HAUSDORFF DIMENSION OF THE SET OF POINTS ON DIVERGENT TRAJECTORIES OF A HOMOGENEOUS FLOW ON A PRODUCT SPACE

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ABSTRACT. In this paper we compute the Hausdorff dimension of the set $D(\varphi_n)$ of points on divergent trajectories of the homogeneous flow φ_n induced by the one-parameter subgroup $\operatorname{diag}(e^t, e^{-t})$ acting by left multiplication on the product space G^n/Γ^n, where $G = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. We prove that $\dim_{\mathrm{H}} D(\varphi_n) = 3n - \frac{1}{2}$ for $n \geq 2$.

1. INTRODUCTION

Let $G = \mathrm{SL}(2,\mathbb{R})$ be the special linear group of two-by-two matrices with real entries and determinant one and let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ be the discrete subgroup formed by those with integer entries. Let $G(n) = G_1 \times \cdots \times G_n$ where $G_i = G$, $\Gamma(n) = \Gamma_1 \times \cdots \times \Gamma_n$ where $\Gamma_i = \Gamma$, and consider the noncompact homogeneous space

$$G(n)/\Gamma(n) = (G_1/\Gamma_1) \times \cdots \times (G_n/\Gamma_n)$$

and the flow induced by the one-parameter subgroup $\varphi_n : \mathbb{R} \to G(n)$,

$$t \to (g_t, \ldots, g_t) \quad \text{where} \quad g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

acting by left multiplication. The forward trajectory $(x_t)_{t\geq 0}$ of a point $x \in G(n)/\Gamma(n)$ is said to be *divergent* if it eventually leaves every compact set, i.e. for any compact subset $K \subset G(n)/\Gamma(n)$ there is a time T such that $x_t \notin K$ for all $t > T$. Let

$$D(\varphi_n) := \{x \in G(n)/\Gamma(n) : x_t \to \infty \quad \text{as} \quad t \to \infty\}$$

be the set of points whose forward trajectories are divergent; we note that $D(\varphi_n)$ is the union of all forward divergent trajectories of φ_n. The Hausdorff dimension of a subset of $G(n)/\Gamma(n)$ is defined with respect to any metric induced by a right invariant metric on $G(n)$, the choice being

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irrelevant since Hausdorff dimension depends only on the Lipschitz class of a metric.

In this paper we compute the Hausdorff dimension of the set $D(\varphi_n)$.

Theorem 1.1. *For any* $n \geq 2$, $\dim_{\mathrm{H}} D(\varphi_n) = \dim G(n) - \frac{1}{2}$.

Theorem 1.1 was motivated by certain analogies between partially hyperbolic homogeneous flows on finite volume noncompact spaces and the Teichmüller flow on moduli spaces of holomorphic quadratic differentials. The Teichmüller flow can be defined as the restriction to the diagonal subgroup (ϕ_t) of a certain action of $\mathrm{SL}(2,\mathbb{R})$ on moduli space. In this context, Masur [Ma] showed that for any $\mathrm{SL}(2,\mathbb{R})$-orbit X the Hausdorff dimension of the set of points on divergent trajectories of the Teichmüller flow is at most $\frac{1}{2}$; moreover, for a generic $\mathrm{SL}(2,\mathbb{R})$-orbit, it was shown by Masur-Smillie [MS] that the Hausdorff dimension is in fact positive.

For any partially hyperbolic homogeneous flow φ on a finite volume noncompact homogeneous space, Dani [Da] showed that the set $D(\varphi)$ contains a countable union of submanifolds of positive codimension that consists of points lying on *degenerate* divergent trajectories. Moreover, he showed that in the case of $\mathbb{R}$-rank one, all divergent trajectories are degenerate, while in the higher rank situation there are *nondegenerate* divergent trajectories. (See also [We] for related questions pertaining to the notion of degeneracy.) Theorem 1.1 shows that in the case of φ_n for $n \geq 2$ the set of points that lie on degenerate divergent trajectories form a subset of $D(\varphi_n)$ of positive Hausdorff codimension, supporting the idea that nondegenerate divergent trajectories are more abundant than degenerate ones. We also note that for $n = 1$, we have $\dim_{\mathrm{H}} D(\varphi_1) = \dim G - 1 = 2$.

For further results concerning the trajectories (bounded or divergent) of partially hyperbolic homogeneous flows and their applications to number theory we refer the reader to [Bu], [Kl], [KM], [St], and [We].

To obtain the upper bound in Theorem 1.1 we shall in fact show that for some sufficiently large compact set $K \subset G(n)/\Gamma(n)$ the points whose forward trajectory under eventually stays outside K form a set $S = S(K)$ of positive Hausdorff codimension. The compact set will depend on a parameter $\delta > 0$ such that the upper bound on $\dim_{\mathrm{H}} S(K)$ tends to $3n - \frac{1}{2}$ as $\delta \to 0$.

Outline. In §2, we use standard methods to reduce the computation of the Hausdorff dimension of $D(\varphi_n)$ to that of the set E_n^* of endpoints of nondegenerate divergent trajectories. The set E_n^* may naturally be thought of as a subset of $\mathbb{R}^n$ and in §3 we give a characterisation of this set in terms of an encoding that uses continued fractions. Using this encoding, we compute the lower bound on $\dim_{\mathrm{H}} D(\varphi_n)$ in §4. Then we introduce the notion of a self-similar covering in §5 as a convenient device for presenting the upper bound calculation, which is presented in §6.

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2. Endpoints of divergent trajectories

In this section we consider the set of endpoints of divergent trajectories and identify it with a subset of $\mathbb{R}^n$. We assume the notation already established in the introduction.

Ideal boundary and Bruhat decomposition. The forward trajectories of two points in $G(n)$ are asymptotic if and only if they belong to the same right coset of the parabolic subgroup $P(n) = P_1 \times \cdots \times P_n$ where $P_i = P$ for $i = 1, \ldots, n$ and

$$P = \{p \in G : g_t p g_{-t} \text{ stays bounded as } t \to \infty\}.$$

Thus, the set $D(\varphi_n)$ is a union of right cosets of $P(n)$.

The ideal boundary, whose points are asymptotic classes of trajectories of φ_n, is represented by the right coset space

$$P(n)\backslash G(n) = (P_1\backslash G_1) \times \cdots \times (P_n\backslash G_n).$$

The unipotent radicals of P and $P(n)$ are respectively given by

$$N = \{u \in G : g_t u g_{-t} \to e \text{ as } t \to \infty\}$$

and $N(n) = N_1 \times \cdots \times N_n$ where $N_i = N$. The Bruhat decomposition establishes a one-to-one correspondence between right cosets of $P(n)$ that differ from $P(n)$ itself with elements of $N(n)$:

$$G(n) - P(n) = \bigcup_{u \in N(n)} P(n) w u$$

where $w = (w_1, \ldots, w_n)$ and $w_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for $i = 1, \ldots, n$. Let E_n be the subset of $N(n)$ such that

$$(D(\varphi_n) - P(n)) = \bigcup_{u \in E_n} P(n) w u.$$

Rational points. A right coset of P_i is a Γ_i-*rational point* if its stabiliser under the action of G_i by right multiplication contains a maximal parabolic subgroup of Γ_i. Similarly, a right coset of $P(n)$ is $\Gamma(n)$-*rational* if its stabiliser under the $G(n)$ action on $P(n)\backslash G(n)$ contains a maximal parabolic subgroup of $\Gamma(n)$. We note that $P(n)$ is a $\Gamma(n)$-rational point and also that the set of $\Gamma(n)$-rational points forms a single orbit under the action of $\Gamma(n)$ on $P(n)\backslash G(n)$ by right multiplication.

We shall say a right coset of the form $P(n)wu$ is *rational in the ith coordinate* if its stabiliser under the action of $G(n)$ on $P(n)\backslash G(n)$ contains the subgroup of $G(n)$ consisting of elements of the form $(x_1, \ldots, x_n)$ where x_j is the identity of G_j for $j \neq i$ and $x_i \in P_i$. Note that $P(n)wu$ is $\Gamma(n)$-rational if and only if it is rational in the ith coordinate for $i = 1, \ldots, n$. We say $P(n)wu$ is *totally irrational* if it is not rational in the ith coordinate for any $i \in \{1, \ldots, n\}$. Let E'_n be the set of points in E_n corresponding to right cosets of $P(n)$ that a totally irrational.

