

# **A genus 2 characterisation of translation surfaces with the lattice property**

(joint work with Howard Masur)

## Outline

1. Translation surface
2. Translation flows
3.  $SL(2, \mathbb{R})$  action
4. Veech surfaces
5. Main Theorem
6. Sketch of Proof

## Translation surfaces

A translation surface may be defined

**Analytically:** as a holomorphic 1-form on a Riemann surface (i.e. an Abelian differential)

Away from the zeros there is an atlas of charts with changes of coordinates of the form

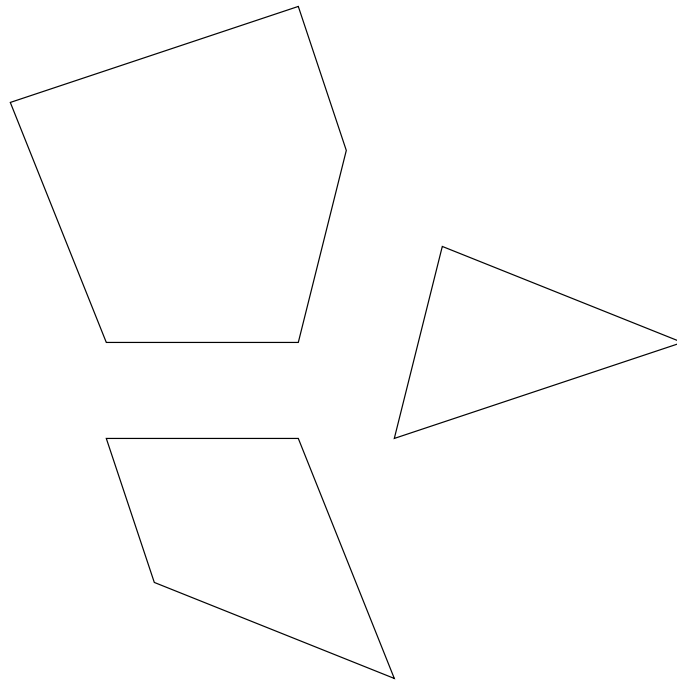
$$z \rightarrow z + c$$

**Geometrically:** as a flat metric with isolated conical singularities and trivial linear holonomy.

Trivial linear holonomy implies all cone angles are integer multiples of  $2\pi$ .

## Make your own translation surface!

Let  $P_j, j = 1, \dots, n$  be disjoint polygons in the plane. Assume the sides of the polygons come in parallel pairs that can be glued together by a translation.



Every compact translation surface can be obtained by glueing together polygons as above.

## Unfolding a Rational Polygon

For any a polygon  $P$  in the plane define

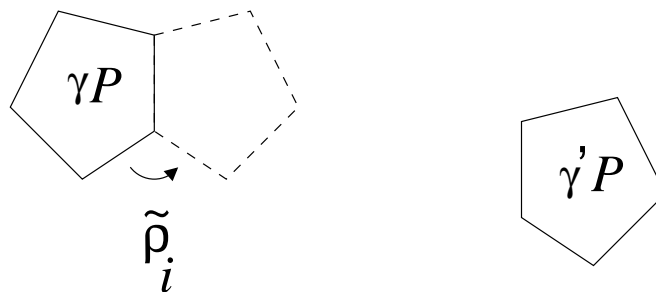
$$G(P) = \langle D\rho_i \rangle \subset O(2)$$

where  $\rho_i$  is the reflection in the  $i$ th side. This group is finite iff all angles of  $P$  are rational multiples of  $\pi$ .

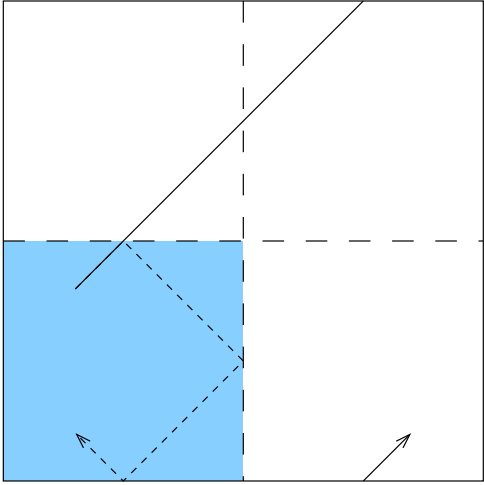
Now glue the reflected copies of  $P$  together by following the recipe on the previous slide:

