Exploring Domains of Approximation in $\mathbb{R}^2$: Expository Essay

Nicolay Postarnakevich

August 12, 2013

1 Introduction

In this paper I explore the concept of the Domains of Best Approximations. These structures are of interest in the study of Diophantine Approximation as they encode the inequalities that govern how efficiently an irrational point can be approximated by a rational point with a given bound on the (lowest common) denominator[2]. The problem of best approximation can be formulated as follows. Given a rational point $p \in \mathbb{Q}^d$, and certain constraints, what are all the points $r \in \mathbb{R}^d$ that point p best approximates. The best approximation is defined by constraints on a distance between p, r and other points of $\mathbb{Q}^d$. Precise definition will be discussed below. In this paper I investigate the case when $d = 2$. Nosratieh[1] has shown that there are points in $\mathbb{Q}^2$ for which the best approximation set is not convex. She gave a specific example of a point $(\frac{17}{33}, \frac{8}{33})$ which generates a non-convex domain of best approximation. One of the goals of this paper is to empirically find more points with non-convex domains of approximation. Of particular interest will be such points with the smallest denominator. Before considering the case $d = 2$, I looked at the concept of best approximation for $d = 1$, points in $\mathbb{R}$.

2 Best approximation in $\mathbb{R}$

For $p \in \mathbb{R}$ the best rational approximation is defined as a point $q = \frac{a}{b}, b > 0$ such that it is closer to p than any other approximation of denominator b or smaller. This approximation has been studied by a number of prominent mathematicians[4,5] including Lagrange and is well understood when described in terms of continued fractions. Continued fractions are then used to derive convergents, which coincide with the rationals that are the best approximation to p.

A continued fraction is a representation of a real number p as:

$$p = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

where a_0 is an integer and all other a_n are positive integers.

Continued fractions have a number of important properties. Any real number can be expressed as a continued fraction. Continued fraction representations of rational numbers are finite. Irrational numbers have infinite continued fraction representations. Furthermore, irrational numbers can be represented in precisely one way using a continued fraction. The quantities a_n are called partial quotients, see [4] for further discussion. For irrational representations the first n of the partial quotients can be used resulting in a rational approximation. Such approximations are called convergents. A canonical example of continued fraction is π:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \ddots}}}}}$$

The first few convergents for π are $3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}$.

For $d = 1$ the best approximants are characterized by continued fractions which are well understood. In higher dimension the problem becomes more complex.

3 Domains of best approximation in $\mathbb{R}^2$

Let $x = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_d}{b_d}) \in \mathbb{Q}^d$ such that each component is in lowest terms. Then we can express x in a unique way as $x = (\frac{c_1}{q}, \frac{c_2}{q}, ..., \frac{c_d}{q})$, such that $v = (c_1, c_2, ..., c_d, q) \in \mathbb{Z}^d \times \mathbb{Z}_{>0}$, and $\gcd(v) = 1$. This can be accomplished by letting $q = lcm(b_1, b_2, ..., b_d)$. Then letting $x \in \mathbb{Q}^d, x = (\frac{c_1}{q}, \frac{c_2}{q}, ..., \frac{c_d}{q}), q \in \mathbb{Z}$, $p = (c_1, c_2, ..., c_d)$, and $v = (p, q), \gcd(v) = 1$, x is best approximant to $\alpha \in \mathbb{R}^d$ if[1]:

(i) $|q\alpha - p| < |n\alpha - m|$ for any $(m, n) \in \mathbb{Z}^{d+1}, m \in \mathbb{Z}^d, 0 < n < q$,

(ii) $|q\alpha - p| \leq |q\alpha - p'|$ for any $p' \in \mathbb{Z}^d$.