We shall identify $N(n)$ with $\mathbb{R}^n$ and E'_n with the corresponding subset of $\mathbb{R}^n$. Since $\dim P(n) = 2n$, the product formula for Hausdorff dimension (see (2) in §4) gives

$$\dim_{\mathrm{H}}(P(n)E'_n) = 2n + \dim_{\mathrm{H}} E'_n.$$

Since $D(\varphi_n) - P(n)E'_n$ is contained in a countable union of submanifolds of $G(n)$ of dimension strictly less than $\dim G(n)$, its Hausdorff dimension is bounded above by $3n-1$ (see Lemma 5.4 in §5.) Therefore, the proof of Theorem 1.1 reduces to the statement

$$\dim_{\mathrm{H}} E'_n = n - \frac{1}{2} \quad \text{for} \quad n \geq 2.$$

The identification of $N(n)$ with $\mathbb{R}^n$ can be made explicit as follows. The action of G on the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ by Möbius transformations extends continuously to the boundary $\mathbb{R} \cup \{\infty\}$; we assume this action is given as a right action. The boundary is naturally identified with $P\backslash G$ such that the right coset P corresponds to the ideal point ∞. Similarly, the group $G(n)$ acts on the product space $(\mathbb{H}^2)^n$ and the ideal boundary $P(n)\backslash G(n)$ is naturally identified with the topological boundary of $(\mathbb{H}^2)^n$ as a subset of $(\hat{\mathbb{C}})^n$ where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Under this identification, the set of right cosets of the form $P(n)wu$ with $u \in N(n)$ corresponds to the subset $\mathbb{R}^n \subset (\hat{\mathbb{C}})^n$. Moreover, a right coset is $\Gamma(n)$-rational (resp. rational in the ith coordinate) if and only if every coordinate (resp. the ith coordinate) of the corresponding point in $\mathbb{R}^n$ is rational. It is totally irrational if and only if every coordinate of the corresponding point in $\mathbb{R}^n$ is irrational.

3. Encoding via piecewise linear functions

In this section we associate a piecewise linear function $W_{\mathbf{x}} : \mathbb{R} \to \mathbb{R}$ to every $\mathbf{x} \in \mathbb{R}^n$ and use it to characterise the set E_n'.

We shall draw freely upon the standard results of continued fraction theory. All the results we use can be found in [Kh].

For any $x \in \mathbb{R}$, let $W_x : \mathbb{R} \to \mathbb{R}$ be the piecewise linear function determined by the conditions

(1) The function W_x is continuous and nonnegative,
(2) has slopes ± 1 whenever defined,
(3) each local minimum of W_x is a zero, and
(4) the zeroes of W_x are enumerated by $(2 \log q_k)$ where (q_k) is the sequence of heights formed by the convergents of x.

(Recall that the height of a rational is the smallest positive integer that multiplies it into the integers.)

It is well-known that an integer q is the height of a convergent of x if and only if the distance of qx to the nearest integer is minimal among the set of integer multiples $x, 2x, \ldots, qx$. The function $W_x(t)$ is broken at a finite number of points if and only if x is rational.

For any nonempty closed discrete subset Z of $\mathbb{R}^2$ we define

$$\ell_\infty(Z) := \min\{\max(|a|, |b|) : (a, b) \in Z, (a, b) \neq (0, 0)\}$$

to be the length of the shortest nonzero vector in Z with respect to the sup norm. Let

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \quad \text{and} \quad h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Theorem 3.1. *There is a universal constant $C > 0$ such that*

$$0 \leq |-2 \log \ell_\infty(g_t h_x^{-1} \mathbb{Z}^2) - W_x(t)| \leq C$$

for any $x \in \mathbb{R}$ and for any $t \in \mathbb{R}$.

Proof. For any nonempty closed discrete subset $Z \subset \mathbb{R}^2$ the function

$$W(t) = -2 \log \ell_\infty(g_t Z)$$

is a continuous piecewise linear function with slopes ± 1 and isolated critical points, and in particular, for the set $Z = h_x^{-1}\mathbb{Z}^2$. Observe that $W(t)$ has constant slope -1 for all $t < 0$ since the shortest nonzero vector in $g_t Z$ is realised by $(1, 0)$.

Suppose that $W(t)$ has a local minimum at a time $t \geq 0$. Then there is a pair of vectors $v, v' \in Z$ and a square S centered at the origin containing $g_t v$ on one of its vertical edges and $g_t v'$ on one of its horizontal edges such that S contains no nonzero vectors of $g_t Z$ in its interior. The vectors have the form $v = (qx-p, q)$ and $v' = (q'x-p', q')$ for some $p, q, p', q' \in \mathbb{Z}$ and we may assume that $q \geq 0$, $q' \geq 1$, and if $q' = 1$ then p' can be chosen so that $|q'x - p'| \leq 1/2$. The side of the square is

$$2e^{-W(t)/2} = 2e^{t/2}|qx - p| = 2e^{-t/2}q'.$$

By Minkowski's theorem, the area of the square is at most 4 so that

$$e^{-W(t)} = q'|qx - p| \leq 1$$

from which it follows that $|qx-p| \leq 1/2$, i.e. the distance to the nearest integer $||qx|| = |qx - p|$. Since $g_{-t}S$ contains no nonzero vectors of Z in its interior, it follows that q' is the height of some convergent of x. The corresponding zero of $W_x(t)$ occurs at time $2\log q'$ so that

$$W(t) - W_x(2\log q') = -\log q'||qx|| = t - 2\log q'.$$

Since $q \leq q'$ and $|q'x - p'| \leq |qx - p|$, the area of the parallelogram spanned by v and v' is at most $2q'||qx||$; and since Z is a unimodular lattice, the area is at least one. Therefore, each local minimum of $W_x(t)$ is shifted upwards and to the right relative to a zero of $W(t)$ by an equal amount of at most $\log 2$ in both directions.

Conversely, given any zero of $W(t)$, occuring at time $2\log q$, there is a convergent p/q of x such that $(qx - p, q) \in Z$ lies on the top of a rectangle R symmetric with respect to the origin and whose interior contains no nonzero vectors of Z. We may assume R is chosen largest possible so that it contains a point of Z on one of its vertical edges. Since its horizontal are vertical sides are respectively at most one and at least one, there is a (unique) time $t \geq 0$ when g_tR is a square and $W(t)$ has a local minimum at this time.

It follows easily that

$$W_x(t) \leq W(t) \leq W_x(t) + 2\log 2$$

so that the statement of the theorem holds with $C = 2\log 2$. □

Remark 1. The encoding of geodesics on the modular surface is well-known and dates back to E. Artin [Ar]. Theorem 3.1 may be thought of as a refinement of this encoding that also involves the parametrisation of the geodesics.

Remark 2. The set of oriented unimodular lattices in $\mathbb{R}^2$ may naturally be identified with G/Γ and the function $-2\log \ell_2(Z)$, where $\ell_2(Z)$ denotes the Euclidean norm of the shortest nonzero vector in Z, lifts to a K-invariant proper function on G/Γ. The induced function on the hyperbolic triangle

$$\Delta := \{z : \text{Im } z > 0, |\text{Re } z| \leq 1/2, |z| \geq 1\}$$

coincides with the function $z \to \operatorname{Im} z$.

For any $\mathbf{x} \in \mathbb{R}^n$ we let $W_{\mathbf{x}}(t) : \mathbb{R} \to \mathbb{R}$ be the function given by

$$W_{\mathbf{x}}(t) = \max(W_{x_1}(t), \ldots, W_{x_n}(t)).$$

Note that $W_{\mathbf{x}}(t)$ has infinitely many local minima if and only if every coordinate of $\mathbf{x}$ is irrational. It follows by Mahler's criterion and Theorem 3.1 that $\mathbf{x} \in E_n$ if and only if $W_{\mathbf{x}}(t) \to \infty$ as $t \to \infty$, which in turn holds if and only if

$$W_{\mathbf{x}}(t_j) \to \infty \quad \text{as} \quad j \to \infty$$

where (t_j) is the sequence of local minimum times. In particular, if (q_k) and (q'_l) are the sequences of heights formed by the convergents of irrationals x and y, respectively, then $(x, y) \in E'_2$ if and only if

$$|\log(q_k/q'_l)| \to \infty \quad \text{as} \quad \min(q_k, q'_l) \to \infty;$$

in other words, for any $\rho > 1$ there exists $R > 1$ such that for any indices k, l such that $q_k > R$ and $q'_l > R$, we have

$$\max\left(\frac{q'_l}{q_k}, \frac{q_k}{q'_l}\right) > \rho.$$

4. Lower bound calculation

In this section we prove that $\dim_{\mathrm{H}} E'_n \geq n - \frac{1}{2}$ for $n \geq 2$. We begin by recalling some results that we need for this calculation.

