), we have $\overline{I(w_{j+1})} \subset I(w_j)$ for j large enough. Hence, $\cap_{j \geqslant j_0} I(w_j) \neq \emptyset$ for some $j_0 > 0$. Since $|w_{j+1}| > |w_j|$ and $W \subset V$ is discrete, we have $\lim |w_j| = \infty$ so that $\lim |I(w_j)| = 0$, which implies the intersection consists of a single direction and the vectors $|w_j|^{-1} w_j \in I(w_j)$ converge to it. Thus, θ is well-defined; moreover, $\theta \in I(w_j)$ for j large enough.

To define $\Lambda(t)$, we first note by (iii) there exists a $j_1 > 0$ such that $t_j < T_j$ for all $j > j_1$, while $T_j < t_{j+1}$ for all $j \geqslant 0$ since $|w_{j+1}| > |w_j|$. Let $\Lambda(t)$ be the continuous piecewise linear function whose graph is broken precisely at the points (T_j, M_j) and (t_{j+1}, m_{j+1}) for $j \geqslant j_1$; it is uniquely determined by requiring its slope be $+1$ for $t \leqslant T_{j_1}$. Hence, each linear piece of $\Lambda(t)$ has slope ± 1 since $T_j - t_j = M_j - m_j$ and $t_{j+1} - T_j = M_j - m_{j+1}$.

For j large enough we have $\theta \in I(w_{j+1})$ so that Lemma 3.4 implies (18) holds with $v = w_{j+1}$. The hypotheses of Lemma 3.1, with w_j and w_{j+1} in place of v and v', are easily verified using (21) and $|w_{j+1}| > |w_j|$. Hence, we conclude by Lemmas 3.3 and 3.1 that the points (T_j, M_j) lie within a uniform bounded distance of the graph of $f(t) := -\log \ell(g_t^\theta V)^2$.

Observe that $f(t)$ is 1-Lipschitz. Indeed, there exist a sequence of vectors v_k in V and a corresponding sequence of intervals I_k whose nonoverlapping union is all of $\mathbb{R}$ such that for all k, $f(t) = -\log |g_t^\theta v_k|^2$ for all $t \in I_k$. It is readily seen from (8) that f' is monotone on each I_k with absolute value $\leqslant 1$. The proof of the proposition is now complete once we show: **Claim:** the points (t_j, m_j) lie within a uniform bounded distance of the graph of $f(t)$.

To prove the claim we shall apply Lemma 3.2 to the vectors w_j and w_{j+1}. The first hypothesis $|w_j| < |w_{j+1}|$ follows by (i). We record here the second and third hypotheses for later reference:

$$|w_{j+1} \times \theta| \leqslant |w_j \times \theta| / \sqrt{2} \quad \text{and} \quad |w_{j+1} \cdot \theta| \geqslant \sqrt{2} |w_j \cdot \theta|. \tag{22}$$

Using (21), the triangle inequality and then $|w_{j+1}| > |w_j|$ we have

$$|w_{j+1} \times \theta| < \frac{|w_j \times \theta| + |w_{j+1} \times \theta|}{2g}$$

which implies the second hypothesis since $g \geqslant 2$. Next, using $|w_{j+1} \times \theta| < |w_j \times \theta|$ and $|w_{j+1}| \geqslant \sqrt{2} |w_j|$, which holds by (i) again, we obtain the third hypothesis. Lemma 3.2 now implies (t_{j+1}, m_{j+1}) lies within bounded distance of the point (t, m) determined by

$$e^{-m/2} = |g_t^\theta w_j| = |g_t^\theta w_{j+1}|.$$

By definition, we have $\ell(g_t^\theta V) \leqslant e^{-m/2}$. To get an inequality in the other direction, let ϕ be the angle between $g_t^\theta w_j$ and $g_t^\theta w_{j+1}$ and h the height of the isosceles triangle formed by them. Since $w_{j+1} = w_j + g v'$ we have

$$h = e^{-m/2} \cos(\phi/2) \qquad \text{and} \qquad |g_t^\theta v'| = (2 e^{-m/2} / g) \sin(\phi/2). \tag{23}$$

Observe that the distance between any two lines parallel to $g_t^\theta v'$ that intersect $g_t^\theta Z$ is an integer multiple of $1/|g_t^\theta v'|$ and the same statement holds if W is replaced by Z. Hence, the length of any vector in $g_t^\theta V$ which is not a multiple of $g_t^\theta v'$ is at least h, provided $h|g_t^\theta v'| \leqslant 1/2$, but this holds because g_t^θ is area-preserving so that $h|g_t^\theta v'| = |w_j \times w_{j+1}|/2g \leqslant 1/2$ by (ii). Therefore, $\ell(g_t^\theta V) \geqslant \min(|g_t^\theta v'|, h)$. Now (22) implies $\sin\phi \geqslant 1/3$, i.e. ϕ is bounded away from 0 and π. Hence, from (23) we see there is a universal constant $c > 0$ such that $\ell(g_t^\theta V) \geqslant ce^{-m/2}/g$. It follows that $|m - f(t)| \in O(\log g)$ and since g is fixed, this completes the proof of the proposition. ☐

4 Density of primitive lattice points

The main result of this section is Corollary 4.5. It will be needed in §5 to find, given a vector $w \in W$, vectors $w' \in W$ such that $w' = w + gv$ for some $v \in Z$ satisfying certain given inequalities on $|v|$ and $|w \times v|$.

4.1 Continued fractions in vector form

Recall each $\alpha \in \mathbb{R}$ admits an expansion of the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} \qquad a_0 \in \mathbb{Z}, a_1, a_2, \dots \in \mathbb{N} \tag{24}$$

whose terms are uniquely determined except for a two-fold ambiguity when α is rational; e.g. $22/7 = 3 + 1/7 = 3 + 1/(6 + 1/1)$. The kth *convergent* of α is the reduced fraction p_k/q_k that results upon simplifying the expression obtained by truncating (24) so that the last term is a_k. The convergents of α satisfy the recurrence relations

$$\begin{array}{ll} p_{k+1} = a_{k+1}p_k + p_{k-1} & p_0 = a_0,\ p_{-1} = 1 \\ q_{k+1} = a_{k+1}q_k + q_{k-1} & q_0 = 1,\ \ q_{-1} = 0 \end{array}, \tag{25}$$

the identity

$$p_k q_{k+1} - p_{k+1} q_k = (-1)^{k+1} \tag{26}$$

and the inequalities

$$\frac{1}{q_k(q_{k+1} + q_k)} < \left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}}. \tag{27}$$

A rational p/q is said to be a *best approximation of the second kind* if

$$|q\alpha - p| \leqslant |n\alpha - m| \qquad \text{for all } m \in \mathbb{Z},\ n = 1, \dots, q \tag{28}$$

and this property characterises the convergents of α modulo the 0th convergent a_0, which is a best approximation to α iff the fractional part of α is $\leqslant 1/2$. The following is a useful test for a rational to be a convergent:

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{2q^2},\ \ \gcd(p,q) = 1 \quad \Rightarrow \quad (p,q) = \pm(p_k, q_k) \text{ for some } k \geqslant 0. \tag{29}$$

It will be convenient for us to recast the above facts in vector form. Setting $v_k := \langle p_k, q_k \rangle$, the recurrence relations (25) and the identity (26) become

$$v_{k+1} = a_{k+1} v_k + v_{k-1} \qquad v_0 = \langle a_0, 1 \rangle, \quad v_{-1} = \langle 1, 0 \rangle \tag{30}$$

and

$$v_k \times v_{k+1} := p_k q_{k+1} - p_{k+1} q_k = (-1)^{k+1}. \tag{31}$$

Although (27) can easily be rewritten in vector notation, the resulting expression looks awkward because of the distinguished nature of the coordinate directions. Instead, we shall use the following analog of (27) which is expressed in terms of the vector $w := \langle \alpha, 1 \rangle$ and its Euclidean length $|w|$:

$$\frac{1}{|v_{k+1} + v_k|} < \frac{|w \times v_k|}{|w|} < \frac{1}{|v_{k+1}|} \tag{32}$$

To see (32) recall that convergents alternate on both sides of α and (27) follows from the fact that the rational $(p_k + p_{k+1})/(q_k + q_{k+1})$ always lies on the same side of α occupied by p_k/q_k. (32) follows similarly from a comparison of the components of $v_{k+1} + v_k$, w and v_{k+1} in the direction perpendicular to v_k.