Given the definition of best approximant, the domain of best approximation, $\triangle(v), v \in \mathbb{Q}^d$, can be defined as[2]:

$$\triangle(v) = \{x \in \mathbb{R}^d : v \text{ is best approximant to } x\}.$$

This section investigates the case when $d = 2$. We adopt the following notation. Let $\dot{v} = (\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2$, such that $v = (p_1, p_2, q) \in \mathbb{Z}^2 \times \mathbb{Z}_{>0}$ and $\gcd(v) = 1$. We define $|v| := q$. Then $\dot{v}$ is best approximant to $(a, b) \in \mathbb{R}$, relative to some norm $||\cdot||$, if

(1) $||q(a,b) - (p_1, p_2)|| = \min\{||n(a,b) - (m_1, m_2)|| : (\frac{m_1}{n}, \frac{m_2}{n}) \in \mathbb{Q}^2, 1 \leq n \leq q\}$.

Given a $\dot{v}$ and an $x \in \triangle(v)$, we can interpret the condition (1) graphically in the following way. Scale $\dot{v}$ by q, which will make $q\dot{v} = (p_1, p_2) \in \mathbb{Z}^2$. Then, generate a sequence of q points, $\{x, 2x, 3x, ..., qx\}$, by scaling point x by integer n for $1 \leq n \leq q$. These are the points $n(a, b)$ of (1). Compare each of the generated sequence points against $\mathbb{Z}^2$ using some given norm. The points of $\mathbb{Z}^2$ are then the points (m_1, m_2) of (1). Then $x \in \triangle(v)$ implies that the distance between qx and $q\dot{v}$ is shorter or equal to any distance between other sequence points nx and points $(m_1, m_2) \in \mathbb{Z}^2$, see Figure 1.

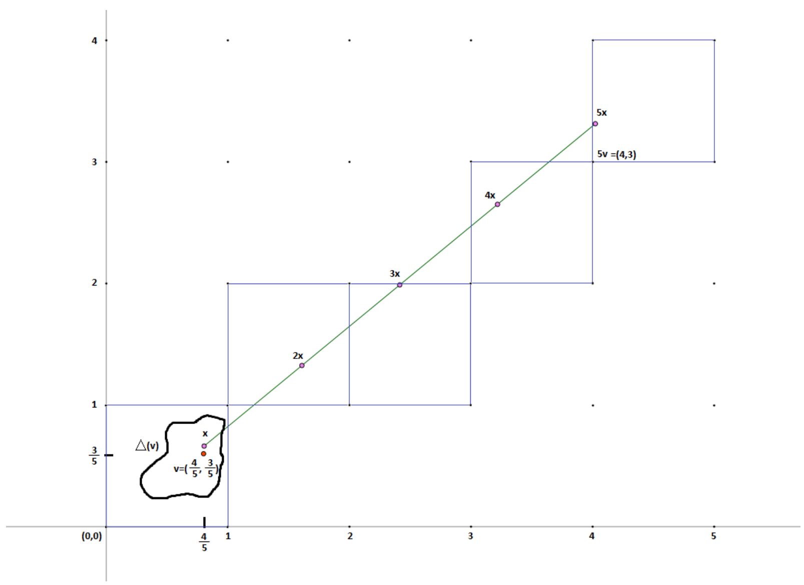

Figure 1: Geometric representation of $x \in \triangle(v)$, $v = (\frac{4}{5}, \frac{3}{5})$. The green line represents the trajectory of the scaled points $xn, 1 \leq n \leq 5$. The vertices of the blue rectangles represent the nearest points of $\mathbb{Z}^2$ to points xn.

To understand the shape of $\triangle(v)$ we will use a related set $\triangle_u(v)$:

$$\triangle_u(v) = \{x \in \mathbb{R}^2 : d(x, \dot{v}) < \frac{|u|}{|v|} d(x, \dot{u})\} .$$

where $d(,)$ is the metric induced by some norm $\| \cdot \|$. In [2] it was observed that $\triangle(v)$ can be represented as

$$\triangle(v) = \left(\bigcap_{|u|<|v|} \triangle_u(v) \right) \bigcap \left(\bigcap_{|u|<|v|} \overline{\triangle_u(v)} \right).$$

Since the definition of best approximant uses a norm, a particular choice of one has a direct influence on the shape of $\triangle(v)$. In the case of Euclidean norm, the $\triangle_u(v)$'s are circles and the finite intersection of them results in a convex set for $\triangle(v)$. However, if the metric is chosen to be the sup norm

$$||(v, w)|| = \max\{|v_x - w_x|, |v_y - w_y|\}$$

then the sets $\triangle_u(v)$ may be non-convex, resulting in a non-convex $\triangle(v)$. The existence of such non-convex sets, as well as an example of one, was the topic of Nosratieh's thesis. Her results are summarized in the following section.