Hausdorff dimension estimates. Consider a Cantor set $F \subset \mathbb{R}$ defined as a nested intersection

$$F = \bigcap_{j \geq 0} F_j$$

where F_0 is a closed interval and each F_j is a disjoint union of (finitely) many closed intervals. Let $(m_j)_{j \geq 1}$ be a sequence of positive integers and $(\varepsilon_j)_{j \geq 1}$ a sequence of real numbers that tend to zero monotonically and suppose that for each $j \geq 1$ there are at least m_j intervals of F_j contained in each interval of F_{j-1} and these intervals are separated by gaps of length at least ε_j. Then the Hausdorff dimension of F satisfies the lower bound (see [Fa], Example 4.6)

$$\dim_{\mathrm{H}} F \geq \limsup_{j \to \infty} \frac{\log(m_1 \cdots m_{j-1})}{-\log m_j \epsilon_j}. \tag{1}$$

The following product formula (see [Fa], Corollary 7.4) holds for subsets $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$,

$$\dim_{\mathrm{H}}(E \times F) = \dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F \tag{2}$$

as soon as the Hausdorff and Minkowski dimensions of one of the sets coincide. Thus, it is enough to prove $\dim_{\mathrm{H}} E_2' \geq \frac{3}{2}$.

We shall also need the following result.

Lemma 4.1. *([Fa], Corollary 7.12) Let $F \subset \mathbb{R}^2$ and E its projection to the x-axis. For each $x \in E$, let L_x be the line $\{(x, y) : y \in \mathbb{R}\}$. If $\dim_{\mathrm{H}}(F \cap L_x) \geq t$ for all $x \in E$ then*

$$\dim_{\mathrm{H}} F \geq t + \dim_{\mathrm{H}} E.$$

Counting rationals. We shall also need the following result.

Theorem 4.2. *([Ch1], Theorem 2) There is a constant $c > 0$ such that for any interval I of the form $I = [x - d, x + d]$ and for any $h > 0$ the number of rationals in I whose height lies between h and $2h$ is at least $ch^2|I|$ provided x has a convergent whose height q satisfies $(hd)^{-1} \leq q \leq h$.*

Remark. Theorem 2 in [Ch1] is stated under an addititonal technical hypothesis that can be shown to be redundant and has therefore been omitted here. See [Ch2] for other variations and improvements.

We now show that $\dim_{\mathrm{H}} E_2' \geq \frac{3}{2}$. The proof of the next lemma will be omitted, since it is essentially contained in the proof of a classical result of Jarnik-Besicovitch.

Lemma 4.3. *Let A_δ be the set of irrationals with the property that the sequence (q_k) of heights formed by its convergents satisfy*

$$q_k^{1+\delta} \leq q_{k+1} \leq 2q_k^{1+\delta} \tag{3}$$

for k large enough. Then $\dim_{\mathrm{H}} A_\delta \geq \frac{1}{2+\delta}$ for any $\delta > 0$.

Lemma 4.4. *Let B_x be the set of $y \in \mathbb{R}$ such that $(x, y) \in E_2'$. Then $\dim_{\mathrm{H}} B_x = 1$ for any $x \in A_\delta$.*

Proof. Let (q_k) be the sequence of heights formed by the convergents of some given $x \in A_\delta$ and fix k_0 such that (3) holds for all $k \geq k_0$. We shall choose $k_0 \gg 1$ so that certain estimates that arise in the course of the proof will hold. Our goal is to construct a set of $y \in B_x$ with the property that for each $k \geq k_0$ there is a pair of consecutive convergents of y whose heights $q < q'$ satisfy (up to a constant factor)

$$\min\left(\frac{q'}{q_k}, \frac{q_k}{q}\right) > \log q_k.$$

This set will be realised as a decreasing intersection $\cap F_j$ where each F_j is a disjoint union of closed intervals. Let us fix the following parameters for the construction:

$$h_j = \left[\frac{q_{k_0+j}}{\log q_{k_0+j}}\right], \quad \varepsilon_j = \frac{1}{8h_j^2}, \quad \text{and} \quad d_j = \frac{1}{q_{k_0+j}^2}$$

where $[\cdot]$ denotes the greatest integer function. The intervals of F_j will all have length $2d_j$ and the length of the gap between any two of them will be at least ε_j; moreover we shall also require each interval of F_j to be centered about a rational whose height is between h_j and $2h_j$.

Let F_0 be an interval of length $2d_0$ centered about a rational of height h_0. Given F_j we shall define F_{j+1} by specifying, for each interval I of F_j, the intervals of F_{j+1} that are contained in I. Let J be an interval of F_j where $j \geq 0$. By the induction hypothesis, J is of the form

$$J = \left[\frac{p}{q} - d_j, \frac{p}{q} + d_j\right]$$

where p/q is a rational whose height q is between h_j and $2h_j$. Note that by construction, every $x \in J$ has p/q as a convergent since

$$\left|x - \frac{p}{q}\right| \leq d_j = \frac{1}{q_{k_0+j}^2} \leq \frac{1}{8h_j^2} \leq \frac{1}{2q^2} \tag{4}$$

where in the third step we used $k_0 \gg 1$. Moreover, if q' is the height of the next convergent then

$$\frac{1}{2qq'} < \left|x - \frac{p}{q}\right| < \frac{1}{qq'}. \tag{5}$$

Let I be the interval of length $\frac{d_j}{3}$ centered about

$$x' = \frac{p}{q} + \frac{d_j}{2}$$

and note that an interval of length $2d_{j+1}$ centered about any point in I is entirely contained in J provided $k_0 \gg 1$. The interval I is of the form $[x' - d, x' + d]$ where $d = \frac{d_j}{6}$ and the first inequality in (5) implies

$$q' > \frac{1}{2qd_j} \geq \frac{1}{4h_j d_j} > (h_{j+1}d)^{-1} \tag{6}$$

assuming $k_0 \gg 1$. For any point $y \in I$ the distance to p/q is at least $\frac{d_j}{3}$ so that the second inequality in (5) implies

$$q' < \frac{3}{d_j q} \leq \frac{3q_{k_0+j}^2}{h_j} < 6q_{k_0+j} \log q_{k_0+j} < \frac{q_{k_0+j+1}}{\log q_{k_0+j+1}} \leq h_{j+1}$$

where in the fourth step we used $k_0 \gg 1$ again. Thus, Theorem 4.2 applied to the interval I implies there are at least

$$m_{j+1} = (c/3)h_{j+1}^2 d_j$$

rationals in I with heights between h_{j+1} and $2h_{j+1}$. We define the intervals of F_{j+1} contained in J to be any closed interval of length $2d_{j+1}$ centered about a rational just constructed and let $F = \cap F_j$ be the resulting Cantor set.

Now we compute the Hausdorff dimension of F. The distance between the centers of two intervals of F_j is at least $2\varepsilon_j$ because the height the rationals at the centers are both at most $2h_j$. Since $2d_j < \varepsilon_j$, the length of the gap between any two intervals of F_j is at least ε_j. Before applying the formula (1), we need to develop some estimates on the parameters of the construction. From (3) we have

$$q_{k_0}^{(1+\delta)^j} \leq q_{k_0+j} \leq 2^j q_{k_0}^{(1+\delta)^j}$$

so that

$$\log q_{k_0+j} = (1+\delta)^j \log q_{k_0} + O(j).$$

Let us write $A \asymp B$ if we have $A/C \leq B \leq AC$ for some explicitly computable $C > 0$ whose value is otherwise irrelevant for the purposes of this calculation. Then, using (3) again, we have

$$m_{j+1} \asymp h_{j+1}^2 d_j \asymp \frac{q_{k_0+j}^{2\delta}}{(\log q_{k_0+j+1})^2}$$

so that

$$\log m_{j+1} = 2\delta(1+\delta)^j \log q_{k_0} + O(j).$$

Therefore,

$$\begin{aligned} \log(m_1 \cdots m_j) &= \sum_{i=0}^{j-1} 2\delta \log(1+\delta)^i \log q_{k_0} + O(j^2) \\ &= 2(1+\delta)^j \log q_{k_0} + O(j^2) \end{aligned}$$

and since $m_{j+1}\varepsilon_{j+1} \asymp d_j$ we have

$$-\log m_{j+1}\varepsilon_{j+1} = 2(1+\delta)^j \log q_{k_0} + O(j)$$

so that the formula (1) yields $\dim_{\mathrm{H}} F = 1$.

It remains to show that $F \subset B_x$. Given any $y \in F = \cap F_j$ and any $k = k_0 + j$ for some $j \geq 0$ we let p/q be the rational at the center of the interval J of F_j that contains y. Since $y \in J$, (4) implies p/q is a convergent of y whose height q satisfies

$$q < 2h_j < \frac{2q_k}{\log q_k}.$$

Now let $I \subset J$ be the interval of F_{j+1} that contains y. Since $y \in I$, the first two inequalities in (6) imply

$$q' > \frac{1}{4h_j d_j} > \frac{q_k \log q_k}{4}$$

from which it follows that $(x, y) \in E_2'$. Therefore, $F \subset B_x$ and the lemma follows. □

Lemmas 4.1, 4.3 and 4.4 now imply that $\dim_{\mathrm{H}} E_2' \geq \frac{3}{2}$.

5. Self-similar coverings

The main technical device for obtaining upper bounds on Hausdorff dimension is the notion of a self-similar covering.

