

UNIQUE ERGODICITY OF TRANSLATION FLOWS

YITWAH CHEUNG AND ALEX ESKIN

ABSTRACT. This preliminary report contains a sketch of the proof of the following result: a slowly divergent Teichmüller geodesic satisfying a certain logarithmic law is determined by a uniquely ergodic measured foliation.

1. INTRODUCTION

Let (X, ω) be a translation surface. This means X is a closed Riemann surface and ω a holomorphic 1-form on X . The line element $|\omega|$ induces a flat metric on X which has cone-type singularities at the zeroes of ω where the cone angle is an integral multiple of 2π . A *saddle connection* in X is a geodesic segment with respect to the flat metric that joins a pair of zeroes of ω without passing through one in its interior. Our main result is a new criterion for the unique ergodicity of the vertical foliation \mathcal{F}_v , defined by $\operatorname{Re} \omega = 0$.

Teichmüller geodesics. The complex structure of X is uniquely determined by the atlas $\{(U_\alpha, \varphi_\alpha)\}$ of natural parameters away from the zeroes of ω specified by $d\varphi_\alpha = \omega$. The evolution of X under the Teichmüller flow is the family of Riemann surfaces X_t obtained by post-composing the charts with the \mathbb{R} -linear map $z \rightarrow e^{t/2}\operatorname{Re} z + ie^{-t/2}\operatorname{Im} z$. It defines a unit-speed geodesic with respect to the Teichmüller metric on the moduli space of compact Riemann surfaces normalised so that Teichmüller disks have constant curvature -1 . The Teichmüller map $f_t : X \rightarrow X_t$ takes rectangles to rectangles of the same area, stretching in the horizontal direction and contracting in the vertical direction by a factor of $e^{t/2}$. By a *rectangle* in X we mean a holomorphic map a product of intervals in \mathbb{C} such that ω pulls back to dz . All rectangles are assumed to have horizontal and vertical edges.

Date: September 17, 2006.

1991 Mathematics Subject Classification. 32G15, 30F30, 30F60, 37A25.

Key words and phrases. Abelian differentials, Teichmüller geodesics, Boshernitzan's criterion, Masur's criterion, slow divergence.

The first author acknowledges the Fields Institute for financial support to attend the Workshop on *Partially hyperbolic dynamics, laminations and Teichmüller flow*, Toronto, Jan. 5-9, 2006.

Let $\ell(X_t)$ denote the length of the shortest saddle connection. Let $d(t) = -2 \log \ell(X_t)$. In this note we give a sketch of the proof of the following result.

Theorem 1.1. *There is an $\varepsilon > 0$ such that if $d(t) < \varepsilon \log t + C$ for some $C > 0$ and for all $t > 0$, then \mathcal{F}_v is uniquely ergodic.*

The hypothesis of Theorem 1.1 can be formulated in terms of the flat metric on X without appealing to the forward evolution of the surface. Let $h(\gamma)$ and $v(\gamma)$ denote the horizontal and vertical components of a saddle connection γ , which are defined by

$$h(\alpha) = \left| \operatorname{Re} \int_{\gamma} \omega \right| \quad \text{and} \quad v(\alpha) = \left| \operatorname{Im} \int_{\gamma} \omega \right|.$$

It is not hard to show that the following statements are equivalent.

- (a) There is a $C > 0$ such that for all $t > 0$, $d(t) < \varepsilon \log t + C$.
- (b) There is a $c > 0$ such that for all $t > 0$, $\ell(X_t) > c/t^{\varepsilon/2}$.
- (c) There are constants $c' > 0$ and $h_0 > 0$ such that for all saddle connections γ satisfying $h(\gamma) < h_0$,

$$(1) \quad h(\gamma) > \frac{c'}{v(\gamma)(\log v(\gamma))^{\varepsilon}}.$$

For any $p > 1/2$ there are translation surfaces with nonergodic \mathcal{F}_v whose Teichmüller geodesic X_t satisfies the sublinear slow rate of divergence $d(t) \leq Ct^p$. See [Ch2]. Our main result asserts a logarithmic slow rate of divergence is enough to ensure unique ergodicity of \mathcal{F}_v .