4 An example of a non-convex $\triangle(\mathbf{v})$

In [1] it is shown that with respect to sup norm, there are cases when $\triangle(v)$ is not convex. Then an example of a $v = (\frac{p_1}{q}, \frac{p_2}{q})$ generating a non-convex set is given.

Nosratiah proceeds to outline a criterion for $\triangle_u(v)$ to be convex. The requirement is stated in Corollary 2.2 in [1], showing that $\triangle_u(v)$ is convex if and only if $\lambda \leq \frac{1-\tau}{1+\tau}$ or $\tau = 1$, where $\lambda = \frac{|u|}{|v|}, \tau = \min\left(|s|, \frac{1}{|s|}\right)$ and s is the slope between $\dot{u}$ and $\dot{v}$.

Then she showed a sufficient condition under which $\triangle(v)$ is an intersection of only two $\triangle_u(v)$ sets. The condition depends on the lattice

$$\mathcal{L}(v) := \{\left(a - \tfrac{cp_1}{q}, b - \tfrac{cp_2}{q}\right) : a, b, c \in \mathbb{Z}\}.$$

Then, by Theorem 3.1 in [1], given $L \in \mathcal{L}(v)$ such that $|L|_\infty < \frac{|v|^{1/2}}{2\sqrt{2}}$, then $\triangle_u(v) = \triangle_{u+} \cap \triangle_{u-}$, where $\boldsymbol{\pi_v}(u_+), \boldsymbol{\pi_v}(u_-) \in \mathcal{L}(v)$ are two shortest vectors, under $\boldsymbol{\pi_v}(u) = (\frac{m_1-np_1}{q}, \frac{m_2-np_2}{q})$ a parallel projection. Specifically, u_+ and u_- are unique vectors such that $L = u_\pm \wedge v$ and $|u_\pm| \leq |v|$, Lemma 2.10 in [2].

Finally, Nosratiah shows that if one of the sets is non-convex, then the intersection is also non-convex.

The example of existence of such a set is given by $v = (17, 8, 33)$, see Figure 2.

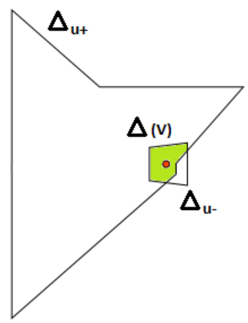

Figure 2: Example of point $(\frac{17}{33}, \frac{8}{33})$ provided by Nosratieh for a non-convex $\triangle(v)$.

5 Generating arbitrary $\triangle$(v) computationally

For this project, I proceeded to write a program that generates and displays the domain of approximation $\triangle(v)$ for any arbitrary v, not just the ones qualifying for Theorem 3.1 in [1]. The norm of choice is the maximum norm. The program first computes a set of the potential candidate elements $u \in \mathbb{Q}^2$ that may contribute $\triangle_u(v)$ that are relevant for the $\triangle(v)$ that is the result of the intersection of all of the $\triangle_u(v)$ candidates. To recall (1), the best approximant $(\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2$ to $(x, y) \in \mathbb{R}^2$ is such that

$$||q(x, y) - (p_1, p_2)|| = \min\{||n(x, y) - (m_1, m_2)|| : (\frac{m_1}{n}, \frac{m_1}{n}) \in \mathbb{Q}^2, 1 \leq n \leq q\}.$$