$$S = \bigsqcup \{ \gamma P : \gamma \in G(P) \} / \sim$$

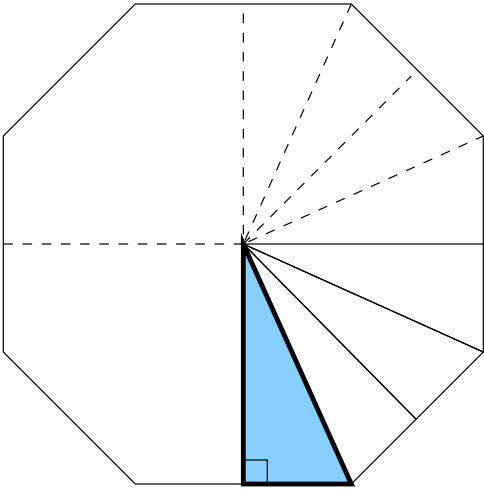
where  $\gamma P$  and  $\gamma' P$  are glued along their  $i$ th sides by translation iff  $\gamma' = D\tilde{\rho}_i \circ \gamma = \gamma \circ D\rho_i$ .



Billiard in a square



Billiard in a  $\pi/8$  right triangle



## Translation flows

Let  $S$  be a compact translation surface.

For each  $\theta \in [0, 2\pi)$  there is an area-preserving flow along parallel lines of slope  $\tan \theta$  generated by a constant field of unit tangent vectors.

We shall say  $\theta$  is *periodic* (likewise *minimal*, *uniquely ergodic*) if every orbit is periodic (resp. dense, uniformly distributed).

If  $S$  is the standard flat torus  $\mathbb{R}^2/\mathbb{Z}^2$  then

**Theorem.** (*Kronecker-Weyl*) *The flow in the direction  $\theta$  is periodic (resp. uniquely ergodic) iff  $\tan \theta$  is rational (resp. irrational).*

A fundamental result is the following

**Theorem.** (*Kerckhoff-Masur-Smillie*) *For any compact translation surface the flow in direction  $\theta$  is uniquely ergodic for almost every  $\theta$ .*

In general, one can show that periodic orbits always exist (although the corresponding direction need not be periodic). Thus, we see that the dynamics in the case of the flat torus is in some sense the simplest possible.

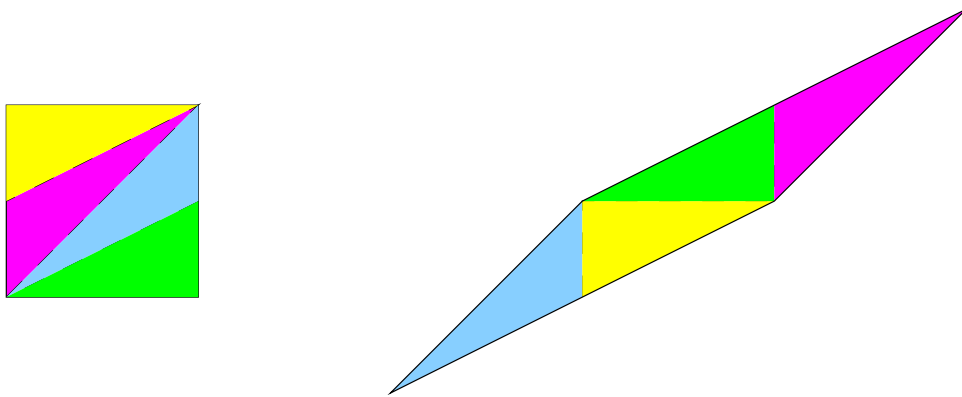
Next, we shall describe a broader class of translation surfaces that generalises the case of the flat torus without losing the simplicity of its dynamics.

## Action by $SL(2, \mathbb{R})$

Given a matrix  $A \in SL(2, \mathbb{R})$  and a translation surface  $S$  (thought of as a bunch of polygons glued together) we can apply  $A$  to the polygons and then glue the new polygons together to obtain a new translation surface, denoted by

$$A(S)$$

Two translation surfaces are equivalent if there is a diffeomorphism between them that respects the translation structure. For example



There is an induced action on the moduli space

$$\Omega\mathcal{M}_g = \{\text{genus } g \text{ translation surfaces}\} / \sim$$

## Lattice property and Veech dichotomy

Let  $\Gamma(S) \subset \mathrm{SL}(2, \mathbb{R})$  denote the stabiliser of  $S$ . For “most”  $S$  one expects  $\Gamma(S)$  to be finite; however, in some cases, e.g. the flat torus, we get a lattice. (Recall that a lattice is a discrete subgroup of finite covolume.)