To illustrate our techniques, we give a new proof of the following well-known result, which is a special case of Masur's criterion for unique ergodicity [Ma] which asserts that \mathcal{F}_v is uniquely ergodic as soon as X_t has an accumulation point in the moduli space of compact Riemann surfaces.

Theorem 1.2. *If $\ell(X_t)$ does not approach zero as $t \rightarrow \infty$ then \mathcal{F}_v is uniquely ergodic.*

Remark. Theorem 1.2 can also be proven using Boshernitzan's criterion for interval exchange transformations, [Ve1], [Ve2].¹ On the other hand, if (X, ω) is the suspension of a minimal interval exchange transformation satisfying Boshernitzan's criterion one can show that $\ell(X_t)$ does not tend to zero.

It is not hard to give nonergodic examples of \mathcal{F}_v where the "lim inf" of $d(t)$ goes to infinity arbitrarily slowly.

¹It should be pointed out that Masur's result is misquoted in [Ve2].

Theorem 1.3. *Given any function $r(t) \rightarrow \infty$ as $t \rightarrow \infty$ there exists a Teichmüller geodesic X_t with nonergodic \mathcal{F}_v and $\liminf d(t)/r(t) < 1$.*

Outline. After introducing some notation and terminology in §2 we prove Theorem 1.2 in §3. Then we explain what modifications are necessary to obtain a proof of Theorem 1.1. A proof of Theorem 1.3 is included in §5 and can be read independent of the other sections.

2. GENERIC POINTS

If ν is a (normalised) ergodic invariant measure transverse to the vertical foliation \mathcal{F}_v then for any horizontal arc I there is a full ν -measure set of points $x \in X$ satisfying

$$(2) \quad \lim \frac{\#I \cap L_x}{|L_x|} = \nu(I) \quad \text{as } |L_x| \rightarrow \infty$$

where L_x represents a vertical segment having x as an endpoint. Given I , the set $E(I)$ of points satisfying (2) for *some* ergodic invariant ν has full Lebesgue measure. We refer to the elements of $E(I)$ as *generic points* and the limit in (2) as the *ergodic average* determined by x .

Convention. When passing to a subsequence $t_n \rightarrow \infty$ along the Teichmüller geodesic X_t we shall suppress the double subscript notation and write X_n instead of X_{t_n} . Similarly, we write f_n instead of f_{t_n} .

Lemma 2.1. *Let $x, y \in E(I)$ and suppose there is a sequence $t_n \rightarrow \infty$ such that for every n the images of x and y under f_n lie in a rectangle $R_n \subset X_n$ and the sequence of heights h_n satisfy $\lim h_n e^{t_n/2} = \infty$. Then x and y determine the same ergodic averages.*

Proof. One can reduce to the case where $f_n(x)$ and $f_n(y)$ lie at the corners of R_n . Let n_- (resp. n_+) be the number of times the left (resp. right) edge of $f_n^{-1}R_n$ intersects I . Observe that n_- and n_+ differ by at most one so that since $h_n e^{t_n/2} \rightarrow \infty$, the ergodic averages for x and y approach the same limit. \square

Ergodic averages taken as $T \rightarrow \infty$ are determined by fixed fraction of the tail: for any given $\lambda \in (0, 1)$

$$\frac{1}{T} \int_0^T f \rightarrow c \quad \text{implies} \quad \frac{1}{(1-\lambda)T} \int_{\lambda T}^T f \rightarrow c.$$

This elementary observation is the motivation behind the following.

Definition 2.2. *A point x is K -visible from a rectangle R if the vertical distance from x to R is at most K times the height of R .*

We have the following generalisation of Lemma 2.1.