Let $v = (x, y) \in \mathbb{R}^2$ and consider a point $p \in \mathbb{Z}^2$ such that $||v-p|| \leq ||v-q||, q \in \mathbb{Z}^2$. Then let $x_l = \lfloor x \rfloor, x_h = \lceil x \rceil, y_l = \lfloor y \rfloor, y_h = \lceil y \rceil$, and define points $p_1 = (x_l, y_l), p_2 = (x_l, y_h), p_3 = (x_h, y_h), p_4 = (x_h, y_l)$. A geometric interpretation can be seen in Figure 1 for a point $2x$ and a blue square with points p_i surrounding it. Then for a point $p' \in \mathbb{Z}^2, p' \notin \{p_1, p_2, p_3, p_4\}$, $d(p_i, v) \leq d(p', v)$ for at least one $p_i \in \{p_1, p_2, p_3, p_4\}$. Hence, we can deduce that $p \in \{p_1, p_2, p_3, p_4\}$. In case when either x, y are in $\mathbb{Z}$, then some of the p_i will be equal. The set of u candidates is then the set of all such p_1, p_2, p_3, p_4 for each scaled version of v, i.e., $n \cdot v$ for $1 \leq n \leq q$.

Once the candidate set is derived, I compute $\triangle_u(v)$ for each candidate u, as follows.

Consider two points $A, B \in \mathbb{R}^2$, where $A = (A_x, A_y), B = (B_x, B_y)$. Define $\triangle_x = B_x - A_x$ and $\triangle_y = B_y - A_y$. And let $d_{x1} = \frac{f \cdot \triangle_x}{1+f}$, $d_{x2} = \frac{f \cdot \triangle_x}{1-f}$, and $d_{y1} = \frac{f \cdot \triangle_y}{1+f}$, $d_{y2} = \frac{f \cdot \triangle_y}{1-f}$, where $f \in \mathbb{R}$ and $0 < f < 1$. Then we can define two regions, $T_x, T_y \in \mathbb{R}^2$, to be the interior of following two trapezoids:

T_x		
P_1	$=$	$(A_x + d_{x1}, A_y - d_{x1})$
P_2	$=$	$(A_x + d_{x1}, A_y + d_{x1})$
P_3	$=$	$(A_x - d_{x2}, A_y + d_{x2})$
P_4	$=$	$(A_x - d_{x2}, A_y - d_{x2})$

T_y		
P_1	$=$	$(A_x - d_{y2}, A_y - d_{y2})$
P_2	$=$	$(A_x + d_{y2}, A_y - d_{y2})$
P_3	$=$	$(A_x + d_{y1}, A_y + d_{y1})$
P_4	$=$	$(A_x - d_{y1}, A_y + d_{y1})$

For the case when $A_x < B_x$ and $A_y > B_y$, a graphical interpretation of the two sets can be seen in Figure 3.

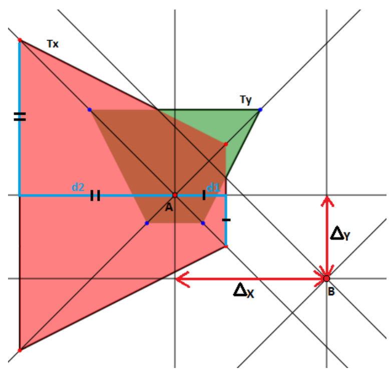

Figure 3: Sets T_x and T_y when $A_x < B_x, A_y > B_y$.

Theorem 1. *For* $v = (v_x, v_y), v_x, v_y \in \mathbb{R}, T_x \cup T_y = \triangle_u(v)$*, where* T_x, T_y *defined as above.*