Definition 4.1. If $\theta = |w|^{-1} w$ where $w = \langle \alpha, 1 \rangle$ as above, then we define

$$\operatorname{Spec}(\theta) = \{v_0, v_1, \dots\} = \{v_k\}_{k \geqslant 0}$$

and call the vectors v_k the *convergents* of θ. The definition is extended to all unit vectors by requiring

$$\operatorname{Spec}(1,0) = \{\langle 1, 0 \rangle\}, \quad \operatorname{Spec}(-\theta) = -\operatorname{Spec}(\theta)$$

and to all nonzero vectors by $\operatorname{Spec}(w) = \operatorname{Spec}(|w|^{-1} w)$. We shall also denote by $\operatorname{spec}(w)$ the sequence of Euclidean lengths of vectors in $\operatorname{Spec}(w)$.

The next lemma was motivated by (29) and will be needed in §5.

Lemma 4.2. *Let w be a vector that makes an angle ϕ with the y-axis. Then for any $v \in \mathbb{Z}^2$ such that $|v| \cos\phi > 1$ we have*

$$\frac{|w \times v|}{|w|} \leqslant \frac{1}{2|v|}, \quad \gcd(v) = 1. \quad \Rightarrow \quad \pm v \in \operatorname{Spec}(w). \tag{33}$$

Proof. Let $P = P(v, w)$ be the closed parallelogram that has $\pm v$ as two of its vertices, one pair of sides parallel to w and the other pair parallel to the x-axis. The characterisation of convergents given in (28) is equivalent to the statement that every nonzero $u \in P \cap \mathbb{Z}^2$ belongs to the union of the two sides of P parallel to w. Hence, it is enough to verify this statement under the given hypotheses.

Apply Lemma 3.3 with $\theta = |w|^{-1} w$ (and V the set of integer lattice points which are not scalar multiples of v) to conclude there is some T for which $g_T^\theta v$

is the shortest vector in $V' = g_T^\theta(\mathbb{Z}^2 - 0)$. Let E be the inverse image under g_T^θ of the largest closed disk centered at the origin whose interior is disjoint from V'. The boundary of E is an ellipse passing through the points $\pm v$ while the interior contains no integer lattice points other than the origin.

Without loss of generality, we assume w lies in the first quadrant and v in the upper half plane. There are two cases. First, if v lies to the right of w, then E contains P and we are done. Now, if v lies to the left of w, then let x be the length of a horizontal side of P and y the vertical distance between v and its reflection in the line $\mathbb{R}w$. It is enough to show $x < 1$ and $y < 1$. If z is the distance between v and its reflection then $z = 2|v| \sin \angle wv = 2|w \times v|/|w| \leqslant 1/|v| < \cos\phi$ so that $x = z \sec\phi < 1$ and $y = z \sin\phi < 1$. □

4.2 Density of rationals in intervals

For any $\Omega \subset \mathbb{R}^2$ with $0 < \text{area}(\Omega) < \infty$ let

$$\text{dens}(\Omega) = \frac{{}^{\#}Z \cap \Omega}{\text{area}(\Omega)}.$$

Lemma 4.3. *If Ω is a compact convex subset of the first quadrant containing $(0,0)$, $(1,0)$ and $(0,1)$ but not $(1,1)$ then* $\text{dens}(2\Omega \setminus \Omega) > 8/27$.

Proof. By convexity there is a function $y = f(x), 0 \leqslant x \leqslant 1$ whose graph is contained in $\partial\Omega$ and $f(0) \geqslant 1$. Similarly, there is a $x = g(y), 0 \leqslant y \leqslant 1$ whose graph is contained in $\partial\Omega$ and $g(0) \geqslant 1$. Without loss of generally, assume

$$f(1/2) \geqslant g(1/2).$$

Let $\Omega_1 \subset \Omega$ be the part below the graph of f and $\Omega_2 = \Omega \setminus \Omega_1$. Since Ω_1 lies below any tangent line at $(1/2, f(1/2))$, $\text{area}(\Omega_1) \leqslant f(1/2)$. There are three cases. First, if $f(1/2) < 1$ then the assumption above implies $g(0) < 3/2$ so that $\text{area}(\Omega_2) \leqslant 1/8$. Since $(1,1) \in 2\Omega \setminus \Omega$ we have

$$\text{dens}(2\Omega \setminus \Omega) \geqslant \frac{1}{3}\left(\frac{1}{f(1/2) + 1/8}\right) > 8/27.$$

Second, if $f(1/2) \geqslant 1$ and $f(1) < 1/2$ then it still follows that $g(0) < 3/2$ while the number of vectors in $2\Omega \setminus \Omega$ of the form $(1, n)$ is $\lfloor 2f(1/2) \rfloor$ so that

$$\text{dens}(2\Omega \setminus \Omega) > \frac{1}{3}\left(\frac{2f(1/2) - 1}{f(1/2) + 1/8}\right) \geqslant 8/27.$$

Finally, if $f(1/2) \geqslant 1$ and $f(1) \geqslant 1/2$ then the assumption above implies $g(0) < 2$ so that $\text{area}(\Omega_2) \leqslant 1/2$. Apart from points of the form $(1, n)$ we also have $(2,1) \in 2\Omega \setminus \Omega$. Hence,

$$\text{dens}(2\Omega \setminus \Omega) > \frac{1}{3}\left(\frac{2f(1/2)}{f(1/2) + 1/2}\right) \geqslant 4/9 > 8/27.$$

□

Note if $\Omega_a = \{(x, y) : x + y \leqslant a, x \geqslant 0, y \geqslant 0\}$ then

$$\begin{aligned}&\operatorname{dens}(2\Omega_a \setminus \Omega_a) = 2/3a^2 \quad \text{for} \quad 1 \leqslant a < 3/2 \quad \text{and}\\&\operatorname{dens}(\gamma\Omega_1 \setminus \Omega_1) = 0 \quad \text{for} \quad \gamma < 2\end{aligned}$$

show that the constants in the preceding lemma are sharp.

Let $S^1(\mathbb{Q})$ be the set of unit vectors of the form $v/|v|$ for some $v \in Z$ and $S^1_b(\mathbb{Q})$ the subset formed by those with $|v| \leqslant b$. Let $\mathcal{I}_b$ denote the collection of intervals in S^1 with endpoints in $S^1_b(\mathbb{Q})$. For any interval $I \subset S^1$ let

$$\Omega(I, b) = \{v \in \mathbb{R}^2 : \mathbb{R}_+ v \cap I \neq \emptyset, |v| \leqslant b\}.$$

Proposition 4.4. *For any $I \in \mathcal{I}_b$,* $\operatorname{dens}(2\Omega(I, b) \setminus \Omega(I, b)) > 8/27$.

Proof. First we show if I is minimal, i.e. $S^1_b(\mathbb{Q}) \cap \operatorname{int} I = \emptyset$, then its endpoints correspond to a pair of vectors $v, v' \in Z$ such that $|v \times v'| = 1$. Indeed, there is a linear map γ that sends v to $(0, 1)$ and v' to $(a, a') \in \mathbb{Z}^2$ with $0 \leqslant a' < a = |v \times v'|$. If $a > 1$ then $\gamma^{-1}(1, 1) \in S^1_b(\mathbb{Q}) \cap \operatorname{int} I$; hence, $a = 1$. Now observe $\gamma\Omega(I, b)$ is a compact convex set satisfying the hypothesis of Lemma 4.3 and since γ preserves density, the proposition holds for minimal I in $\mathcal{I}_b$. Since every interval in $\mathcal{I}_b$ is a (finite) disjoint union of minimal ones, this completes the proof. □

Theorem 3. *Let $\Omega = \Omega(I, b)$ where $I = \{\theta' \in S^1 : \sin\angle\theta\theta' < \varepsilon/b\}$, $\theta \in S^1$, $\varepsilon > 0$ and $b \geqslant 1$. Then* $\operatorname{spec}(\theta) \cap [\varepsilon^{-1}, b] \neq \emptyset$ *implies* $\operatorname{dens}(2\Omega \setminus \Omega) > 4/27\pi$.