**Theorem.** (Veech) *If  $\Gamma(S)$  is a lattice then the flow in any given direction is either periodic or uniquely ergodic.*

A translation surface satisfying the hypothesis of the theorem is called a *Veech surface*. The dynamical property in its conclusion is often referred to as *Veech dichotomy*.

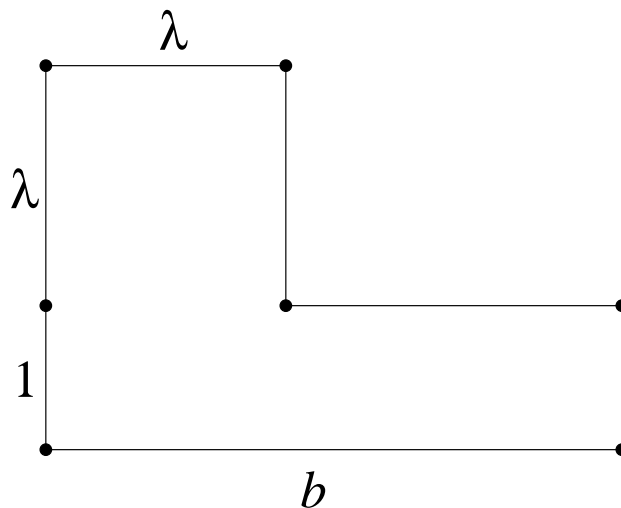
**Question:** If a translation surface has the dichotomy property, is it necessarily Veech ?

## Classification in genus 2

A translation surface is *primitive* if it is not a branched cover of a lower genus surface.

Primitive Veech surfaces of genus 2 have been classified by McMullen. They are all accounted for (up to equivalence under  $SL(2, \mathbb{R})$ ) by

- a countably family of  $L$ -shaped billiards



- regular decagon with opposite sides glued

## Genus 2 characterisation

Assume  $S$  is a translation surface of genus 2.

**Main Theorem.** *If  $S$  is not Veech then there are uncountably many directions  $\theta$  for which the translation flow in direction  $\theta$  is minimal but not uniquely ergodic.*

**Corollary.** *A genus two translation surface is Veech iff it has the dichotomy property.*

Unfortunately, there is an example in genus 5 due to Hubert-Schmidt that is not Veech but has the dichotomy property. (This was pointed out to us by Barak Weiss.)

*Remark.* McMullen gave a *topological* converse to Veech dichotomy by exhibiting neither periodic nor dense orbits. The main theorem gives a stronger *ergodic* converse.

## Splittings

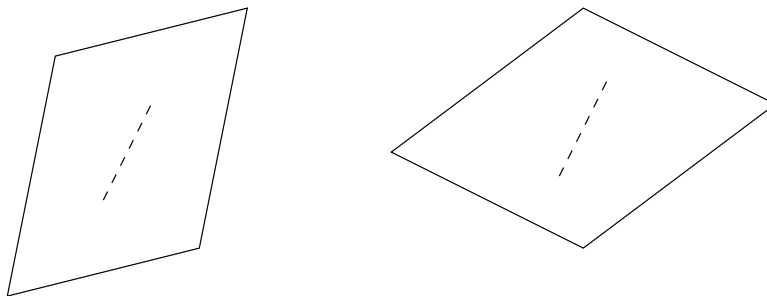
A flat torus with an embedded geodesic arc removed is called a *slit torus*, represented by

$$(\mathbb{C}/L, w)$$

where  $L$  is a lattice in  $\mathbb{C}$  and  $w$  the holonomy vector of the geodesic arc.

*Connected sum:* A pair of flat tori slit along a common arc and then glued together to form a (genus 2) translation surface.

**Theorem.** (McMullen) *Every translation surface of genus 2 splits as a connected sum in an infinite number of ways.*



A splitting may be represented by  $(L_1, L_2, w)$ .