Lemma 2.3. *If $x, y \in E(I)$, $t_n \rightarrow \infty$ and $K > 0$ are such that for every n the images of x and y under f_n are K -visible from some rectangle whose height h_n satisfies $h_n e^{t_n/2} \rightarrow \infty$, then x and y determine the same ergodic averages.*

3. NETWORKS

To prove unique ergodicity we shall show that all ergodic averages converge to the same limit. The ideas in this section were motivated by the proof of Theorem 1.1 in [Ma].

Definition 3.1. *We say two points are K -reachable from each other if there is a rectangle R from which both are K -visible. We also say two sets are K -reachable from each other if every point of one is K -reachable from every point of the other.*

Definition 3.2. *Given a collection \mathcal{A} of subsets of X , we define an undirected graph $\Gamma_K(\mathcal{A})$ whose vertex set is \mathcal{A} and whose edge relation is given by K -reachability. A subset $Y \subset X$ is said to be K -fully covered by \mathcal{A} if every $y \in Y$ is K -reachable from some element of \mathcal{A} . We say \mathcal{A} is a K -network if $\Gamma_K(\mathcal{A})$ is connected and X is K -fully covered by \mathcal{A} .*

To prove Theorem 1.2 we first need a proposition.

Proposition 3.3. *If $K > 0$, $N > 0$, $\delta > 0$ and $t_n \rightarrow \infty$ are such that for all n , there exists a K -network \mathcal{A}_n in X_n consisting of at most N squares, each having measure at least δ , then \mathcal{F}_v is uniquely ergodic.*

Proof. Suppose \mathcal{F}_v is not uniquely ergodic. Then we can find a distinct pair of ergodic invariant measures ν_0 and ν_1 and a horizontal arc I such that $\nu_0(I) \neq \nu_1(I)$.

We construct a finite set of generic points as follows. By allowing repetition, we may assume each \mathcal{A}_n contains exactly N squares, which shall be enumerated by $A(n, i)$, $i = 1, \dots, N$. Let $A_1 \subset X$ be the set of points whose image under f_n belongs to $A(n, 1)$ for infinitely many n . Note that A_1 has measure at least δ because it is a descending intersection of sets of measure at least δ . Hence, A_1 contains a generic point; call it x_1 . By passing to a subsequence we can assume the image of x_1 lies in $A(n, 1)$ for all n . By a similar process we can find a generic point x_2 whose image belongs to $A(n, 2)$ for all n . When passing to the subsequence, the generic point x_1 retains the property that its image lies in $A(n, 1)$ for all n . Continuing in this manner, we obtain a finite set F consisting of N generic points x_i with the property that the image of x_i under f_n belongs to $A(n, i)$ for all n and i .

Given a nonempty proper subset $F' \subset F$ we can always find a pair of points $x \in F'$ and $y \in F \setminus F'$ such that $f_n(x)$ and $f_n(y)$ are K -reachable from each other for infinitely many n . This follows from the fact that $\Gamma_K(\mathcal{A}_n)$ is connected. By Lemma 2.3, the points x and y determine the same ergodic averages for any horizontal arc I . Since F is finite, the same holds for any pair of points in F .

Now let z_j be a generic point whose ergodic average is $\nu_j(I)$, for $j = 0, 1$. Since X_n is K -fully covered by \mathcal{A}_n , z_j will have the same ergodic average as some point in F , which contradicts $\nu_0(I) \neq \nu_1(I)$. Therefore, \mathcal{F}_v must be uniquely ergodic. \square

Proof of Theorem 1.2. Fix $\delta > 0$ such that every saddle connection in X_t has length greater than 2δ along some subsequence $t_n \rightarrow \infty$. Note that any immersed square of side at most δ is embedded so that its Lebesgue measure is equal to its area. Consider the Delaunay triangulation of X_n . (See [MS].) It has 2ν triangles and 3ν edges where ν depends only on the genus and the number of singularities. Each triangle is contained in an immersed circumscribing disk of radius at least δ . It is not hard to see that every point in the disk is 1-reachable from the central square of side δ located at the circumcenter. Let $\mathcal{A}_n = \{A_i\}_{i=1}^{2\nu}$ be the collection of central squares and note that X_n is 1-fully covered by \mathcal{A}_n .