Proof. Let $v_k \in \operatorname{Spec}(\theta)$ be the convergent with length $|v_k| = \max \operatorname{spec}(\theta) \cap [\varepsilon^{-1}, b]$. Then the RHS of (32) implies (the direction of) v_k lies in I. Without loss of generality, assume v_k lies to the left of θ. Let θ' be the right endpoint of I and $v'_l \in \operatorname{Spec}(\theta')$ the convergent with length $|v'_l| = \max \operatorname{spec}(\theta) \cap [1, b]$.

Note that v_k does not lie strictly between v'_l and v'_{l+1} since the length of any such vector in Z is at least $|v'_{l+1}| > b$. Nor can $v_k = v'_l$ since the RHS of (32) would imply v'_l lies strictly to the right of v_k. Therefore, v'_l lies strictly to the right of v_k. Let $\Omega' = \Omega(J, b)$ where J is the interval with left endpoint v_k and right endpoint v'_l.

The interval I has length $|I| = 2\sin^{-1}(\varepsilon/b) \leqslant \pi\varepsilon/b$. If $|v'_l| \geqslant 2\varepsilon^{-1}$ then the RHS of (32) implies $|J| > \sin^{-1}(\varepsilon/b) - \sin^{-1}(\varepsilon/2b) > \varepsilon/2b$ (we may assume $\varepsilon < b$ for otherwise the theorem is easily seen to hold) while if $|v'_l| < 2\varepsilon^{-1}$ then $|J| > \sin\angle v_k v'_l \geqslant \frac{1}{|v_k|\,|v'_l|} > \varepsilon/2b$. In either case, we have $|J| > |I|/2\pi$.

If v'_l lies to the left of θ' or $|v'_{l+1}| > 2b$ then $Z \cap (2\Omega' \setminus \Omega') \subset Z \cap (2\Omega \setminus \Omega)$ since any vector strictly between v'_l and v'_{l+1} has length greater than $2b$. In this case, Proposition 4.4 implies $\operatorname{dens}(2\Omega \setminus \Omega) > 4/27\pi$. If $|v'_l| \geqslant 2\varepsilon^{-1}$ then arguing as before we see the angle between v_k and v'_{l+1} is at least $\varepsilon/2b > |I|/2\pi$ so that we may again conclude that $\operatorname{dens}(2\Omega \setminus \Omega) > 4/27\pi$. Therefore, we may assume v'_{l+1} lies between v_k and θ', $|v'_{l+1}| \leqslant 2b$ and $|v'_l| < 2\varepsilon^{-1}$ in the remaining. Assuming $\varepsilon < b/\sqrt{2}$ as we may, since the theorem is easily seen to hold otherwise, we obtain the following criterion for a vector left of θ' to lie in I:

$$\frac{|v \times \theta'|}{|v|} < \frac{\sqrt{2}\varepsilon}{b} \tag{34}$$

Note that a vector of the form $av'_l + v_{l-1}$ lies to the left of θ' if $a \leqslant a_{k+1}$. We will show the number of vectors of length $\geqslant b/2$ satisfying the above conditions is at least $c_0 b\varepsilon$ for some absolute constant $c_0 > 0$. This will complete the proof since between any two vectors of length $\geqslant b/2$ (but $\leqslant b$) there is a vector whose length is $> b$ and $\leqslant 2b$. Using the RHS of (32) we have

$$|v \times \theta'| = |v'_{l+1} \times \theta' - (a_{l+1} - a)v'_l \times \theta'| \leqslant \frac{1 + |a_{l+1} - a|}{|v'_{l+1}|}. \tag{35}$$

Assuming $|v| \geqslant b/2$ ($v = av'_l + v_{l-1}$) we see (34) holds for $\lfloor b\varepsilon/\sqrt{2} \rfloor$ integers $a \leqslant a_{l+1}$. Among these there are at least $b/2|v'_l| \geqslant b\varepsilon/4$ which make $|v| \geqslant b/2$. Since $\text{area}(\Omega) < \pi b\varepsilon/2$, we conclude $\text{dens}(2\Omega \setminus \Omega) > 1/6\pi$. □

4.3 Density in a strip

Theorem 4. *Let* $\Sigma = \{v \in \mathbb{R}^2 : |v \times \theta| < \varepsilon, b < |v| \leqslant 2b, v \cdot \theta > 0\}$ *where* $\theta \in S^1$, $0 < \varepsilon \leqslant 1$ *and* $b \geqslant 1$. *Then* $\text{spec}(\theta) \cap [\varepsilon^{-1}, b] \neq \emptyset$ *implies* $\text{dens}(\Sigma) > 2/27\pi$.

Proof. Let $v_k \in \text{Spec}(\theta)$ with $|v_k| = \max \text{spec}(\theta) \cap [\varepsilon^{-1}, b]$. There are two cases. If $|v_k| \geqslant 2\varepsilon^{-1}$ let $\Omega = \Omega(\theta, \varepsilon, b)$ be the region in Theorem 3 and put $\Omega' = \Omega(\theta, \varepsilon/2, b)$. Then Theorem 3 implies $\text{dens}(2\Omega' \setminus \Omega') > 4/27\pi$. Since $2\Omega' \setminus \Omega' \subset \Sigma$ and $\text{area}(\Omega') > b\varepsilon/2$ it follows there are at least $2b\varepsilon/9\pi$ vectors in $Z \cap \Sigma$. If $|v_k| < 2\varepsilon^{-1}$ let $v = av_k + v_{k-1}$ and use (35) for θ and the hypothesis $|v_k| \geqslant \varepsilon^{-1}$ to deduce

$$|v \times \theta| \leqslant \frac{1 + |a_{k+1} - a|}{|v_{k+1}|} < \frac{1 + |a_{k+1} - a|}{a_{k+1}|v_k|} \leqslant \varepsilon$$

for all a such that $1 \leqslant a < 2a_{k+1}$. Since $|v_k| < 2\varepsilon^{-1}$ the number v with $b < |v| \leqslant 2b$ is $\geqslant b\varepsilon/2$. In either case, we have found $2b\varepsilon/9\pi$ vectors $Z \cap \Sigma$. By an elementary analysis we obtain

$$3b\varepsilon/2 \leqslant 2b\varepsilon \left(1 - \frac{\varepsilon^2}{4b^2}\right) \leqslant \text{area}(\Sigma) \leqslant 2b\varepsilon \left(1 + \frac{\varepsilon^2}{2b^2}\right) \leqslant 3b\varepsilon. \tag{36}$$

Using the RHS we get $\text{dens}(\Sigma) > 2/27\pi$. □

Corollary 4.5. *There are universal constants* $\rho_1 > 0$ *and* $c_1 > 0$ *such that* $\text{spec}(w) \cap [\varepsilon^{-1}, b] \neq \emptyset$ *(and* $0 < \varepsilon \leqslant 1 \leqslant b$*) implies there are* $\rho_1 b\varepsilon$ *vectors* $v \in Z$ *satisfying the inequalities*

$$w \cdot v > 0, \quad b \leqslant |v| \leqslant 2b, \quad c_1\varepsilon|w| \leqslant |w \times v| \leqslant \varepsilon|w|.$$

Proof. Let $\Sigma = \Sigma(\varepsilon, b)$ be as in Theorem 4 with $\theta = |w|^{-1}w$. First we claim $\mathrm{dens}(\Sigma)$ is bounded above by some universal constant. Indeed, from the cross product formula we see there's a universal constant $C > 1$ such that the largest (resp. smallest) angle between two vectors in $Z \cap \Sigma$ is $< C\varepsilon/b$ (resp. $> C^{-1}/b^2$). Hence, the number of vectors in $Z \cap \Sigma$ is at most $C^2 b\varepsilon$ so that the LHS of (36) implies the claim.

Now let $\Sigma_1 = \Sigma(c_1\varepsilon, b)$ and $\Sigma_2 = \Sigma \setminus \Sigma_1$. Observe that the claim implies for any $\rho < 2/27\pi$, c_1 can be chosen sufficiently small so that $\mathrm{dens}(\Sigma_2) > \rho$. Since (36) implies $\mathrm{area}(\Sigma_2) \geqslant cb\varepsilon$ for some universal $c > 0$, the corollary follows. ☐

5 Nonergodic directions and sublinear growth

In this section we prove Theorem 1.