Definition 5.1. *Let $\mathcal{B}$ be a countable covering of a subset $E \subset \mathbb{R}^n$ by bounded subsets of $\mathbb{R}^n$ and let σ be a function from the set $\mathcal{B}$ to the set of all nonempty subsets of $\mathcal{B}$. We say $(\mathcal{B}, \sigma)$ is a* self-similar covering *of E if there exists $\lambda, 0 < \lambda < 1$ such that for every $x \in E$ there is a sequence B_j of elements in $\mathcal{B}$ satisfying*

1. *$\cap B_j = \{x\}$,*
2. *$\operatorname{diam} B_{j+1} < \lambda \operatorname{diam} B_j$ for all j, and*
3. *$B_{j+1} \in \sigma(B_j)$ for all j.*

We shall need a slightly more general notion which is notationally more cumbersome but offers added flexibility in applications.

Definition 5.2. *Let $\mathcal{B}$ be a countable covering of a subset $E \subset \mathbb{R}^n$ by bounded subsets of $\mathbb{R}^n$ and assume that it is indexed by some countable set J; let σ be a function from the set J to the set of all nonempty subsets of J. For any $\alpha \in J$ we write $B(\alpha)$ for the element of $\mathcal{B}$ indexed by α. We say $(\mathcal{B}, J, \sigma)$ is an* indexed self-similar covering *of E (the indexing function $\iota : J \to \mathcal{B}$ being implicit) if there exists a $\lambda, 0 < \lambda < 1$ such that for every $x \in E$ we have a sequence (α_j) of elements in J satisfying*

1. *$\cap B(\alpha_j) = \{x\}$,*
2. *$\operatorname{diam} B(\alpha_{j+1}) < \lambda \operatorname{diam} B(\alpha_j)$ for all j, and*
3. *$\alpha_{j+1} \in \sigma(\alpha_j)$ for all j.*

Remark. We shall carry out the calculations using self-similar coverings by indexed families of sets. Although the calculations can also be done using ordinary self-similar coverings, the advantage of using indexed families will become clear, especially in the case $n > 2$.

The goal of this section is to prove the following.

Theorem 5.3. *Let $(\mathcal{B}, J, \sigma)$ be an indexed self-similar covering of a subset $E \subset \mathbb{R}^n$ and suppose there is an $s > 0$ such that for every $\alpha \in J$*

$$\sum_{\alpha' \in \sigma(\alpha)} (\operatorname{diam} B(\alpha'))^s \leq (\operatorname{diam} B(\alpha))^s. \tag{7}$$

Then $\dim_{\mathrm{H}} E \leq s$.

For the convenience of the reader, we briefly recall the definition of the Hausdorff dimension of a subset $E \subset \mathbb{R}^n$. A countable covering $\mathcal{U}$ of a set $E \subset \mathbb{R}^n$ is said to be an ε-*cover* if the diameter of each element is less than ε. Given an ε-cover $\mathcal{U}$, we let $\mu^s(\mathcal{U})$ denote the sum of the sth powers of the diameters of the elements in $\mathcal{U}$, and set

$$\mu^s_\varepsilon(E) = \inf \{\mu^s(\mathcal{U}) : \mathcal{U} \text{ is an } \varepsilon\text{-cover of } E\ \}.$$

The s-dimensional measure of E is defined as the limit

$$\mu^s(E) := \lim_{\varepsilon \to 0^+} \mu^s_\varepsilon(E)$$

which as a function of $s \geq 0$ has a critical value $h \geq 0$ with the property $\mu^s(E) = \infty$ for all $s < h$ while $\mu^s(E) = 0$ for all $s > h$. This value may be taken as the definition of the Hausdorff dimension of E.

The following is a basic property enjoyed by Hausdorff dimension.

Lemma 5.4. *(*[Bi]*, Theorem 4.1) For any countable collection $\{E_j\}$ of subsets of $\mathbb{R}^n$,*

$$\dim_{\mathrm{H}}(\cup E_j) = \sup \dim_{\mathrm{H}} E_j.$$

Proof of Theorem 5.3. For each $x \in E$ we have a sequence of elements in J satisfying 1.-3. in the definition of a self-similar covering; we shall think of this sequence as an infinite word $w(x)$ in a language over the alphabet J. If $w(x) = (\alpha_1, \alpha_2, \dots)$ then we say $(\alpha_1, \dots, \alpha_k)$ is a *prefix* of $w(x)$ of length k. Let T be the collection of words formed by all possible prefixes of $w(x)$ as x ranges over E. For any $w \in T$, let E_w be the subset of E formed by those x for which the infinite word $w(x)$ has w as a prefix. Since T is countable and $E = \bigcup_{w \in T} E_w$ it suffices to show that $\dim_{\mathrm{H}} E_w \leq s$ for every $w \in T$. For $w \in T$ we shall also write $B(w)$ for the element of $\mathcal{B}$ indexed by the last letter of w.

Fix $w \in T$ and consider any $\varepsilon > 0$ with $\varepsilon < \operatorname{diam} B(w)$. For each $x \in E_w$, let $j(x)$ be the smallest integer j such that $\operatorname{diam} B(\alpha_j(x)) < \varepsilon$

and let $w'(x)$ be the prefix of $w(x)$ of length $j(x)$. Let $A \subset T$ be the set of all $w'(x)$ as x ranges over E_w. Then $\mathcal{U} = \{B(w') : w' \in A\}$ is an ε-cover of E_w. We claim that $\mu^s(\mathcal{U}) \leq (\operatorname{diam} B(w))^s$ from which it follows that $\mu^s_\varepsilon(E_w) \leq (\operatorname{diam} B(w))^s$ and since $\varepsilon > 0$ can be made arbitrarily small, we have $\mu^s(E_w) < \infty$ so that $\dim_{\mathrm{H}} E_w \leq s$.

To prove the claim, we consider the subset $T_w \subset T$ consisting of all those words which have w as a prefix. A subset $A' \subset T_w$ is an *anti-chain* if no word in A' is a prefix of some other word in A'. By construction, A is an anti-chain contained in T_w. Hence, the claim is a consequence of the following assertion: *for any anti-chain* $A' \subset T_w$

$$\sum_{w' \in A'} (\operatorname{diam} B(w'))^s \leq (\operatorname{diam} B(w))^s. \tag{8}$$

Let $T_k \subset T_w$ be the subset formed by all words of length at most $k + l$ where l is the length of w. Since $T_0 = \{w\}$, (8) holds for all $A' \subset T_0$. Now suppose that (8) holds for all anti-chains contained in T_k and let A' be an anti-chain contained in T_{k+1}. Write $A' = A_0 \cup A_1$ where $A_0 = A \cap T_k$ and $A_1 = A' - A_0$. Let $A'' = A_0 \cup \pi(A_1)$ where $\pi : A_1 \to T_k$ is the map that sends w' to the word $\pi(w')$ obtained by dropping the last letter of w'. Note that $A_0 \cap \pi(A_1) = \emptyset$ and that A'' is an anti-chain contained in T_k. If $w' \in \pi(A_1)$, $w'' \in \pi^{-1}(w')$, and α is the last letter of w', then w'' is obtained from w' by adding a suffix $\alpha' \in \sigma(\alpha)$ where α is the last letter of w'. Moreover, it is obvious that $w'' \in \pi(A_1)$ is uniquely determined by α'. Therefore, (using the notation $|\cdot|$ for the diameter of a set)

$$\begin{aligned}
\sum_{w' \in A'} |B(w')|^s &= \sum_{w' \in A_0} |B(w')|^s + \sum_{w' \in \pi(A_1)} \sum_{w'' \in \pi^{-1}(w')} |B(w'')|^s \\
&\leq \sum_{w' \in A_0} |B(w')|^s + \sum_{w' \in \pi(A_1)} \sum_{\alpha' \in \sigma(\alpha)} |B(\alpha')|^s \\
&\leq \sum_{w' \in A''} |B(w')|^s
\end{aligned}$$

which proves (8) for all anti-chains contained in T_k for some $k \geq 0$. Since any $A' \subset T_w$ can be written as an increasing union of the sets $A'_k = A' \cap T_k$, each of which is an anti-chain if A' is, it follows that (8) holds for all anti-chains contained in T_w as well, proving the assertion and hence the theorem. □

As a special case of Theorem 5.3 we obtain.

Corollary 5.5. *Let $(\mathcal{B},\sigma)$ be a self-similar covering of a subset $E \subset \mathbb{R}^n$ and suppose there is an $s > 0$ such that for every $B \in \mathcal{B}$*

$$\sum_{B' \in \sigma(B)} \left(\frac{\operatorname{diam} B'}{\operatorname{diam} B}\right)^s \leq 1.$$

Then $\dim_{\mathrm{H}} E \leq s$.

We remark that Theorem 5.3 is implied by the corollary in the case when the indexing function $\iota : J \to \mathcal{B}$ is injective.