Let γ_j be a Delaunay edge. It lies on the boundary of exactly two Delaunay triangles, each having circumcenter on the perpendicular bisector of γ_j . Pick any square of side δ centered at some point on γ_j and divide it into 9 nonoverlapping squares of side $\delta/3$. Two of the four corner squares are disjoint from γ_j . Let A'_j (resp. A''_j) be the corner square that is disjoint from γ_j and lies on the same side of one of the circumcenters (resp. the other circumcenter), possibly with $A'_j = A''_j$. One verifies easily that by adding these squares to the collection \mathcal{A}_n we obtain a 1-network of at most 8ν squares, each with area at least $\delta^2/9$. The theorem now follows from Proposition 3.3. \square

4. BUFFERED SQUARES

Our interest lies in the case where $\ell(X_t) \rightarrow 0$ as $t \rightarrow \infty$. To prove of Theorem 1.1 we shall need an analog of Proposition 3.3 that applies to a continuous family of networks \mathcal{A}_t whose squares have dimensions going to zero. We also need to show that the slow rate of divergence gives us some control on the rate at which the small squares approach zero.

Recall that in the proof of Theorem 1.2 it was essential that the measure of the squares in the networks stay bounded away from zero.

This allowed us to find generic points that persist in the squares of the networks along a subsequence $t_n \rightarrow \infty$. If the area of the squares tend to zero slowly enough, one can still expect to find persistent generic points, with the help of the following result from probability theory.

Lemma 4.1. (Paley-Zygmund [PZ]) *If A_n be a sequence of measurable subsets of a probability space satisfying*

- (i) $|A_n \cap A_m| \leq K|A_n||A_m|$ for all $m > n$, and
- (ii) $\sum |A_n| = \infty$

then $|A_n \text{ i.o.}| \geq 1/K$.

Definition 4.2. *We say a rectangle is δ -buffered if it can be extended in the vertical direction to a larger rectangle that overlaps itself at most once and has area at least δ . Here, we only require the product of the dimensions of the larger rectangle be at least δ , which obviously holds if its Lebesgue measure exceeds δ .*

Proposition 4.3. *Suppose that for every $t > 0$ we have a δ -buffered square A_t embedded in X_t with side $\sigma_t > c/t^{\varepsilon/2}$, $0 < \varepsilon \leq 1$. Then there exists $t_n \rightarrow \infty$ such that conditions (i) and (ii) of Lemma 4.1 are satisfied by the sequence $f_n^{-1}A_n$ of rectangles in X for some K depending only on c and δ .*

Proof. Let (t_n) be any sequence satisfying the recurrence relation

$$t_{n+1} = t_n + \varepsilon \log(t_{n+1}) \quad t_0 > 1.$$

Note that the function $y = y(x) = x - \varepsilon \log x$ is increasing for $x > \varepsilon$ and has inverse $x = x(y)$ is increasing for $y > 1$, from which it follows that (t_n) is increasing. We have

$$\sigma_m e^{t_m - t_n} > (ct_m^{-\varepsilon/2})t_{n+1}^\varepsilon \cdots t_m^{\varepsilon/2} > ct_n^{\varepsilon/2}.$$