Let $e_0 > \max(d_0, 2)$ be given. The construction involves the choice of a sequence $(\delta_j)_{j\geqslant 0}$ descending to zero at some prescribed rate as required by Lemmas 5.3 and 5.4 below. For concreteness we set

$$\delta_j := \frac{e_0}{j+1} \quad \text{for all } j \geqslant 0.$$

In addition, we also fix a constant

$$C := \max(2g + 1, c_1^{-1} e^{2e_0})$$

needed in the statement of Lemma 5.5 below.

Our goal is to find sequences $(w_j)_{j\geqslant 0}$ in W satisfying

$$|w_j|^{1+\delta_j} \leqslant |w_{j+1}| \leqslant C|w_j|^{1+\delta_j}, \quad \frac{C^{-1}}{\log|w_j|} \leqslant |w_j \times w_{j+1}| \leqslant \frac{C}{\log|w_j|} \tag{37}$$

and

$$w_{j+1} = w_j + gv' \quad \text{for some } v' \in Z \text{ with } \quad |v'| > |w_j| \text{ and } w_j \cdot v' > 0 \tag{38}$$

for all $j \geqslant 0$.

Definition 5.1. If w_j and w_{j+1} satisfy (37) and (38) then we say w_{j+1} is a *child* of w_j. More precisely, (37) and (38) define a family $\{\prec_j\}$ of binary relations on W and to say w_{j+1} is a child of w_j is equivalent to the statement $w_j \prec_j w_{j+1}$.

Definition 5.2. We say $(w_j)_{j\geqslant 0}$ is *admissible* if $|w_0| > 1$, $w_0 \pm (x_0, y_0) \in g\mathbb{Z}^2$ and for all $j \geqslant 0$, w_{j+1} is a child of w_j. A finite sequence $(w_0, \ldots, w_k)$ is admissible if the latter condition holds for $0 \leqslant j < k$.

The choice of (δ_j) was motivated by the next two lemmas, which are stated more generally for a sequence of positive δ_j.

Lemma 5.3. *If $(\delta_j)_{j\geqslant 0}$ is a sequence of positive real numbers such that*

$$\liminf j\delta_j > 1 \quad \textit{and} \quad e_0 = \limsup j\delta_j < \infty$$

and $(w_j)_{j\geqslant 0}$ is an admissible sequence then $\lim |w_j|^{-1}w_j$ is a slowly divergent nonergodic direction whose sublinear rate is at most $1 - 1/e_0$.

Proof. First note that the hypotheses imply (i) $\lim \delta_j = 0$ and (ii) $\sum \delta_j = \infty$. Let $S_j = \sum_{i<j} \delta_i$ and $R_j = \sum_{i<j} \log(1+\delta_i)$. Claim: $\lim R_j/S_j = 1$. Indeed, for any $x \in [0,1]$ we have $x(1-x) \leqslant \log(1+x) \leqslant x$. Using (i) we may fix $K > 1$ so that $R_j \leqslant S_j \leqslant KR_j$ for all j. Then $\lim R_j = \infty$ by (ii). Let $c > 1$ be given. Using (i) again we fix j_0 large enough so that $\delta_j \leqslant \sqrt{c}\log(1+\delta_j)$ for all $j \geqslant j_0$. Now $S_j \leqslant KR_{j_0} + \sqrt{c}R_j \leqslant cR_j$ for all large enough j. Since $c > 1$ was arbitrary, this poves the claim.

From the first inequality in (37) we have $\log|w_j| \geqslant (\log|w_0|)\prod_{i<j}(1+\delta_i)$. Now $\liminf j\delta_j > 1$ implies for some $p > 1$ and $C' > 0$ we have $S_j \geqslant p\log j - C'$ for all $j > 0$. The same statement for R_j holds by the preceding claim. Hence, $\prod_{i<j}(1+\delta_i) \geqslant c_1 j^p$ for some $c_1 > 0$. The last two inequalities in (37) imply the cross products form a summable series while (38) implies $|w_j|$ is increasing. Hence, $\lim |w_j|^{-1}w_j$ is a nonergodic direction, by Lemma 2.4.

It is clear from the preceding that $\lim |w_j| = \infty$; however, a much stronger statement holds. First, there is some $p > 1$ and $c_2 > 0$ such that (for all $j > 0$)

$$\delta_j \log|w_j| \geqslant (c_2 \log|w_0|)j^{p-1}. \tag{39}$$

Now the second inequality in (37) implies $\log|w_j| \leqslant (\log|w_0|)\prod_{i<j}(1+\delta_i) + j\log C$. Using $\limsup j\delta_j < \infty$ and arguing as before we find $q > e_0$ and $C'' > 0$ such that $R_j \leqslant q\log j + C''$ for all $j > 0$. Thus $\prod_{i<j}(1+\delta_i) \leqslant c_3 j^q$ some $c_3 > 1$ and $\log|w_j| \leqslant c_3 j^q \log|w_0| + j\log C$. Using $\log(1+x+y) \leqslant \log(1+x) + \log(1+y)$ we conclude: for some $q \geqslant p$ and $C''' > 0$ we have (for all $j > 0$)

$$\log\log|w_j| \leqslant \log\log|w_0| + (q+1)\log j + C'''. \tag{40}$$

It follows from (39) and (40) that $\lim |w_j|^{\delta_j}/\log|w_j| = \infty$.

The hypotheses of Proposition 3.6 are satisfied since (i) is the same as (38) while (37) and (39) imply $\lim |w_j|/|w_{j+1}| = 0$ and $\lim |w_j \times w_{j+1}| = 0$, where the ratio of consecutive cross-product terms is bounded. Propositions 2.1 and 3.6 imply $\lim |w_j|^{-1}w_j$ is slowly divergent since

$$\lim \frac{M_j}{T_j} = \lim \frac{\delta_j \log|w_j| - \log|w_j \times w_{j+1}|}{(2+\delta_j)\log|w_j| - \log|w_j \times w_{j+1}|} = \lim \frac{\delta_j}{2+\delta_j} = 0$$

while the sublinear rate is at most

$$\begin{aligned}\limsup \frac{\log M_j}{\log T_j} &= \limsup \frac{\log(\delta_j \log|w_j| - \log|w_j \times w_{j+1}|)}{\log((2+\delta_j)\log|w_j| - \log|w_j \times w_{j+1}|)} \\ &= 1 - \liminf \frac{-\log\delta_j}{\log\log|w_j|} \leqslant 1 - \frac{1}{q}\end{aligned}$$

because $-\log\delta_j \geqslant \log j + O(1)$ and $R_j \leqslant q\log j + O(1)$. The proof is completed by observing that $q > e_0$ may be chosen arbitrarily close to e_0. $\square$

Lemma 5.4. *Let $(\delta_j)_{j\geqslant 0}$ be a sequence of positive real numbers such that*

$$\liminf j\delta_j > 2 \quad \textit{and} \quad \limsup j\delta_j < \infty$$

and $(w_j)_{j\geqslant 0}$ an admissible sequence. Then for any $\varepsilon > 0$ there exists $L_0 = L_0(\varepsilon)$ such that $|w_0| \geqslant L_0$ implies

$$\sup_{j\geqslant 0} \frac{(\log|w_j|)^2}{|w_j|^{\delta_j\delta_{j+1}}} \leqslant \varepsilon. \tag{41}$$

Proof. Repeating the arguments in the preceding proof with the stronger hypotheses we find there are constants $p > 2$, $c_2 > 0$, $q \geqslant p$ and $C''' > 0$ such that (39) and (40) hold for all $j > 0$. It follows that the difference

$$\delta_j\delta_{j+1}\log|w_j| - 2\log\log|w_j| \geqslant \\ (c_2'\log|w_0|)j^{p-2} - 2\log\log|w_0| - 2(q+1)\log j - 2C''' =: \beta(j)$$

for some $c_2' > 0$. The function $\beta(j)$ is increasing for $j \geqslant 1$ and by choosing L_0 large enough we have $\beta(1) \geqslant -\log\varepsilon$. By choosing L_0 even larger so that $(\log|w_0|)^2/|w_0|^{\delta_0\delta_1} \leqslant \varepsilon$ we obtain (41). □

Lemma 5.5. *Let $(w_0,\ldots,w_j)$ be a admissible sequence and suppose*

$$\operatorname{spec}(w_j) \cap [e^t|w_j|\log|w_j|, |w_j|^{1+\delta_j}] \neq \emptyset \tag{42}$$

for some $t \in [0, 2e_0]$. Then w_j has at least $\rho_1 e^{-2e_0}|w_j|^{\delta_j}/\log|w_j|$ children and these vectors satisfy

$$\operatorname{spec}(w_{j+1}) \cap [e^{t-\delta_j}|w_{j+1}|\log|w_{j+1}|, |w_{j+1}|^{1+\delta_{j+1}}] \neq \emptyset \tag{43}$$

provided $|w_0|$ is large enough.