6. Upper bound calculation

In this section we prove the following.

Theorem 6.1. *Let $E_n(\delta), \delta > 0$ be the set of points $\mathbf{x} \in \mathbb{R}^n$ satisfying*

1. *every coordinate of $\mathbf{x}$ is irrational, and*
2. *$W_{\mathbf{x}}(t) > -\log\delta$ for all sufficiently large t.*

Then for any $\delta \leq 2^{-7}$ we have

$$\dim_{\mathrm{H}} E_2(\delta) \leq \frac{3}{2} + 16\delta + O(\delta^2)$$

and for $n > 2$, there are constants $\delta_n > 0$ and $C_n > 0$ such that

$$\dim_{\mathrm{H}} E_n(\delta) \leq n - \frac{1}{2} + C_n\sqrt{\delta} + O(\delta)$$

for any $\delta \leq \delta_n$.

Let $0 < \delta < 1$ be fixed. We shall first develop some convenient notation. Let

$$Q = \{(p,q) \in \mathbb{Z}^2 : \gcd(p,q) = 1, q > 0\}$$

and for any $v, w \in Q$ such that $v = (p,q)$ and $w = (p',q')$ set

$$\dot{v} = \frac{p}{q}, \quad |v| = q, \quad v \times w = pq' - p'q.$$

We shall introduce two relations $\vdash$ and $\models$ on Q as follows. Let $\mathcal{C}$ be the collection of sequences $\mathbf{v} : \mathbb{N} \cup \{0\} \to Q$ with the property that $|v_0| = 1$ and for each $k \geq 0$,

$$v_{k+1} = a v_k + v_{k-1}$$

for some $a \in \mathbb{N}$ with the understanding that when $k = 0$ this holds for some $a \geq 2$ and some choice of $v_{-1} = \pm(1,0)$. Define $u \vdash v$ if there exists a sequence $(v_k) \in \mathcal{C}$ and $k \geq 0$ such that $u = v_k$ and $v = v_{k+1}$; similarly, define $u \models v$ if there exists a sequence $(v_k) \in \mathcal{C}$, $k \geq 0$ and $l \geq 0$ such that $u = v_k$ and $v = v_{k+l}$.

It follows from well-known properties of continued fractions that a sequence $(v_k)_{k\geq 0}$ in Q belongs to $\mathcal{C}$ if and only if $(\dot{v}_k)_{k\geq 0}$ is the sequence of convergents of some irrational number. It is easy to see that $|v_k| < |v_{k+1}|$ for all k and for any sequence (v_k) in $\mathcal{C}$; in other words the sequence of heights formed by the convergents of any real number is strictly increasing. Note also that if $u \vdash v$ then $|u \times v| = 1$ and $|u| < |v|$.

6.1. **Case $n = 2$.** For any $(u, v) \in Q \times Q$, let $B(u, v)$ denote the ball, with respect to the sup metric on $\mathbb{R}^2$, of radius $\frac{1}{|u||v|}$ centered at the rational point $(\dot{u}, \dot{v})$. Let $\mathcal{B}$ be the collection of balls $B(u, v)$ as (u, v) ranges over the elements of the set $J \subset Q \times Q$ consisting of all pairs (u, v) satisfying either of the mutually exclusive conditions

$$|u| < \delta|v| \quad \text{or} \quad |v| < \delta|u|.$$

Given $(u, v) \in J$ satisfying $|u| < \delta|v|$ we define $\sigma(u, v)$ to be the subset of J consisting of all pairs (u', v') satisfying

$$u \vdash u', \quad v \models v', \quad \text{and} \quad |v'| < \delta|u'|;$$

similarly, if $|v| < \delta|u|$ we define $\sigma(u, v)$ to be the subset of J consisting of those pairs (u', v') satisfying $v \vdash v'$, $u \models u'$, and $|u'| < \delta|v'|$.

We now show $(\mathcal{B}, J, \sigma)$ is a self-similar covering of $E_2(\delta)$. Given $\mathbf{x} = (x, y) \in E_2(\delta)$ let $t_j \to \infty$ be the sequence of times formed by the local minima of $W_{\mathbf{x}}$ and choose any j_0 such that $W_{\mathbf{x}}(t_j) > -\log\delta$ for all $j > j_0$. Let

$$\alpha_j = (u_j, v_j) \quad \text{for} \quad j > j_0$$

where $u_j, v_j \in Q$ are defined as follows. The zero of W_x closest to t_j is of the form $2\log q$ for some integer q which is the height of a convergent p/q of x; let u_j be the element in Q such that $\dot{u}_j = p/q$. Similarly, let v_j be the element of Q such that $\dot{v}_j$ is a convergent of y and $2\log|v_j|$ is the zero of W_y closest to t_j.

We claim that for any $j > j_0$

$$t_j = \log(|u_j||v_j|) \quad \text{and} \quad W_{\mathbf{x}}(t_j) = \left|\log\frac{|u_j|}{|v_j|}\right|.$$

Since $W_{\mathbf{x}}(t_j) > -\log\delta$ we either have $|u_j| < \delta|v_j|$ or $|v_j| < \delta|u_j|$, so that the second part of the claim implies $\alpha_j \in J$ for any $j > j_0$.

To see the claim, let t (resp. t') be the time of the local maximum of $W_{\mathbf{x}}$ that immediately precedes (resp. follows) t_j, so that

$$t < t_j < t'.$$

Let $z \in \{x, y\}$ be the coordinate of $\mathbf{x}$ such that $W_{\mathbf{x}}(s) = W_z(s)$ for all $s \in [t, t_j]$; similarly, let z' be the coordinate of $\mathbf{x}$ such that $W_{\mathbf{x}}(s) =$

$W_{z'}(s)$ for all $s \in [t_j, t']$. Then z has a consecutive pair of convergents with heights $p < p'$ such that

$$t = \log(pp') \quad \text{and} \quad W_{\mathbf{x}}(t) = W_z(t) = \log \frac{p'}{p};$$

similarly, z' has a consecutive pair of convergents with heights $q < q'$ such that

$$t' = \log(qq') \quad \text{and} \quad W_{\mathbf{x}}(t') = W_{z'}(t) = \log \frac{q'}{q}.$$

Since $W_z(t_j) = 2\log p' - t_j$ and $W_{z'}(t_j) = t_j - 2\log q$ and both are equal to $W_{\mathbf{x}}(t_j)$ we have

$$t_j = \log(p'q) \quad \text{and} \quad W_{\mathbf{x}}(t_j) = \log \frac{p'}{q}.$$

Now $W_z(t) > W_z(t_j)$ implies $p < q$ while $W_{z'}(t) > W_{z'}(t_j)$ implies $q < q'$ and since $W_{\mathbf{x}}(t_j) > -\log\delta > 0$ we also have $q < p'$ so that altogether we have the inequalities

$$p < q < p' < q'.$$

Since $t_j = \log(p'q)$ and there are no zeroes of W_z lying strictly between $2\log p$ and $2\log p'$, it follows that $2\log p'$ is the zero of W_z closest to t_j; similarly, $2\log q$ is the zero of $W_{z'}$ closest of t_j. Moreover, since $p < q < p' < q'$ we cannot have $z = z'$ so that

$$\text{either} \quad \mathbf{x} = (z, z') \quad \text{or} \quad \mathbf{x} = (z', z).$$

In the first case we have $|u_j| = p'$ and $|v_j| = q$, while in the second case $|u_j| = q$ and $|v_j| = p'$; in both cases, the claim holds.

Next, we verify that the sequence $(\alpha_j)_{j>j_0}$ satisfies the three conditions in the definition of a self-similar covering. First, we show that $\mathbf{x} \in B(\alpha_j)$ for all $j > j_0$. Suppose that $\alpha_j = (u_j, v_j)$ satisfies $|u_j| < \delta|v_j|$. Let $u'_j \in Q$ be the element such that $\dot{u}'_j$ is the convergent of x that immediately follows $\dot{u}_j$. Note that

$$|v_j| < |u'_j|$$

since $t_j < t'$ where $t' = \log(|u_j||u'_j|)$ is the time of the first local maximum of $W_{\mathbf{x}}$ beyond t_j. It follows from continued fraction theory that

$$|x - \dot{u}_j| < \frac{1}{|u_j||u'_j|} < \frac{1}{|u_j||v_j|}$$

and

$$|y - \dot{v}_j| < \frac{1}{|v_j|^2} < \frac{\delta}{|u_j||v_j|}$$

so that $\mathbf{x} \in B(\alpha_j)$, assuming $|u_j| < \delta|v_j|$. If instead $\alpha_j = (u_j, v_j)$ satisfies $|v_j| < \delta|u_j|$, an argument similar to the one just given leads to the same conclusion $\mathbf{x} \in B(\alpha_j)$. Therefore, $\mathbf{x} \in \cap_{j>j_0} B(\alpha_j)$ and since $\operatorname{diam} B(\alpha_j) = 2e^{-2t_j} \to 0$ as $j \to \infty$ it follows that $\cap B(\alpha_j) = \{x\}$, giving the first condition in the definition of a self-similar covering.