Let $B_n \supset A_n$ be the rectangle in X_n that has the same width as A_n and area at least δ . Let $\alpha(B_n)$ denote the product of the dimensions of B_n . Since B_n overlaps itself at most once, $\alpha(B_n) \leq 2|B_n| \leq 2$. Therefore, the height of B_n is $< 2/\sigma_n < (2/c)t_n^{\varepsilon/2}$, which is less than $2/c^2$ times the height of the rectangle $A'_m = f_n \circ f_m^{-1}A_m$, by the choice of t_n . Let A''_m be the smallest rectangle containing A'_m that has horizontal edges disjoint from the interior of B_n . Its height is at most $1 + 4/c^2$ times that of A_m . For each component I of $A_n \cap A''_m$ there is a corresponding component J of $B_n \cap A''_m$ (see Figure 1) so that

$$|A_n \cap A''_m| = \sum |I| \leq \frac{\alpha(A_n)}{\alpha(B_n)} \sum |J| \leq \delta^{-1}|A_n|\alpha(A''_m) < \frac{4 + c^2}{\delta c^2}|A_n||A_m|$$

giving (i). Choose t_0 large enough so that $t_{n+1} < 2t_n$ for all n and suppose that for some $C > 0$ and $n > 1$ we have $t_n < Cn \log n \log \log n$. Then

$$\begin{aligned} t_{n+1} &< t_n + \log t_{n+1} < t_n + \log t_n + \log 2 \\ &< Cn \log n \log \log n + \log n + \log \log n + \log \log \log n + \log 2C \\ &< Cn \log n \log \log n + \log n \log \log n \quad \text{for } n \gg 1 \\ &< C(n+1) \log(n+1) \log \log(n+1) \end{aligned}$$

so that $t_n \in O(n \log n \log \log n)$. Since

$$\sum |A_n| = \sum \sigma_n^2 > \sum \frac{c^2}{t_n^\varepsilon}$$

and $\varepsilon \leq 1$, (ii) follows. \square

There is a simple procedure for constructing buffered squares. Take any saddle connection γ and decompose the surface into a finite number of vertical strips, each of which is a parallelogram with a zero on each of its vertical edges and the remaining pair of edges contained in γ . The number of vertical strips is $\nu = 2g - 1 + r$ where g is the genus of X and r the number of zeroes of ω . Since one of the strips has area $\geq 1/\nu$, any rectangle containing the strip serves as a $1/\nu$ -buffer for any square of the same width contained in it. See Figure 2.

The condition (1) prevents the slopes of saddle connections from being too close to vertical. This allows for some control on the widths of vertical strips.

Proposition 4.4. *There is a constant $c > 0$ depending only on the constants of the logarithmic rate such that for any $t > 1$ and any saddle connection γ in X_t of length at most 1 there is a vertical strip with a pair of edges contained in γ , having area $\geq 1/\nu$ and width $\geq c/t^{\nu\varepsilon}$.*

By an argument similar to the proof of Theorem 1.2 one constructs a network of small buffered squares in X_t whose areas tend to zero no faster than $1/t^{2\nu\varepsilon}$ where ε is the coefficient of the logarithmic slow rate. Proposition 4.4 is used repeatedly on the edges of the Delaunay triangulation to construct the individual buffered squares of the network. (The main technical obstruction is that some of the edges of Delaunay triangulation may be too long so that we cannot apply Proposition 4.4; however, long edges are well understood—they have to cross long thin cylinders.) Invoking Proposition 4.3 we pass to a subsequence where the hypotheses of the lemma of Paley-Zygmund hold. An argument similar to the proof of Proposition 3.3 leads to the conclusion that \mathcal{F}_v is uniquely ergodic. Indeed, by the argument outlined above one can