Proof. Apply Corollary 4.5 with $\varepsilon^{-1} = e^t|w_j|\log|w_j|$ and $b = |w_j|^{1+\delta_j}$ to get $\rho_1 e^{-t}|w_j|^{\delta_j}/\log|w_j|$ vectors $v \in Z$ satisfying the inequalities

$$w_j \cdot v > 0, \quad |w_j|^{1+\delta_j} \leqslant |v| \leqslant 2|w_j|^{1+\delta_j}, \quad \frac{c_1 e^{-t}}{\log|w_j|} \leqslant |w_j \times v| \leqslant \frac{e^{-t}}{\log|w_j|}. \tag{44}$$

The vector $w_{j+1} = w_j + gv$ satisfies (38) by the first inequality in (44), which together with the second inequality implies $|w_{j+1}| \geqslant |v| \geqslant |w_j|^{1+\delta_j}$. The third implies $|w_{j+1}| \leqslant |w_j| + g|v| \leqslant (2g+1)|w_j|^{1+\delta_j}$, which together with the remaining inequalities and $|w_j \times w_{j+1}| = g|w_j \times v|$ implies (37) for the given value of C. Therefore, w_{j+1} is a child of w_j and since $t \leqslant 2e_0$, this proves the first part.

Using $|w_{j+1}| > g|v|$, the last inequality in (44) and $g \geqslant 2$ we have

$$\frac{|w_{j+1} \times v|}{|w_{j+1}|} < \frac{|w_j \times v|}{g|v|} \leqslant \frac{e^{-t}}{g|v|\log|w_0|} < \frac{1}{2|v|}$$

as soon as $\log|w_0| > 1$. Since $v \in Z$, Lemma 4.2 implies $+v$ is a convergent of w_{j+1}, where the sign follows from $w_{j+1} \cdot v > 0$ and the fact that all angle between a vector and its convergents are acute. (See Remark 5.6 below for an explanation of how the hypothesis of Lemma 4.2 is satisfied.)

Let $v' \in \mathrm{Spec}(w_{j+1})$ be the next convergent after v. Using the RHS of (32), the second to last inequality in (44), $t \leqslant 2e_0$, $|w_j| < |w_{j+1}|$ and (41) we have

$$|v'| \leqslant |w_{j+1}||w_j \times v|^{-1} \leqslant c_1^{-1}e^{2e_0}|w_{j+1}|\log|w_{j+1}| \leqslant |w_{j+1}|^{1+\delta_{j+1}}$$

provided $|w_0|$ is chosen large enough as required by Lemma 5.4 for $\varepsilon = c_1 e^{-2e_0}$.

Using the LHS of (32), $|v| < |w_{j+1}|$, $t \geqslant 0$ and $\delta_j \leqslant \delta_0$ we have

$$\begin{aligned}|v'| &\geqslant |w_{j+1}||w_j \times v|^{-1} - |v| \geqslant |w_{j+1}|(e^t \log|w_j| - 1)\\ &\geqslant e^{t-\delta_j}|w_{j+1}|(e^{\delta_j}\log|w_j| - e^{\delta_0}).\end{aligned}$$

On the other hand, since $|w_{j+1}| \leqslant |w_j| + g|v| \leqslant (2g+1)|w_j|^{1+\delta_j}$ we have

$$\log|w_{j+1}| \leqslant e^{\delta_j}\log|w_j| - \delta_j^2 \log|w_j| + \log(2g+1)$$

from which is follows $|v'| \geqslant e^{t-\delta_j}|w_{j+1}|\log|w_{j+1}|$ if $|w_0|$ is chosen large enough as required by Lemma 5.4 for $\varepsilon = (2g+1)^{-1}e^{-e^{\delta_0}}$. □

Remark 5.6. In order to satisfy the hypothesis of Lemma 4.2 one needs to make a minor technical assumption that the angles ϕ_j made between the vectors w_j of an admissible sequence and the y-axis are bounded away from $\pi/2$. This can be ensured by choosing ϕ_0 close to the y-axis, using (37) and the cross product formula to control the angles $\angle w_j w_{j+1}$, and then requiring $|w_0|$ large enough.

It would be desirable if the conclusion of Lemma 5.5 could be strengthened so that the newly constructed vectors satisfy (43) without the "$-\delta_j$" in the exponent, for then we can use the lemma to construct admissible sequences by recursive definition. However, it can be shown that this stronger statement is false. (This uses a result of Boshernitzan–see the appendix to [Ch].) Fortunately, the induction can be rescued by using a slight variation of the condition (42).

Let W_j be the set of $w \in W$ with the following property: for all $t \geqslant \delta_j$,

$$\mathrm{spec}(w) \cap [e^t|w|\log|w|, |w|^{1+t}] \neq \emptyset. \tag{45}$$

Lemma 5.7. *There exists $L_0 > 0$ such that (45) holds for all $t \geqslant e_0$ if $|w| \geqslant L_0$.*

Proof. If (45) does not hold for some $t \geqslant e_0$, then w has convergents v_k and v_{k+1} satisfying $|v_k| < e^t|w|\log|w|$ and $|v_{k+1}| > |w|^{1+t}$. On the one hand we have $|w \times v_k| < |w|/|v_{k+1}| < |w|^{-t}$ by (32); on the other hand we have (3) implies $|w \times v_k| > c_0/|v_k|^{d_0} > c_0 e^{-d_0 t}|w|^{-d_0}(\log|w|)^{-d_0}$. These inequalities contradict each other if

$$t \geqslant \frac{d_0\log|w| + d_0\log\log|w| - \log c_0}{\log|w| - d_0}.$$

Therefore, if L_0 is chosen large enough so that the RHS is $< e_0$ for $|w| \geqslant L_0$, then (45) holds for all $t \geqslant e_0$. □

Proposition 5.8. *There exist $L_0 > 0$ and $\rho_2 > 0$ such that if $w_j \in W_j$ belongs to an admissible sequence $(w_0, \ldots, w_j)$ with $|w_0| \geqslant L_0$, then it has $\rho_2|w_j|^{\delta_j}/\log|w_j|$ children contained in the set W_{j+1}.*

Proof. Let $v_k \in \mathrm{Spec}(w_j)$ be the unique convergent of w_j determined by the condition $|v_k| \leqslant |w_j|^{1+\delta_j} < |v_{k+1}|$ so that

$$|v_k| = e^{t_1}|w_j|\log|w_j| \quad \text{and} \quad |v_{k+1}| = |w_j|^{1+t_2}$$

for some $t_1 \geqslant \delta_j$ and $t_2 \leqslant t_1$. If $t_1 \geqslant e_0 + \delta_j$ then w_j satisfies the hypothesis of Lemma 5.5 with $t = e_0 + \delta_j$ and each child constructed by the lemma satisfies (43), which is easily seen to imply (45) for $t \in [\delta_{j+1}, e_0]$. By Lemma 5.7 it follows that all children constructed lie in the set W_{j+1}. Therefore, the conclusion of the proposition holds in this case for $\rho_2 = \rho_1 e^{-2e_0}$.

Now consider the case $t_1 < e_0 + \delta_j$. This time Lemma 5.5 is applied with $t = t_1$ to obtain the same number of children as before, each satisfying (43) with t replaced by t_1. Let W' consist of those children which do not belong to W_{j+1}. Our goal is to show W' occupies only a small fraction (independent of j) of all the children constructed, provided L_0 is large enough.

Let $\varphi : W' \hookrightarrow Z$ be the function that assigns to any $w_{j+1} \in W'$ the unique convergent $v'' \in \mathrm{Spec}(w_{j+1})$ with maximal Euclidean length $|v''| \leqslant |w_{j+1}|^{1+\delta_j}$. The plan is to show φ is injective then control the cardinality of its image.