Now we check the second and third conditions. Again, suppose that $\alpha_j = (u_j, v_j)$ satisfies $|u_j| < \delta|v_j|$. Let $z \in \{x, y\}$ the coordinate of $\mathbf{x}$ such that $W_{\mathbf{x}}(t) = W_z(t)$ for all $t \in [t_j, t_{j+1}]$. Let t' be the time of the unique local maximum of $W_{\mathbf{x}}$ such that

$$t_j < t' < t_{j+1}.$$

Then $t' = \log(qq')$ where $q < q'$ are the heights of a pair of consecutive convergents of z. From

$$\begin{aligned} W_{\mathbf{x}}(t') - W_{\mathbf{x}}(t_j) &= t' - t_j, \\ W_{\mathbf{x}}(t_j) &= t_j - 2\log|u_j|, \quad \text{and} \\ W_{\mathbf{x}}(t') = W_z(t') &= t' - 2\log q \end{aligned}$$

we see that $q = |u_j|$; and since $t_j < t'$ we also have $q' > |v_j|$. Thus $|v_j|$ lies strictly between q and q' so that $\dot{v}_j$, which is a convergent of y, cannot be a convergent of z; it follows that

$$z = x.$$

And since $\dot{u}_{j+1}$ is a convergent of x with $|u_{j+1}| > |u_j| = q$ and q' is the height of the convergent that immediately follows $\dot{u}_j$, we have $q' \leq |u_{j+1}|$. On the other hand,

$$2\log q' - t_{j+1} = W_{\mathbf{x}}(t_{j+1}) \geq \log \frac{|u_{j+1}|}{|v_{j+1}|}$$

implies that $q' \geq |u_{j+1}|$, so that $q' = |u_{j+1}|$. It follows by the definition of q and q' that $u_j \vdash u_{j+1}$.

We claim $|v_j| \leq |v_{j+1}|$. Indeed, suppose on the contrary that $|v_{j+1}| < |v_j|$. Let q'' be the height of the convergent of y that immediately follows $\dot{v}_{j+1}$. Since $\dot{v}_{j+1}$ precedes $\dot{v}_j$ in the continued fraction expansion of y, we must have $q'' \leq |v_j|$. Let t'' be the time of the local maximum of $W_{\mathbf{x}}$ that immediately follows t_{j+1}. Arguing as before, we see that $t'' = \log(|v_{j+1}|q'')$ and since $t'' > t_{j+1}$ we have $q'' > |u_{j+1}| = q' > |v_j|$, which gives a contradiction, proving the claim.

Since v_j and v_{j+1} are convergents of y, it follows that $v_j \models v_{j+1}$, which together with $u_j \vdash u_{j+1}$ establishes the third condition

$$\alpha_{j+1} \in \sigma(\alpha_j)$$

in the definition of a self-similar covering. From

$$2\log q' - t_{j+1} = W(t_{j+1}) = \left|\log\frac{|u_{j+1}|}{|v_{j+1}|}\right|$$

we see that $q' = \max(|u_{j+1}|, |v_{j+1}|)$. Since $q' = |u_{j+1}|$, as was shown earlier, we have $|v_{j+1}| \le |u_{j+1}|$ and since $\alpha_{j+1} \in J$ we actually have $|v_{j+1}| < \delta|u_{j+1}|$. Therefore, $|u_j| < \delta|v_j| \le \delta|v_{j+1}| < \delta^2|u_{j+1}|$ so that

$$\frac{\operatorname{diam} B(\alpha_{j+1})}{\operatorname{diam} B(\alpha_j)} = \frac{|u_j||v_j|}{|u_{j+1}||v_{j+1}|} < \delta^2 < 1$$

giving the second condition in the definition of a self-similar covering with $\lambda = \delta^2$. As the case where $\alpha_j = (u_j, v_j)$ satisfies $|v_j| < \delta|u_j|$ can be treated by a similar argument, this completes the proof that $(\mathcal{B}, J, \sigma)$ is a self-similar covering of $E_2(\delta)$.

Now we analyse the set $\sigma(u,v)$ for each $(u,v) \in J$. As before, we consider only the case $|u| < \delta|v|$ as the other case is similar.

Let $\pi : \sigma(u,v) \to \mathbb{N}\times\mathbb{N}$ be the function defined as follows. Given $(u',v') \in \sigma(u,v)$ we have $u \vdash u'$, $v \models v'$ and

$$|v'| < \delta|u'|.$$

Since $u \vdash u'$ so that $|u \times u'| = 1$ we have

$$u' = au + \tilde{u}$$

for some positive integer a and some $\tilde{u} \in Q \cup \{\pm(1,0)\}$ satisfying $|\tilde{u}| < |u|$; moreover, both a and $\tilde{u}$ are uniquely determined by u'. Since $v \models v'$, there is a sequence $(v_k) \in \mathcal{C}$ and $k, l \ge 0$ such that $v = v_k$ and $v' = v_{k+l}$. We either have $v' = v$, $v' = v_{k+1}$, or v' is a positive linear combination of v and v_{k+1}. Since $v_{k+1} = a'v + \tilde{v}$ for some uniquely determined integer $a' > 0$ and $\tilde{v} \in Q \cup \{\pm(1,0)\}$ such that $|\tilde{v}| < |v|$, we have in any case

$$v' = bv + c\tilde{v}$$

for some uniquely determined $b > 0$ and $c \ge 0$ such that $c \le b$. (In fact, $\gcd(b,c) = 1$ although this will not be used.) We define

$$\pi(u',v') = (a,b).$$

For each $(u',v') \in \pi^{-1}(a,b)$ we have the inequalities

$$a|u| \le |u'| \le 2a|u| \quad \text{and} \quad b|v| \le |v'| \le 2b|v|$$

so that

$$\frac{1}{4ab} \le \frac{\operatorname{diam} B(u',v')}{\operatorname{diam} B(u,v)} = \frac{|u||v|}{|u'||v'|} \le \frac{1}{ab}.$$

From the inequalities

$$|v| \le b|v| \le |v'| < \delta|u'| \le 2a\delta|u| < 2a\delta^2|v|$$

we obtain the following bounds

$$a > \frac{1}{2\delta^2} \quad \text{and} \quad 1 \le b \le 2a\delta^2.$$

Observe that $\pi^{-1}(a,b)$ contains at most $8b$ elements since there are at most two choices for each of $\tilde{u}$ and $\tilde{v}$ while there are at most $b+1 \le 2b$ choices for c.

We now compute for any $s \in (1.5, 2)$

$$\begin{aligned}\sum_{(u',v')\in\sigma(u,v)} \left(\frac{\operatorname{diam} B(u',v')}{\operatorname{diam} B(u,v)}\right)^s &\le \sum_{a>1/2\delta^2} \sum_{b=1}^{[2a\delta^2]} \frac{8}{a^s b^{s-1}} \\ &\le \frac{8\cdot(2\delta^2)^{2-s}}{2-s} \sum_{a>1/2\delta^2} \frac{1}{a^{2s-2}} \\ &\le \frac{8\cdot(2\delta^2)^{s-1}}{(2s-3)(2-s)} \le \frac{16\delta}{(2s-3)(2-s)}.\end{aligned}$$

The last expression is equal to one if

$$2s^2 - 7s + 6 + 16\delta = 0$$

so that Theorem 5.3 applies to give

$$\dim_{\mathrm{H}} E_2(\delta) \le \frac{7}{4} - \sqrt{\frac{1}{16} - 8\delta} = \frac{3}{2} + 16\delta + O(\delta^2)$$

for any $\delta \le 2^{-7}$.

6.2. **Case $n > 2$.** Let $Q_n = Q^n \times \{1,\ldots,n\} \times \{1,\ldots,n\}$ and for any triple $(\mathbf{c},i,j) \in Q_n$, let $B(\mathbf{c},i,j)$ denote the ball, with respect to the sup metric on $\mathbb{R}^n$, of radius $\frac{1}{|c_i||c_j|}$ centered at the rational point $\dot{\mathbf{c}} = (\dot{c}_1,\ldots,\dot{c}_n)$. Let $\mathcal{B}$ be the collection of balls $B(\mathbf{c},i,j)$ as $(\mathbf{c},i,j)$ ranges over the elements of the set $J \subset Q_n$ consisting of all triples $(\mathbf{c},i,j)$ satisfying

$$|c_j| < \sqrt{\delta}|c_i| \quad \text{and} \quad |c_k| \text{ divides } |c_i||c_j| \text{ for } k = 1,\ldots,n.$$

For each $(\mathbf{c},i,j) \in Q_n$ let $A(\mathbf{c},i,j) \subset B(\mathbf{c},i,j)$ be the subset consisting of those points whose ith coordinate lies within a distance $\frac{1}{|c_i|^2}$ of $\dot{c}_i$. For any $(\mathbf{c},i,j) \in \mathcal{B}$ we define $\sigma(\mathbf{c},i,j)$ to be the subset of J consisting of all triples $(\mathbf{c}',i',j')$ satisfying the following conditions:

(a) $i' = j$ and $c_j \vdash c'_j$,
(b) if $j' = i$ then $c_i \models c'_i$,
(c) $|c_i| < |c'_j|$ and $|c_j| < |c'_{j'}|$,
(d) $A(\mathbf{c},i,j) \cap A(\mathbf{c}',i',j') \neq \emptyset$, and
(e) $|c_i|^2 < |c'_{i'}||c'_{j'}|$.