Proof. Consider the point $P_1 \in T_x$, and consider distances $d(A, P_1)$ and $d(B, P_1)$. We have $d(A, P_1) = |d_{x1}|$ which follows from definition of P_1. Also, $d(B, P_1) = \max\{| \triangle_x | - |d_{x1}|, |B_x - (A_y - d_{x1})|\}$, where $\triangle_x = B_x - A_x$ as above. If $d(B, P_1) = | \triangle_x | - |d_{x1}|$, then we have:

$$\begin{aligned} f \cdot d(B, P_1) &= f \cdot (| \triangle_x | - |d_{x1}|) \\ &= f \cdot | \triangle_x | - f \cdot \tfrac{|\triangle_x|}{f+1} \\ &= \tfrac{f^2 \cdot |\triangle_x| + f \cdot |\triangle_x| - f^2 \cdot |\triangle_x|}{f+1} \\ &= \tfrac{f \cdot |\triangle_x|}{f+1} \\ &= |d_{x1}| \\ &= d(A, P_1) \end{aligned}$$

Note that $0 < f < 1$. On the other hand if $d(B, P_1) = |B_x - (A_y - d_{x1})|$, the other possible value, then we observe, $d(B, P_1) = |B_x - (A_y - d_{x1})| \Rightarrow$

$|B_x - (A_y - d_{x1})| > |\triangle_x| - |d_{x1}| = |d_{x1}|$. Hence, $d(A, P_1) \leq f \cdot d(B, P_1)$. By similar argument it can be shown that the other three points $P_2, P_3, P_4 \in T_x$ also have $d(A, P_2) \leq f \cdot d(B, P_2), d(A, P_3) \leq f \cdot d(B, P_3), d(A, P_4) \leq f \cdot d(B, P_4)$.

Consider the boundary of T_x, specifically the line segments $\overline{P_1P_2}, \overline{P_2P_3}, \overline{P_3P_4}, \overline{P_4P_1}$. Let point $v = (v_x, v_y) = P_1 \cdot t + P_2 \cdot (t-1), 0 < t < 1$ be a point on a line segment $\overline{P_1P_2}$. Then from definition of P_1 and P_2, $d(A, v) = d_{x1}$. Then v has the same x coordinate as P_1, P_2 and its y coordinate is between the y coordinates of P_1 and P_2. From above, $d(A, P_1) \leq f \cdot d(B, P_1)$ and $d(A, P_2) \leq f \cdot d(B, P_2)$ hence $d(A, v) \leq f \cdot d(B, v)$. By a similar argument, it follows that for $v = (v_x, v_y) = P_3 \cdot t + P_4 \cdot (t-1), 0 < t < 1$, where is $\overline{P_3P_4}$ is parallel to $\overline{P_1P_2}$ from definition, we also have $d(A, v) \leq f \cdot d(B, v)$.

Let $v = (v_x, v_y) = P_2 \cdot t + P_3 \cdot (t-1), 0 < t < 1$ be a point on the segment $\overline{P_2P_3}$. Then $slope(\overline{P_2P_3}) = \frac{A_y + d_{x2} - (A_y + d_{x1})}{A_x - d_{x2} - (A_x + d_{x1})} = \frac{d_{x2} - d_{x1}}{-d_{x2} - d_{x1}} = \frac{f \cdot (1+f)\triangle_x - f \cdot (1-f)\triangle_x}{-f \cdot (1+f)\triangle_x - f \cdot (1-f)\triangle_x} = -f$. Similarly $slope(\overline{P_1P_4}) = f$.

Consider a point v such that $d(A, v) = |A_x - v_x|$ and $d(A_x, v_x) < f \cdot d(B_x, v)$. Then, if $v_x < A_x$ we get

$$\begin{aligned} A_x - v_x &< f \cdot (B_x - v_x) \\ A_x - v_x &< f \cdot B_x - f \cdot v_x \\ v_x(f-1) &< f \cdot B_x - A_x \\ v_x &> \frac{f \cdot B_x - A_x}{f-1}, \end{aligned}$$

and when $v_x > A_x$ we get

$$\begin{aligned} v_x - A_x &< f \cdot (B_x - v_x) \\ v_x - A_x &< f \cdot B_x - f \cdot v_x \\ v_x(f+1) &< f \cdot B_x + A_x \\ v_x &< \frac{f \cdot B_x + A_x}{f+1}. \end{aligned}$$