First, we claim every $v'' \in \operatorname{im}\varphi$ satisfies an inequality of the form

$$|w_j \times v'' \pm ga| \leqslant \frac{1}{|w_j|^{(1+\delta_j)\max(\delta_{j+1}, t_1-\delta_j)}} \tag{46}$$

for some positive integer $a < e^{e_0}$ and a choice of the sign on the LHS. Indeed, suppose $v'' = \varphi(w_{j+1})$ and let v''' be the next convergent after v''. Then

$$|v''| = e^{t_3}|w_{j+1}|\log|w_{j+1}| \quad \text{and} \quad |v'''| = |w_{j+1}|^{1+t_4}$$

for some real numbers $t_3 \geqslant t_1 - \delta_j$, since w_{j+1} satisfies (43), and $t_4 > \delta_{j+1}$, by definition of v'''. Note that $t_4 > t_3$ because[2] $w_{j+1} \in W'$. By the RHS of (32) and the first inequality in (37) we have

$$|w_{j+1} \times v''| \leqslant \frac{1}{|w_{j+1}|^{t_4}} \leqslant \frac{1}{|w_j|^{(1+\delta_j)\max(\delta_{j+1}, t_1-\delta_j)}}.$$

Recalling the consecutive pair of convergents v and v' constructed in the proof of Lemma 5.5 we see that $v'' = av' + bv$ for some positive integers a and b, except in the case when $v'' = v'$. In any case $a > 0$ and the definition of v'' implies these are the only possibilities. Recall also that v' is the convergent responsible for (43) and since $t_1 \geqslant \delta_j$ it follows that $|v'| \geqslant |w_{j+1}|\log|w_{j+1}|$. On the other hand, we have $t_3 < e_0$, for otherwise Lemma 5.7 would imply $w_{j+1} \notin W'$; therefore $|v''| < e^{e_0}|w_{j+1}|\log|w_{j+1}|$ and since $|v''| > a|v'|$ (because angles between convergents are acute) we have $a < e^{e_0}$. Finally, observe that $w_{j+1} \times v'' = w_j \times v'' + gv \times v'' = w_j \times v'' \pm ga$, by (31). This proves the claim.

[2] Actually, this requires a short calculation because if (45) fails for some $t \geqslant \delta_{j+1}$, then $t_3 < t < t_4$ holds *only if* we know $e^t|w_{j+1}|\log|w_{j+1}| \leqslant |w_{j+1}|^{\delta_{j+1}}$, but this holds provided L_0 is chosen large enough as required by Lemma 5.4 for $\varepsilon = e^{-2e_0}$, since $t < e_0 + \delta_j \leqslant 2e_0$.

Next, we show φ is injective. Let $v_i'' = \varphi(w_{j+1}^i)$ for $i = 1, 2$ and recall in the proof of Lemma 5.5 it was shown that $w_{j+1}^i = w_j + gv^i$ for some $v^i \in \mathrm{Spec}(w_{j+1}^i)$. Obviously, $v^1 \neq v^2$ since we are assuming $w_{j+1}^1 \neq w_{j+1}^2$. Using (44), we see that

$$\angle v^1 v^2 \geqslant \frac{|v^1 \times v^2|}{|v^1||v^2|} \geqslant \frac{1}{4|w_j|^{2+2\delta_j}}$$

which is approximately a factor of $\log |w_j|$ greater than the angle either vector makes with the corresponding child:

$$\angle v^i w_{j+1}^i = \sin^{-1} \frac{|w_j \times v^i|}{|w_{j+1}^i||v^i|} \gtrapprox \frac{1}{|w_j|^{2+2\delta_j} \log |w_j|}.$$

Since the angle between v_i'' and w_{j+1}^i is even smaller than the above, it follows that the vectors v_1'' and v_2'' are also distinct.

Finally, we bound the number of vectors in $\mathrm{im}\,\varphi$. Let v_i'' for $i = 1, 2$ be two vectors in $\mathrm{im}\,\varphi$ satisfying (46) with the same sign and the same positive integer a. Put $u = v_1'' - v_2''$ and recall the definition of $v_k \in \mathrm{Spec}(w_j)$ at the beginning of the proof. **Claim:** If $|u| \leqslant (1/4)|w_j|^{1+\delta_j}/4$ then $u = \pm d v_k$ for some positive integer $d < 4|w_j|^{\delta_j(1-\delta_{j+1})}$. The claim implies we either have

$$|u| > (1/4)|w_j|^{1+\delta_j} \quad \text{or} \quad |u| < 4e^{2e_0}|w_j|^{1+\delta_j(1-\delta_{j+1})} \log |w_j|.$$

Observing that the former is much greater than the latter, which means according to (46) the vectors in W' are contained in $< 2e^{e_0}$ narrow strips parallel to w_j and within each strip there are $< 2(2g+1)e^{2e_0} \log |w_j|$ clusters[3] each having $< 4|w_j|^{\delta_j(1-\delta_{j+1})}$ vectors. If L_0 is chosen large enough as required by Lemma 5.4 for $\varepsilon = 32^{-1}(2g+1)^{-1}e^{-2e_0}\rho_1$, then it follows that less than half the children constructed lie in W', i.e. assuming the claim, the proposition holds in this case with $\rho_2 = (1/2)\rho_1 e^{-2e_0}$.

To prove the claim, we suppose $|u| \leqslant (1/4)|w_j|^{1+\delta_j}$. Let $d = \gcd(u)$ so that $u = dv$ for some $v \in Z$. By the triangle inequality and the convenient fact that $(1+\delta_j)\delta_{j+1} > \delta_j$ we have

$$|w_j \times u| \leqslant \frac{2}{|w_j|^{(1+\delta_j)\max(\delta_{j+1}, t_1 - \delta_j)}} \leqslant \frac{2}{|w_j|^{\delta_j}} \tag{47}$$

which together with $|v| \leqslant |u| \leqslant (1/4)|w_j|^{1+\delta_j}$ implies

$$\frac{|w_j \times v|}{|w_j|} \leqslant \frac{|w_j \times u|}{|w_j|} \leqslant \frac{2}{|w_j|^{1+\delta_j}} \leqslant \frac{1}{2|u|} \leqslant \frac{1}{2|v|}.$$

Therefore, $v = \pm v_{k'}$ for some convergent $v_{k'} \in \mathrm{Spec}(w_j)$. (See the Remark 5.6 for an explanation of how the hypothesis of Lemma 33 is satisfied.) Since $|v| <$

[3] A rough estimate: $|u| \leqslant 2e^{t_3}|w_{j+1}| \log |w_{j+1}| < 2(2g+1)e^{2e_0}|w_j|^{1+\delta_j} \log |w_j|$ provided $|w_0|$ is large enough. Here, we used $w_{j+1}^i \in W'$ and Lemma 5.7 to get $|v_i''| < e^{e_0}|w_{j+1}| \log |w_{j+1}|$ then apply $|w_{j+1}^i| < (2g+1)|w_j|^{1+\delta_j}$.