We now show $(\mathcal{B}, J, \sigma)$ is a self-similar covering of $E_n(\delta)$. Given $\mathbf{x} \in E_n(\delta)$ let $t_k \to \infty$ be the sequence of times formed by the local maxima of $W_{\mathbf{x}}$ and choose any k_0 such that $W_{\mathbf{x}}(t) > -\log\delta$ for all $t > t_{k_0}$. For each $k \geq k_0$ let

$$(u_k, v_k, i_k) \in Q \times Q \times \{1, \dots, n\}$$

be a triple defined as follows. Let i_k be the index i such that $W_{\mathbf{x}}(t_k) = W_{x_i}(t_k)$. Then $t_k = \log(qq')$ where $q < q'$ are heights of a pair of consecutive convergents of x_i. Let u_k (resp. v_k) be the element of Q such that $\dot{u}_k$ (resp. $\dot{v}_k$) is a convergent of x_i and $|u_k| = q$ (resp. $|v_k| = q'$). For any $k > k_0$ we have

$$W_{\mathbf{x}}(t) = \log\frac{|v_{k-1}|}{|u_k|}$$

at the unique local minimum time t between t_{k-1} and t_k. Since $W_{\mathbf{x}}(t) > -\log\delta$, we have $|u_k| < \delta|v_{k-1}|$ for all $k > k_0$.

Construct a subsequence of (u_k, v_k, i_k) as follows. Given k_l let k_{l+1} be the first (smallest) $k > k_l$ such that

$$|v_{k_l}|^2 < |v_k||u_{k+1}| \tag{9}$$

holds. Let $(k_l)_{l\geq 0}$ be the sequence obtained by recursive definition. By construction, we have

$$|v_{k_l}|^2 < |v_{k_{l+1}}||u_{k_{l+2}}|$$

for all l. Moreover, since (9) does not hold for $k = k_l$, we have

$$|v_{k_l}|^2 \geq |v_{k_{l+1}-1}||u_{k_{l+1}}| > \delta^{-1}|u_{k_{l+1}}|^2$$

for all l. To simply notation a bit, we shall suppress the double subscript and write (u_l, v_l, i_l) instead of $(u_{k_l}, v_{k_l}, i_{k_l})$. This will cause no confusion as we shall have no further use for the original sequence of triples. Thus, for any $l \geq 0$ the triple (u_l, v_l, i_l) satisfies

$$|u_{l+1}| < \sqrt{\delta}|v_l|, \qquad |v_l|^2 < |v_{l+1}||u_{l+2}|, \tag{10}$$

and $\dot{u}_l$ and $\dot{v}_l$ are consecutive convergents of x_{i_l}.

For any $l > 0$ let $\mathbf{c}_l$ be the element $(c_1, \dots, c_n) \in Q^n$ given by

$$c_i = \begin{cases} u_{l+1} & i = i_{l+1} \\ v_l & i = i_l \\ w_i & \text{otherwise} \end{cases}$$

where $w_i \in Q$ is the element such that $\dot{w}_i$ is the rational closest to x_i whose height $|w_i|$ divides $|u_{l+1}||v_l|$. Let $\alpha_l = (\mathbf{c}_l, i_l, i_{l+1})$ and note that $\alpha_l \in J$ by the first condition in (10) and the definition of w_i.

Now we check that $(\alpha_l)_{l>0}$ satisfies the first condition in the definition of a self-similar covering. To avoid double subscript notation, we fix $l > 0$ and let $(\mathbf{c}, i, j) = \alpha_l = (\mathbf{c}_l, i_l, i_{l+1})$. Then

$$c_i = v_l, \qquad c_j = u_{l+1} \tag{11}$$

and for $k \neq i, j$ $\dot{c}_k$ is the rational closest to x_k with height dividing $|c_i||c_j|$. Since $\dot{v}_{l+1}$ is the convergent of x_j that immediately follows $\dot{c}_j$, and $|v_l| < |v_{l+1}|$, we have

$$|x_j - \dot{c}_j| < \frac{1}{|u_{l+1}||v_{l+1}|} < \frac{1}{|c_i||c_j|}.$$

Likewise, $\dot{c}_i = \dot{v}_l$ is a convergent of x_i so that

$$|x_i - \dot{c}_i| < \frac{1}{|v_l|^2} = \frac{1}{|c_i|^2}.$$

By definition of c_k we have (for $k \neq i, j$)

$$|x_k - \dot{c}_k| \leq \frac{1}{2|c_i||c_j|} < \frac{1}{|c_i||c_j|}.$$

It follows that $\mathbf{x} \in A(\alpha_l)$ for all $l > 0$, and since $A(\alpha_l) \subset B(\alpha_l)$ and $\operatorname{diam} B(\alpha_l) = \frac{2}{|u_{l+1}||v_l|} \to 0$ as $l \to \infty$ we conclude that $\cap B(\alpha_l) = \{x\}$.

Now we check that (α_l) satisfies the second and third conditions in the definition of a self-similar covering. Fix $l > 0$ and let $(\mathbf{c}, i, j) = \alpha_l$ and $(\mathbf{c}', i', j') = \alpha_{l+1}$. In addititon to (11) we also have

$$i' = i_{l+l} = j, \quad c'_j = v_{l+1}, \quad \text{and} \quad c'_{j'} = u_{l+2}.$$

We first check that α_{l+1} satisfies the conditions (a)-(e) in the definition of $\sigma(\alpha_l)$. Since $\dot{u}_{l+1}, \dot{v}_{l+1}$ are consecutive convergents of x_j and $|u_{l+1}| < |v_{l+1}|$, we have $u_{l+1} \vdash v_{l+1}$ so that (a) follows. If $j' = i$ then $\dot{v}_l, \dot{u}_{l+2}$ are convergents of x_i and since $\dot{v}_{l+2}$ is the convergent that immediately follows $\dot{u}_{l+2}$, we cannot have $|u_{l+2}| < |v_l|$ (because then $|v_{l+2}| \leq |v_l|$, which is a contradiction); therefore, $|v_l| \leq |u_{l+2}|$ from which it follows that $v_l \models u_{l+2}$, giving (b). The conditions in (c) follows from $|v_l| < |v_{l+1}|$ and $|u_{l+1}| < |u_{l+2}|$, while (d) follows from $\mathbf{x} \in A(\alpha_l) \cap A(\alpha_{l+1})$, and (e) follows from the second condition in (10). Now using (e) and $\alpha_l \in J$ we find

$$\frac{\operatorname{diam} B(\alpha_{l+1})}{\operatorname{diam} B(\alpha_l)} = \frac{|c_i||c_j|}{|c'_{i'}||c'_{j'}|} < \frac{|c_j|}{|c_i|} < \delta^{1/2} < 1$$

so that the second condition in the definition of a self-similar covering holds with $\lambda = \delta^{1/2}$.

Now we analyse the set $\sigma(\alpha)$ for each $\alpha \in J$. Given $\alpha' \in \sigma(\alpha)$ we set $\alpha = (\mathbf{c}, i, j)$ and $\alpha' = (\mathbf{c}', i', j')$ so that

$$\operatorname{diam} B(\alpha) = \frac{2}{|c_i||c_j|} \quad \text{and} \quad \operatorname{diam} B(\alpha') = \frac{2}{|c'_{i'}||c'_{j'}|}.$$

Computing the sum in (7) reduces to analysing the possibilities for the elements $c'_{i'}$ and $c'_{j'}$ of Q. The possibilities for $c'_{i'}$ can be analysed using the same techniques in the case $n = 2$; they can also be used to analyse the possibilities for $c'_{j'}$ if $j' = i$. The main new ingredient is estimating the number of possibilities for $c'_{j'}$ in the case when $j' \neq i$. Since $\operatorname{diam} B(\alpha') < \operatorname{diam} B(\alpha)$ we observe that $\dot{c}'_{j'}$ is a rational that lies within $\operatorname{diam} B(\alpha)$ of $\dot{c}_{j'}$ and is therefore restricted to some interval of length $2 \operatorname{diam} B(\alpha)$ centered at $c_{j'}$. Before continuing the calculation, we pause to state a lemma that will be used to estimate the number of possibilities for $c'_{j'}$ in the case when $j' \neq i$.