But from the definitions of $P_i \in T_x$ we get

$$A_x + d_1 = A_x + \frac{f \cdot \triangle_x}{1+f} = \frac{(1+f)A_x + f(B_x - A_x)}{1+f} = \frac{f \cdot B_x + A_x}{f+1} < v_x \text{ and}$$

$$A_x - d_2 = A_x - \frac{f \cdot \triangle_x}{1-f} = \frac{(1-f)A_x - f(B_x - A_x)}{1-f} = \frac{A_x - f \cdot B_x}{1-f} = \frac{f \cdot B_x - A_x}{f-1} > v_x.$$

Hence the set T_x is bounded in x such that for each point $v \in T_x, |A_x - v_x| \leq |B_x - v_x|$. As above the lines containing segments $\overline{P_2P_3}, \overline{P_4P_1}$ have slopes $-f$ and f, and the boundary points P_1, P_2, P_3, P_4 are all such that $d(P_i, A) \leq d(P_i, B)$, hence we get that for each point $v \in T_x, d(A, v) < f \cdot d(B, v)$. In essence, the set T_x represent all the points v such that $d(B, v)$ is defined by the $x-$component of B, i.e., $d(B, v) = |B_x - v_x|$. A similar proof can be used to show that symmetrically the points $P_i \in T_y$ are such that $d(P_i, A) \leq d(P_i, B)$, and for all points $v \in T_y, d(v, A) \leq d(v, B)$. Symmetrically, the set T_y are all the points v such that the $d(B, v) = |B_y - v_y|$, which can be shown by a similar proof.

For an arbitrary point $v, d(B, v) = \{|B_x - v_x|, |B_y - v_y|\}$, and if $d(A, v) < f \cdot d(B, v)$ then if $d(B, v) = |B_x - v_x| \Rightarrow v \in T_x$, and if $d(B, v) = |B_y - v_y| \Rightarrow v \in T_y$. Hence we have $T_x \cup T_y = \triangle_u(v)$. □

Now $A \in T_x$ and $A \in T_y$ so the sets are connected. The sets T_x, T_y can be empty if either $A_x = B_x$ (T_x is empty) or $A_y = B_y$ (T_y is empty), then the remaining non-empty set is $\triangle_u(v)$. It is also possible that one of the sets can be completely inside the other. That happens when there is sufficient difference between $\triangle_x$ and $\triangle_y$. Geometrically such a case happens when all the points $P_i \in T_i$ are on the same sides of lines with slopes 1 and -1 that run through the point B. Another special case happens when $A_x - B_x = A_y - B_y$. Then two of the corner points of T_x coincide with two corner points of T_y. Different cases can be seen in Figure 4.

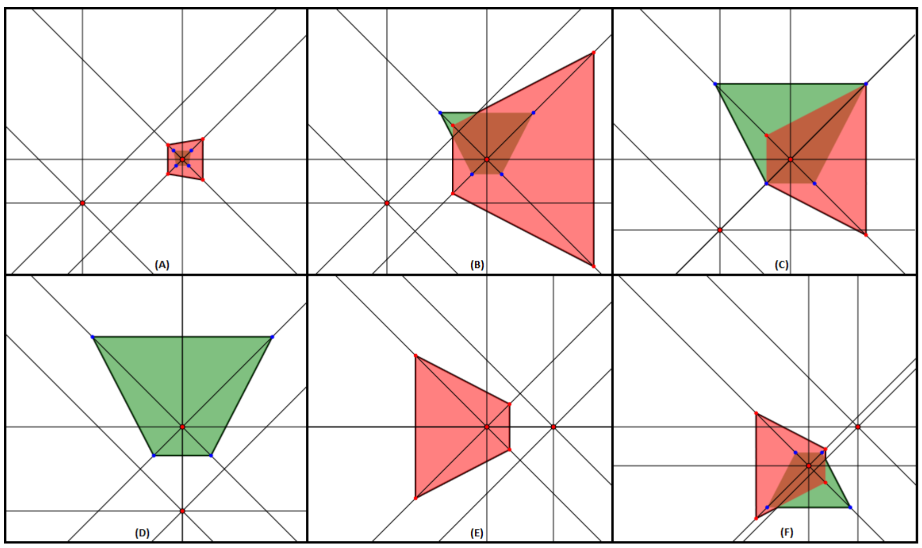

Figure 4: (A) $f = 0.169, T_y \subset T_x$ (B) $f = 0.516$ (C) $|A_x - B_x| = |A_y - B_y|$ (D) $A_x = B_x$ (E) $A_y = B_y$ (F) Non-convex $\triangle(v)$