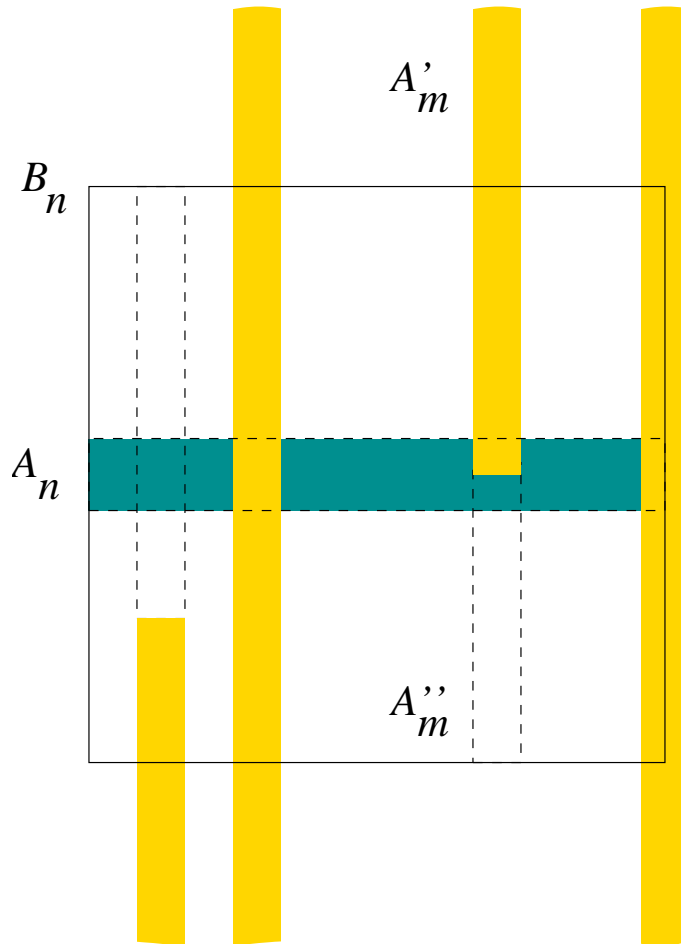


FIGURE 1. For purposes of illustration, the rectangles are represented by their images in X_t where t is the unique time when B_n maps to a square under the composition $f_t \circ f_n^{-1}$ of Teichmüller maps.

take $\varepsilon = 1/(2\nu)$ in Theorem 1.1.

A complete proof of Theorem 1.1 will appear elsewhere.

5. ARBITRARILY SLOW LIMINF DIVERGENCE

The examples with nonergodic vertical foliation are exhibited in a well-known family of branched double covers of tori. For background regarding the construction, we refer the reader to [MT].

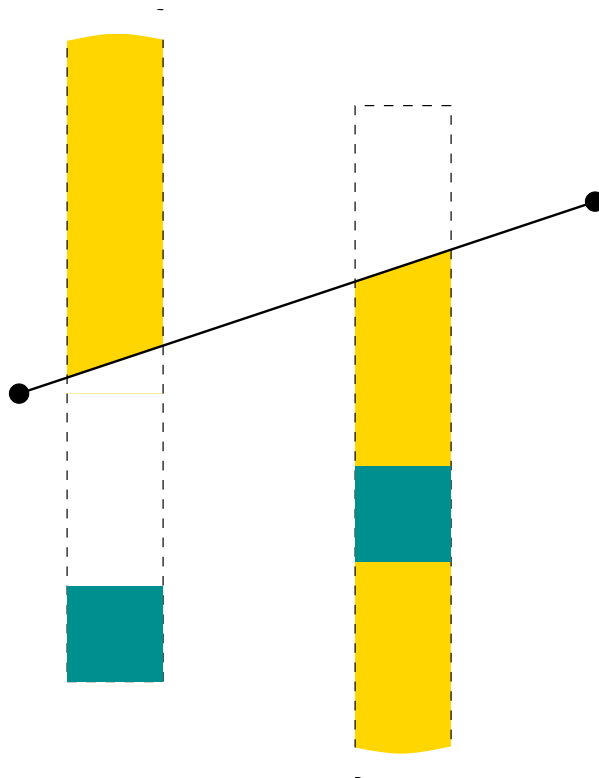


FIGURE 2. Any rectangle contained the strip with largest area serves as a buffer for any square of the same width contained in it.