Lemma 6.2. *Let I be an open interval of the real line and b a positive integer. Suppose that the minimum height of a rational in I is q. Then the number of rationals in I with heights q' satisfying $\lfloor q'/q \rfloor = b$ is at most $2bq^2|I|$.*

Proof. A rational in I satisfying the given condition corresponds to an integer point with relatively prime coordinates in the region $P := \{(x, y) | x/y \in I, bq \leq y < (b+1)q\}$. It suffices to show that P contains at most $(b+1)q^2|I|$ integer points (relatively prime or not).

Let p/q be any rational in I of minimum height. The projection of P onto the x-axis along lines of slope q/p is an open interval of length $(b+1)q|I|$. An integer point in P maps to a rational whose height divides q. Since a line of slope q/p contains at most one integer point in P, it follows there are at most $(b+1)q^2|I|$ integer points in P. $\square$

Let $\sigma_k(\mathbf{c}, i, j)$ be the subset of $\sigma(\mathbf{c}, i, j)$ consisting of those triples $(\mathbf{c}', i', j')$ with $j' = k$. Note that $\sigma_j(\mathbf{c}, i, j) = \emptyset$ since $j' \neq i' = j$. For $k \neq j$, let $\pi_k : \sigma_k(\mathbf{c}, i, j) \to \mathbb{N} \times \mathbb{N}$ be the function that sends $(\mathbf{c}', j, k)$ to the pair (a, b) given by

$$a = \left\lceil \frac{|c'_j|}{|c_j|} \right\rceil \qquad \text{and} \qquad b = \left\lfloor \frac{|c'_k|}{q_k} \right\rfloor$$

where $q_k := |c_i|$ if $k = i$ and is otherwise set equal to the minimum height of a rational in the open interval centered at $\dot{c}_k$ of length $\frac{4}{|c_i||c_j|}$. To see that π_k is well-defined we need to check that $a \geq 1$, which holds by the second condition in (c), and that $b \geq 1$, which, in the case $k = i$, holds with equality and otherwise because the rational $\dot{c}'_k$ lies in the open interval centered at $\dot{c}_k$ of length $\frac{4}{|c_i||c_j|}$ and q_k is the minimum height of a rational in that interval.

For each $(\mathbf{c}', j, k) \in \pi_k^{-1}(a,b)$ we have the inequalities

$$|c'_j| \le a|c_j| \le 2|c'_j| \quad \text{and} \quad \frac{|c'_k|}{2} \le bq_k \le |c'_k|$$

so that

$$\frac{|c_i|}{2abq_k} \le \frac{\operatorname{diam} B(\alpha')}{\operatorname{diam} B(\alpha)} \le \frac{2|c_i|}{abq_k}. \tag{12}$$

Moreover, the first condition in (c) and $\alpha \in J$ implies

$$a > \frac{|c'_j|}{|c_j|} > \frac{|c_i|}{|c_j|} > \frac{1}{\delta}$$

while $\alpha' \in J$ and the first inequality above imply

$$|c'_k| < \sqrt{\delta}|c'_j| < a\sqrt{\delta}|c_j|$$

giving us the following bounds

$$a > \frac{1}{\sqrt{\delta}} \quad \text{and} \quad 1 \le b \le \frac{a\sqrt{\delta}|c_j|}{q_k}. \tag{13}$$

Now we estimate the number of elements in $\pi_i^{-1}(a,b)$. Since each element is of the form $(\mathbf{c}', j, i)$ we need to bound the number of possibilities for $\mathbf{c}' \in Q^n$. By (a) $c_j \vdash c'_j$ from which it follows that $c'_j = a'c_j + u$ for some positive integer a' and some $u \in Q \cup \{\pm(1,0)\}$ satisfying $|c_j \times u| = 1$ and $|u| < |c_j|$. Note that a' is uniquely determined ($a' = a - 1$) while u admits two possibilities, giving rise to 2 choices for c'_j. By (b) $c_i \models c'_i$ from which it follows that $\dot{c}'_i$ is a convergent of $\dot{c}_i$ and therefore is restricted to an open interval of length $\frac{2}{|c_i|^2}$ centered at $\dot{c}_i$. Since $\dot{c}_i$ is the rational of minimum height in this interval and $|c_i| = q_k$, Lemma 6.2 implies there are at most $4b$ choices for c'_i. For any index $k \ne i, j$ the rational $\dot{c}'_k$ is restricted to an open interval of length $\frac{4}{|c_i||c_j|}$ and since its height $|c'_k|$ divides $|c'_i||c'_j|$, the number of choices for c'_k is bounded above by $\frac{4|c'_i||c'_j|}{|c_i||c_j|}$, which, by the first inequality in (12), is bounded by $8ab$. Therefore,

$$^{\#}\pi_i^{-1}(a,b) \le 8^{n-1}a^{n-2}b^{n-1}.$$

Now we estimate the number of elements in $\pi_k^{-1}(a,b)$ for $k \ne i, j$. Again $i' = j$ and there are 2 choices for c'_j. By (d) the rational $\dot{c}'_k$ is restricted to an open interval of length $\frac{4}{|c_i||c_j|}$ and since q_k is the minimum height of a rational in this interval, Lemma 6.2 implies that there are at most $\frac{8bq_k^2}{|c_i||c_j|}$ choices for c'_k. For any index $l \ne j, k$, the rational $\dot{c}'_l$ is restricted to an interval of length $\frac{4}{|c_i||c_j|}$ and since its height $|c'_l|$ divides $|c'_j||c'_k|$, the number of choices for c'_l is bounded above by $\frac{8abq_k}{|c_i|}$. This estimate can be improved by a factor of $\frac{|c_j|}{|c_i|}$ in the case $l = i$ because by (d) and (e)

$$|\dot{c}'_i - \dot{c}_i| < \frac{1}{|c'_{i'}||c'_{j'}|} + \frac{1}{|c_i|^2} \leq \frac{2}{|c_i|^2}$$

so that $\dot{c}'_i$ is further restricted to an open interval whose length is smaller than $2 \operatorname{diam} B(\alpha)$ by this factor. Hence, for $k \neq i, j$, we have

$$^{\#}\pi_k^{-1}(a,b) \leq 2 \cdot 8^{n-1} \left(\frac{q_k}{|c_i|}\right)^n a^{n-2} b^{n-1}.$$

Using (13) we now compute for any $k \neq i, j$, and any $s \in (n-1/2, n)$

$$\begin{aligned}
\sum_{\alpha' \in \sigma_k(\alpha)} \left(\frac{\operatorname{diam} B(\alpha')}{\operatorname{diam} B(\alpha)}\right)^s &\leq \sum_{a > \delta^{-1/2}} \sum_{b=1}^{a\sqrt{\delta}|c_j|q_k^{-1}} \sum_{\alpha' \in \pi_k^{-1}(a,b)} \left(\frac{2|c_i|}{abq_k}\right)^s \\
&\leq 2 \cdot 8^{n-1} \left(\frac{q_k}{|c_i|}\right)^{n-s} \sum_{a > \delta^{-1/2}} \frac{1}{a^{s-n+2}} \sum_{b=1}^{a\sqrt{\delta}|c_j|q_k^{-1}} \frac{1}{b^{s-n+1}} \\
&\leq \frac{2 \cdot 8^{n-1}}{n-s} \left(\frac{\sqrt{\delta}|c_j|}{|c_i|}\right)^{n-s} \sum_{a > \delta^{-1/2}} \frac{1}{a^{2s-2n+2}} \\
&\leq \frac{2 \cdot 8^{n-1}\sqrt{\delta}}{(2s-2n+1)(n-s)}
\end{aligned}$$

where we once again used $\alpha \in J$ in the last step. Note that if $k = i$ this estimate can be improved by a factor of 2. Therefore, summing over $k \neq j$ we get

$$\sum_{\alpha' \in \sigma(\alpha)} \left(\frac{\operatorname{diam} B(\alpha')}{\operatorname{diam} B(\alpha)}\right)^s \leq \frac{(2n-3) \cdot 8^{n-1}\sqrt{\delta}}{(2s-2n+1)(n-s)}.$$

Setting the last expression equal to one with $s = n - \frac{1}{2} + \varepsilon$ we find that

$$2\varepsilon^2 - \varepsilon + (2n-3)8^{n-1}\sqrt{\delta} = 0$$

and applying Theorem 5.3 we now obtain, for $n > 2$,

$$\dim_{\mathrm{H}} E_n(\delta) \leq n - \frac{1}{2} + (2n-3) \cdot 8^{n-1}\sqrt{\delta} + O(\delta)$$

for any $\delta \leq 2^{-6n}(2n-3)^{-2}$.

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