Once all the $\triangle_u(v)$ are computed, I proceed to intersect them to produce $\triangle(v)$. The intersection is computed on pair-wise basis using the Weiler-Atherton[3] algorithm. The following is a brief outline of the algorithm. The algorithm takes two polygons as input and produces a list of polygons that are the result of the intersection of the two input polygons. The output could potentially be more than a single polygon in cases where the input polygons are convex. In my computations, however, no intersection resulted in multiple polygons. For a pair of input polygons, the algorithm first constructs two circular graphs, one for each polygon, where each node represents a vertex of a polygon. Then for each segment of the first polygon, all the intersections against the second polygon are computed. Each found intersection point is inserted into both graphs as a new node in the appropriate order. Then the two nodes, one from each graph, representing this intersection, are linked by an edge. This connects the two graphs through the nodes representing this intersection. In the final step, starting at a random node of first graph, a walk of the graph is performed.

A state variable signifying if the current vertex is inside or outside of the other polygon is kept. When a new node is discovered, if we are currently inside the other graph and this node is not an intersection then it is added to the result list as being part of the intersection polygon. If we are not inside then the node is discarded and next node is chosen. If the node is an intersection, then if we are inside, we switch the walk to the other graph via the edge that connects this intersection to its complement in the other graph. If we were outside, then we change the state variable to being inside, and pick the next node on the same graph. The result is the list of nodes signifying the intersection of both polygons.

6 Exploring the results

One of the main goals of this paper was to empirically find a point $v = (\frac{p_1}{q}, \frac{p_2}{q})$ with smaller denominator than 33 (Nosratieh's example), having a non-convex $\triangle(v)$. The smallest such point found has a denominator of 4, $v = (\frac{1}{4}, \frac{2}{4})$. A figure of $\triangle(v)$ can be seen in Figure 5. In the figure the yellow polygon is the non-convex $\triangle(v)$. The figure also includes outlines of the candidate $\triangle_u(v)$.

Before proceeding to look at other sets, consider Figure 6. It is a plot of all points $p = (\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2, q \leq 33, 0 < \frac{p_1}{q} < 1, 0 < \frac{p_2}{q} < 1$. By definition of best approximant, all these points could affect the shape of $\triangle(v)$. The red point is $(\frac{17}{33}, \frac{8}{33})$. The green points are the candidate u's that I compute to generate the superset of relevant $\triangle_u(v)$ that effect the resulting intersection $\triangle(v)$. This set also includes the only two relevant u's as defined by Nosrtatieh's thesis. Many of the other $\triangle(v)$ are generated by more then two u's. In Figure 6, some points have larger empty neighborhoods around them than others, like $(\frac{1}{2}, \frac{1}{2})$. In general, lower denominator implies larger distance between u candidate points, resulting in larger $\triangle(v)$ sets.

Figure 7 shows $\triangle(v)$ for points $p = (\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2, q \leq 9, 0 < \frac{p_1}{q} < 1, 0 < \frac{p_2}{q} < 1$. The sets are drawn with partial transparency to show overlap. Overlapping can only occur between sets generated by points with different denominators.

Figure 8 shows the same points as Figure 7, but only the sets with denominator of 9 are shown. The four points that have large sets that are overlapping others are those that have $\gcd(p_1, p_2, 9) \neq 1$ such as $(\frac{3}{9}, \frac{3}{9}) = (\frac{1}{3}, \frac{1}{3})$.

Figure 9 shows several polygons with the most edges found, which is 15. The search was done on $v = (\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2, q \leq 100$. Polygons with least edges found had 6.