Proof of Theorem 1.3. Let X be the connected sum of $T^2 = (\mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), dz)$ with itself along a horizontal arc I of irrational length $\lambda \in (0, 1)$. Let $W = \{\lambda + 2m + 2ni : m, n \in \mathbb{Z}, n > 0\}$ and note that each element in W is a vector with irrational slope. The holonomy of a saddle connection γ in X joining the two endpoints of the slit I is of the form $\lambda + m + ni$ for some $m, n \in \mathbb{Z}$ and since λ is irrational, every such number is the holonomy of some saddle connection in X provided $n \neq 0$. It is convenient to think of elements in W as saddle connections. By concatenating γ with its image under the hyperelliptic involution, which interchanges the endpoints of I , we obtain a simple closed curve Γ . The complement of Γ has two connected components if and only if m and n are both even integers; thus, every element of W determines a partition of X into two sets of equal measure. For more details, see [Ch1].

Fix a summable series of positive terms $\sum a_j < \infty$. Let $w_0 \in W$. Given $w_j \in W$ we can find an arbitrarily long $v_j \in \mathbb{Z} + i\mathbb{Z}$ such that $|w_j \times v_j| < a_j$. (Here, $|z \times w| := \text{Im}(\bar{z}w)$.) We set $w_{j+1} = w_j + 2v_j$ and note that $w_{j+1} \in W$ and that $|w_j \times w_{j+1}| < 2a_j$. The symmetric difference between the partitions determined by w_j and w_{j+1} is a union of parallelograms whose total area is bounded above by the cross product. Applying Masur-Smillie's criterion for non-ergodicity [MS], we see that the directions of w_j converge to a nonergodic direction θ .

Rotate so that θ is vertical and let X_t be the evolution of X under the Teichmüller flow. Let w_j^t be the image of w_j in X_t . Let t_j be the unique time $t = t_j$ when w_j^t and w_{j+1}^t have the same length. By choosing v_j so that the vertical (resp. horizontal) component of w_{j+1} is at least twice (resp. at most half) that of w_j we can ensure that the angle between w_j^t and w_{j+1}^t is at least $\cos^{-1}(4/5)$, the angle between the vectors $1 + 2i$ and $2 + i$. Hence, $\ell(X_t)^2 \leq |w_j^t||w_{j+1}^t| \leq (5/3)|w_j \times w_{j+1}|$ so that $d(t_j) \leq -\log |w_j \times w_{j+1}| + C$. By choosing v_j long enough we can make $d(t_j)/r(t_j) < 1/2$, which completes the proof. \square

REFERENCES

- [Ch1] Cheung, Yitwah. *Hausdorff dimension of the set of nonergodic directions*. Ann. of Math., **158** (2003), 661–678.
- [Ch2] Cheung, Yitwah. *Slowly divergent trajectories in moduli space*, Conform. Geom. Dyn., **8** (2004), 167-189.
- [Ma] Masur, Howard. *Hausdorff dimension of the set of nonergodic foliations of a quadratic differential* Duke Math. Journal, **66** (1992), 387-442.
- [MS] Masur, Howard and Smillie, John. *Hausdorff dimension of sets of nonergodic measured foliations*, Ann. of Math., **134** (1991), 455-543.
- [MT] Masur, Howard and Tabacknikov, Serge. *Rational billiards and flat structures*, in Handbook of Dynamical Systems **1A** (2002), 1015-1089.
- [PZ] Paley, R.E.A.C and Zygmund, A. *Proc. Camb. phil. Soc.* **26** (1930), 337-357, 458-474 and **28** (1932), 190-205.
- [Ve1] Veech, William A. *Boshernitzan's criterion for unique ergodicity of an interval exchange transformation*. Ergodic Theory Dynam. Systems, **7** (1987), 149–153.
- [Ve2] Veech, William A. *Measures supported on the set of uniquely ergodic directions of an arbitrary holomorphic 1-form*. Ergodic Theory Dynam. Systems, **19** (1999), 1093–1109.

SAN FRANCISCO STATE UNIVERSITY, SAN FRANCISCO, CALIFORNIA
E-mail address: cheung@math.sfsu.edu

UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS
E-mail address: eskin@math.uchicago.edu