$|w_j|^{1+\delta_j}$ by hypothesis, $k' \leqslant k$ by definition of v_k. In fact, we must have equality for if $k' < k$ then using the LHS of (32), the fact that Euclidean lengths of convergents form an increasing sequence, and $t_1 \leqslant 2e_0$, we have

$$|w_j \times u| \geqslant |w_j \times v| \geqslant \frac{|w_j|}{|v_{k'+1} + v_{k'}|} \geqslant \frac{|w_j|}{2|v_k|} \geqslant \frac{1}{2e^{2e_0} \log |w_j|}$$

which contradicts (47) if L_0 is chosen large enough as required by Lemma 5.4 for $\varepsilon = (1/4)e^{-2e_0}$. Using (47) and the preceding facts about continued fractions once again, we get

$$d = \frac{|w_j \times u|}{|w_j \times v|} < \frac{2|v_{k+1} + v_k|}{|w_j|^{1+(1+\delta_j)\max(\delta_{j+1}, t_1 - \delta_j)}} < 4|w_j|^{t_2 - (1+\delta_j)\max(\delta_{j+1}, t_1 - \delta_j)}$$

which shows $d < 4|w_j|^{\delta_j(1-\delta_{j+1})}$ because

$$t_2 - (1+\delta_j)\max(\delta_{j+1}, t_1 - \delta_j) \leqslant t_2 - t_1 + \delta_j - \delta_j\delta_{j+1} \leqslant \delta_j(1 - \delta_{j+1}).$$

This proves the claim, and hence the proposition. □

Proof of Theorem 1. Since $\delta_0 = e_0$ Lemma 5.7 implies any vector in W with large enough Euclidean length belongs to W_0. Choose any $w_0 \in W_0$ with $|w_0|$ greater than the value of L_0 given by Proposition 5.8. (In addition, require that the initial direction be chosen as in Remark 5.6.) Applying Proposition 5.8 inductively we construct an infinite number of admissible sequences (w_j) with the property $w_j \in W_j$ for all $j \geqslant 0$. Moreover, Lemma 5.3 implies the directions of the vectors in each sequence converge to a slowly divergent nonergodic direction with sublinear rate $\leqslant 1 - 1/e_0$. Any vector w_j occuring at the jth stage of the construction has at least

$$m_j = \rho_2 |w_j|^{\delta_j} / \log |w_j|$$

children for which the inductive process may be continued indefinitely. The angle between the directions of these children are at least

$$\varepsilon_j = \frac{c}{|w_j|^{2(1+\delta_j)}}$$

where $c > 0$ is constant depending only on g. Using [Fa, Example 4.6], we see the Hausdorff dimension of the set of directions constructed is at least

$$\liminf_{j\to\infty} \frac{\log m_0 \cdots m_{j-1}}{-\log m_j \varepsilon_j} = \liminf_{j\to\infty} \frac{\sum_{i=0}^{j-1} \delta_i \log |w_i|}{(2+\delta_j)\log |w_j|} = \lim_{j\to\infty} \frac{1}{2}\left(1 - \prod_{i=0}^{j-1} \frac{1}{1+\delta_i}\right) = \frac{1}{2}.$$

□

6 Arbitrarily slowly divergent directions

To prove Theorem 2 we shall show given any function $R(t)$ with $R(t) \to \infty$ as $t \to \infty$ there exists a sequence (w_j) satisfying the hypotheses of Propostion 3.6 together with $M_j \leqslant R(T_j)$ for j large enough and $\lim m_j = \infty$. Note that the length of the shortest simple closed curve on (X_t^θ, q_t^θ) is at most $2\ell(g_t^\theta V)$ since it consists of at most two saddle connections, both corresponding to the same vector in $g_t^\theta V$. Therefore, $\lim m_j = \infty$ implies $\lim |w_j|^{-1} w_j$ is divergent.

The notation $A \asymp B$ means $A/C \leqslant B \leqslant AC$ for some implicit universal constant $C > 0$. Also, $A \ll B$ means $A \leqslant B\varepsilon$ for some implicit constant $\varepsilon > 0$ that may be chosen as small as desired at the beginning of the construction. $B \gg A$ is equivalent to $A \ll B$.

Proposition 6.1. *If $r(t)$ increases to infinity and $r(t)r'(t) \to 0$ as $t \to \infty$, then there is a sequence $(w_j)_{j \geqslant 0}$ satisfying the hypotheses of Proposition 3.6 such that for all j large enough*

(a) $m_j \leqslant r(t_j) + C$, where $C := \log 2$,

(b) either $|m_{j+1} - r(t_{j+1})| \leqslant C$ or $m_{j+1} \geqslant m_j$,

(c) if $m_{j+1} < r(t_{j+1}) - C$ then $m_{j+1} > m_j + c_0 e^{-m_j}$ for some $c_0 > 0$, and

(d) $|w_{j+1}| \asymp |w_j| e^{m_j}$.

Proof of Theorem 2 assuming Proposition 6.1. Observe the hypotheses of the proposition is satisfied by the logarithm of any smooth Lipschitz function increasing to infinity. Therefore, given $R(t)$ and any $\varepsilon > 0$ one can readily find $r(t)$ satisfying the hypotheses and $R(t) \leqslant (2+\varepsilon) r(t)$.

Given $r(t)$, (a)-(c) implies $m_j \to \infty$ as $j \to \infty$. (d) implies $M_j = m_j + m_{j+1} + O(1) \leqslant r(t_j) + r(t_{j+1}) + O(1)$. Using $r(t)r'(t) \to 0$ and $t_{j+1} - T_j = m_j + O(1)$ we have $r(t_{j+1}) \leqslant r(T_j) + O(1)$ while $r(t_j) \leqslant r(T_j)$ since $r(t)$ is increasing. Hence, $M_j \leqslant 2r(T_j) + O(1) \leqslant R(T_j)$. ☐

Before proving Proposition 6.1 we need a lemma. Note for any pair $(w, v) \in W \times Z$ there exists a unique vector $u \in Z$ satisfying

$$|w \times u| < \frac{1}{2}|w \times v|, \quad |u \times v| = 1, \quad \text{and} \quad w \cdot u > 0.$$

Here, we used the fact that vectors in W have irrational slope. The next lemma estimates the length of u.

Lemma 6.2. *If $|w| > b|v|$ and $|w \times v| \leqslant \varepsilon \leqslant 1/2\sqrt{2}$ then*

$$(1 - \varepsilon/b)\, |w|\, |w \times v|^{-1} \leqslant |u| \leqslant (1 + \varepsilon/b)\, |w|\, |w \times v|^{-1}. \tag{48}$$

Proof. First, consider the case where w lies between u and v. Then $u + v$ lies between w and v so that comparing the component of the vectors u, w and $u+v$ orthogonal to v we obtain

$$\frac{1}{|u+v|} \leqslant \frac{|w \times v|}{|w|} \leqslant \frac{1}{|u|}. \tag{49}$$

Let $a, b', c > 0$ be given by $|v| = a|u|$, $|v| = b'|w|$ and $c = |w \times v|$. Using $|u+v| \leqslant (1+a)|u|$ we note the LHS implies

$$|w| \leqslant |u+v|c \leqslant (1+a)c|u|$$

But then $|v| \leqslant (1+a)b'c|u|$ so that $|u+v| \leqslant (1+(1+a)b'c)|u|$. Repeating the above argument starting with new estimate on $|u+v|$ we get $|u+v| \leqslant (1+(1+(1+a)b'c))b'c|u|$. By induction, we find $|u+v| \leqslant |u|/(1-b'c)$. Since $bb' < 1$ this gives the LHS of (48) while the RHS holds trivially.

Now consider the case where u lies between w and v. Then w lies between u and $u-v$ so that comparing the component of u w and $u-v$ orthogonal to v we get

$$\frac{1}{|u|} \leqslant \frac{|w \times v|}{|w|} \leqslant \frac{1}{|u-v|}. \tag{50}$$

Using $|u-v| \geqslant (1-a)|u|$ we note the RHS implies

$$|w| \geqslant |u-v|c \leqslant (1-a)c|u|$$

But then $|v| \leqslant (1-a)b'c|u|$ so that $|u-v| \geqslant (1-(1-a)b'c)|u|$. Repeating the above argument starting with new estimate on $|u-v|$ we get $|u-v| \geqslant (1-(1-(1-a)b'c))b'c|u|$. By induction, we find $|u-v| \geqslant |u|/(1+b'c)$. Since $bb' < 1$ this gives the RHS of (48) while the LHS holds trivially.

There are no other cases because Lemma 3.4 implies $|u| > |v|$ so that v does not lie between w and u. □

Proof of Proposition 6.1. Let $w_0 \in W$ be arbitrary. Since w_0 has irrational slope, we may choose $w_1 = w_0 + gv_1$ for some $v_1 \in Z$ so that $|w_0 \times w_1| = g|w_0 \times v_1| \ll 1$, and in particular, $\leqslant 1/2\sqrt{2}$. Since $r(t)$ is slowly increasing, the choice can be made so that $m_1 \gg r(t_1)$.