## Area exchanges

Fix a translation surface  $S$  of genus 2.

**Theorem.** (Masur-Smillie) Let  $(L_1^j, L_2^j, w^j)$  be a sequence of splittings of  $S$  such that

- (a) the vectors  $w^j$  have a limit direction  $\theta$
- (b)  $\text{area}(\mathbb{C}/L_i^j)$  stays bounded away from zero
- (c) the series formed by the areas of exchange between consecutive splittings is summable.

Then  $\theta$  is a nonergodic direction.

Task: find splittings that permit arbitrarily close approximations by nearby splittings

## Irrational Splittings

**Definition.** A splitting  $(L_1, L_2, w)$  is *irrational* if the slope of  $w$  is irrational with respect to one (or both) of the lattices.

**Lemma.** *If  $S$  is not Veech, then it admits an irrational splitting.*

**Proposition.** *Every irrational splitting can be approximated arbitrarily close by nearby irrational splittings.*

The proof of the proposition is in 2 steps:

- Construct new splittings via Dehn twists
- Apply Ratner's theorem to get irrationality

## Twist conditions

Given an irrational splitting  $(L_1, L_2, w)$  we look for a pairs of simple closed curves, one from each torus summand, that cross the slit at a common point, then concatenate them to form a simple closed curve  $\alpha$  in  $S$ .

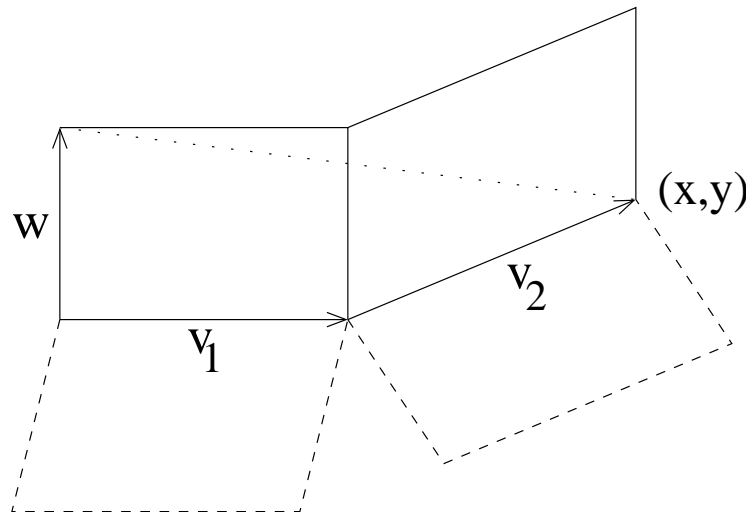
Its holonomy is given by  $v_1 + v_2$  for a pair of primitive vectors in  $L_1 \times L_2$ . If  $w$  is twisted  $k$  times about  $\alpha$ , the new curve is realised by a single geodesic segment if and only if

$$|v_1 \times v_2| < \frac{1}{k} \max(|v_1 \times w|, |v_2 \times w|)$$

(Note that this is invariant under  $SL(2, \mathbb{R})$ .)

Goal: find pairs  $(v_1, v_2)$  with  $k = 3$ .

Normalise using  $SL(2, \mathbb{R})$  invariance



To get  $k$  left twists we need to have

$$y < \frac{1}{k} |w|$$

or

$$\frac{|v_1 \times v_2|}{|v_1|} < \frac{|v_1 \times w|}{k|v_1|}$$

(Note: In the figure, we assumed  $v_1 \times v_2 > 0$ .)

## Final Step

Let  $G = \mathrm{SL}(2, \mathbb{R})$ . It acts on vectors as well as on pairs of lattices.

$$\mathrm{Stab}(w) \hookrightarrow N_{\Delta} \subset G \times G$$

By Ratner's theorem we have

$$\overline{N_{\Delta}(L_1, L_2)} = H(L_1, L_2)$$

where  $H = N_{\Delta}, N \times N, G_{\Delta}, N \times G, G \times N$  or  $G \times G$ . It now only remains to

- Verify  $\exists (L'_1, L'_2) \in H(L_1, L_2)$  containing a pair  $(v'_1, v'_2)$  satisfying the twist conditions
- Approximate by pairs in  $N_{\Delta}$ -orbit to obtain the desired pair  $(v_1, v_2)$