Given $(w_j, v_j) \in W \times Z$ with $|w_j \times v_j| < 1/2\sqrt{2}$ let u_j be the unique vector in Z satisfying

$$|w_j \times u_j| < \frac{1}{2}|w_j \times v_j|, \quad |u_j \times v_j| = 1, \quad \text{and} \quad w_j \cdot u_j > 0 \tag{51}$$

and define $v^i_{j+1}, i = 0, 1, 2$ by

$$v^1_{j+1} = u_j + \sigma v_j, \qquad v^2_{j+1} = 2u_j + \sigma v_j \qquad \text{and} \qquad v^0_{j+1} = u_j - \sigma v_j \tag{52}$$

where $\sigma = +1$ if w_j lies between u_j and v_j and -1 otherwise. We note here

$$\frac{1}{2}|w_j \times v_j| \leqslant |w_j \times v^1_{j+1}| \leqslant |w_j \times v_j| \leqslant |w_j \times v^0_{j+1}| \leqslant 2|w_j \times v_j|$$

and $|w_j \times v^2_{j+1}| \leqslant |w_j \times v^1_{j+1}|$.

The next pair (w_{j+1}, v_{j+1}) will be chosen among the three possibilities (w^i_{j+1}, v^i_{j+1}) where $w^i_{j+1} = w_j + gv^i_{j+1}$. Note that u_j and v_j are uniquely determined by w_j. Hence, we may let $\delta = \delta(w_j) \in (0, 1/2)$ be defined by

$$|w_j \times u_j| = \delta|w_j \times v_j|. \tag{53}$$

The index $i \in \{0, 1, 2\}$ is determined according to the following rule:

(A) if $m_j > r(t_j) + C$ set $i = 0$;

(B) otherwise, choose any $i \in \{0, 1\}$ satisfying $|m^i_{j+1} - r(t^i_{j+1})| \leqslant C$ where

$$t^i_{j+1} = \frac{1}{2} \log \frac{|w^i_{j+1}|^2}{|w_j \times w^i_{j+1}|} \quad \text{and} \quad m^i_{j+1} = \frac{1}{2} \log \frac{1}{|w_j \times w^i_{j+1}|}$$

if possible; if not

(C) let $i = 1, 2$ be the index realizing the larger of $\delta(w^1_{j+1})$ and $\delta(w^2_{j+1})$.

The choice made in the case of ambiguity will not matter.

Note the choice $i = 0$ implies

$$m_j - C \leqslant m_{j+1} \leqslant m_j$$

while the choice $i = 1$ implies

$$m_j \leqslant m_{j+1} \leqslant m_j + C.$$

Similarly, for either choice $i = 1, 2$, we have $m_{j+1} \geqslant m_j$. If $m_j \leqslant r(t_j) + C$ for $j = j_0$ then the same holds for all $j \geqslant j_0$. Since $r(t)$ increases to infinity and $m_{j+1} \leqslant m_j$ whenever (A) is used to choose the next vector, $m_j \leqslant r(t_j) + C$ for some j. This proves (a).

Since (A) is used to choose the next vector for at most finitely many j, from some point on the only situation when $m_{j+1} < m_j$ is if $i = 0$ in (B) is used to choose the next vector, but then $|m_{j+1} - r(t_{j+1})| \leqslant C$. This proves (b).

By choosing v_1 so that $r(t_1) \gg 1$ we can ensure that $m_j \gg 1$ for all $j \geqslant 1$, since $r(t)$ is increasing. In other words, for any $\varepsilon > 0$ we can choose v_1 so that the sequence of pairs (w_j, v_j) constructed satisfy $|w_j \times v_j| \leqslant \varepsilon$. We have $|w_j| \in O(\varepsilon|u_j|)$ by Lemma 6.2 so that $|v_{j+1}| \asymp |u_j| \asymp |w_j|e^{m_j}$ for ε small enough. This proves (d) and using $|v_{j+1}| \gg |w_j|$ it is readily verified that (w_j) satisfies the hypotheses of Proposition 3.6.

It remains to prove (c). The hypothesis implies (C) is used to choose the next vector with either $i = 1, 2$. In this case,

$$m_{j+1} = m_j + \log 1/(1 - i\delta(w_j)) > m_j + \delta(w_j).$$

Let $\delta = \delta(w_j)$, $\delta_i = \delta(w^i_{j+1})$ and set $\delta' = \max(\delta_1, \delta_2)$. We need to show $\delta' \leqslant \delta$ implies $\delta > c_0 e^{-m_j}$.

Let $\Delta u = u^2_{j+1} - u^1_{j+1}$ where u^i_{j+1} is the unique vector in Z satisfying (51) with (w^i_{j+1}, v^i_{j+1}) in place of (w_j, v_j). By definition, we have

$$|w^i_{j+1} \times u^i_{j+1}| = \delta_i |w_j \times v^i_{j+1}| = \delta_i (1 - i\delta)|w_j \times v_j|$$

and

$$w^i_{j+1} \times u^i_{j+1} = w_j \times u^i_{j+1} + g v^i_{j+1} \times u^i_{j+1}.$$

Note that

$$u^i_{j+1} \times v^i_{j+1} = \operatorname{sgn}(w^i_{j+1} \times v^i_{j+1}) = \operatorname{sgn}(w_j \times v^i_{j+1})$$

does not depend on $i = 1, 2$ (since $\delta < 1/2$). Therefore,

$$|w_j \times \Delta u| = |w^2_{j+1} \times u^2_{j+1} - w^1_{j+1} \times u^1_{j+1}| \tag{54}$$
$$\leqslant (\delta_2(1-2\delta) + \delta_1(1-\delta))|w_j \times v_j| < 2\delta|w_j \times v_j|. \tag{55}$$

Let $\Delta u = du$ where $d = \gcd(\Delta u)$ and $u \in Z$. We show $u \neq \pm u_j$ provided ε was chosen small enough at the beginning. Indeed, using $|v_j| \in O(\varepsilon|u_j|)$

$$\begin{aligned} |u^2_{j+1}| &= (1 + O(\varepsilon))\ |w^2_{j+1}|\ |w_j \times v^2_{j+1}|^{-1} \\ &\geqslant (g + O(\varepsilon))\ |v^2_{j+1}|\ |w_j \times v_j|^{-1} \\ &\geqslant (2g + O(\varepsilon))\ |u_j|\ |w_j \times v_j|^{-1} \end{aligned}$$

and using $|w_j| \in O(\varepsilon|v^1_{j+1}|)$ and $|v_j| \in O(\varepsilon|u_j|)$

$$\begin{aligned} |u^1_{j+1}| &= (1 + O(\varepsilon))\ |w^1_{j+1}|\ |w_j \times v^1_{j+1}|^{-1} \\ &\leqslant \frac{g + O(\varepsilon)}{1 - 2\delta}\ |v^1_{j+1}|\ |w_j \times v_j|^{-1} \\ &\leqslant (3g/2 + O(\varepsilon))\ |u_j|\ |w_j \times v_j|^{-1} \end{aligned}$$

from which it follows $|\Delta u| > |u_j|$ (provided ε is small enough) so that $d \geqslant 2$. Hence, $u = \pm u_j$ contradicts (55).

From (55) we see that the vector $u' = 2u_j + \Delta u$ lies between u_j and w_j. Therefore,

$$\frac{|w_j \times u_j|}{|w_j|} \geqslant \frac{|u' \times u_j|}{|u'|} \geqslant \frac{1}{3|\Delta u|}$$

and since $|\Delta u| \in O(|w_j||w_j \times v_j|^{-2})$ it follows that $\delta \geqslant c_0 e^{-m_j}$. □

References

[Ch] Y. Cheung, *Hausdorff dimension of the set of nonergodic directions*, Ann. of Math. **158** (2003), 661-678.

[CE] Y. Cheung and A. Eskin, in preparation.

[Fa] K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, John Wiley & Sons Ltd., Chichester, 1990.

[Ma92] H. Masur, *Hausdorff dimension of the set of nonergodic foliations of a quadratic differential*, Duke Math. J.,**66** (1992), 387-442.

[Ma93] H. Masur, *Logarithmic law for geodesics in moduli space* Contemp. Math. **150** (1993), 229-245.

[MMY] S. Marmi, P. Moussa, J.-C. Yoccoz, *On the cohomological equation for interval exchange maps*, arXiv:math.DS/0304469, **1** (2003), 1-11.

[MS] H. Masur and J. Smillie, *Hausdorff dimension of sets of nonergodic measured foliations*, Ann. of Math., **134** (1991), 455-543.

[Ve] W. Veech, *Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem mod 2*, Trans. Amer. Math. Soc. **140** (1969), 1-34.