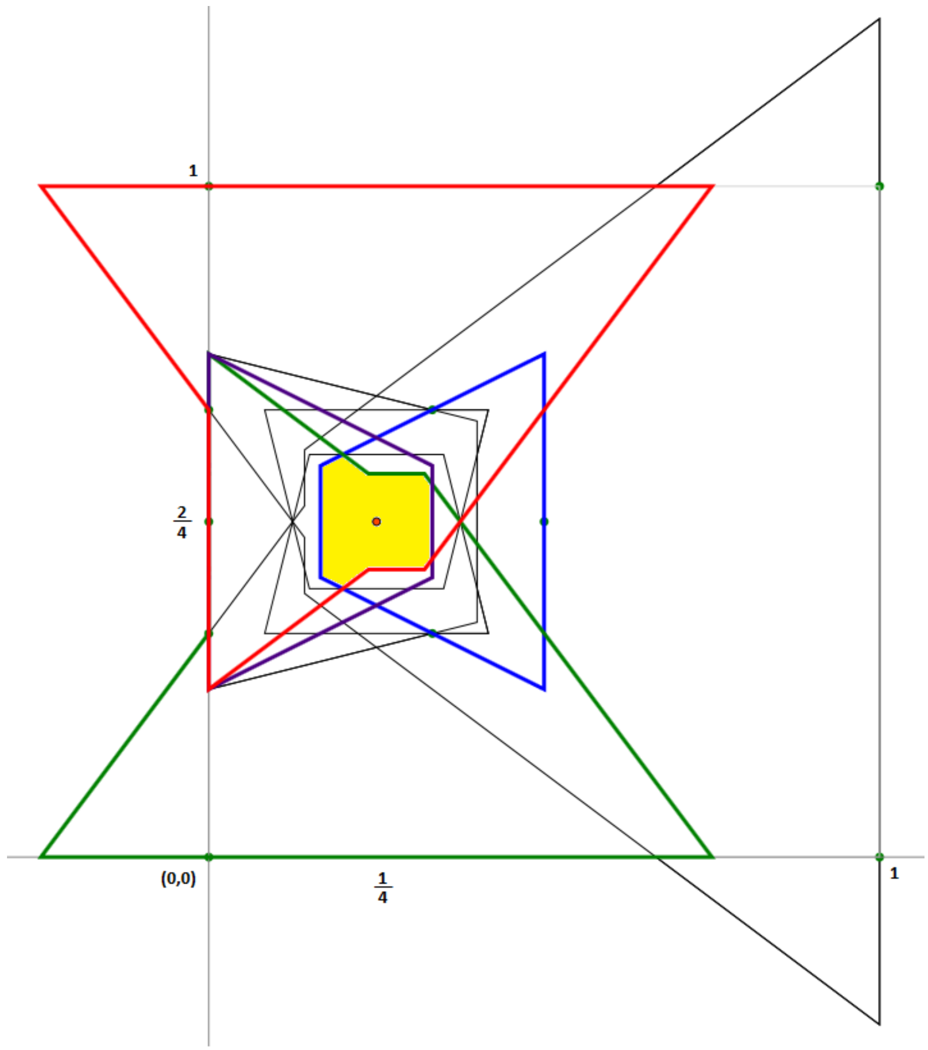

Figure 5: Example of the point (red) $(\frac{1}{4}, \frac{2}{4})$ with non-convex $\triangle(v)$. The denominator 4 is the smallest exhibiting such behavior. Green points signify candidates for $\triangle_u(v)$, outlines of each candidate are included. Black outlines are (trivial) candidates that did not contribute to the $\triangle(v)$. Color outlines are for relevant $\triangle_u(v)$: $(\frac{1}{3}, \frac{2}{3})$ = green, $(\frac{1}{3}, \frac{1}{3})$ = red, $(\frac{0}{2}, \frac{1}{2})$ = blue, $(\frac{1}{2}, \frac{1}{2})$ = indigo. The yellow region is the $\triangle(v)$.

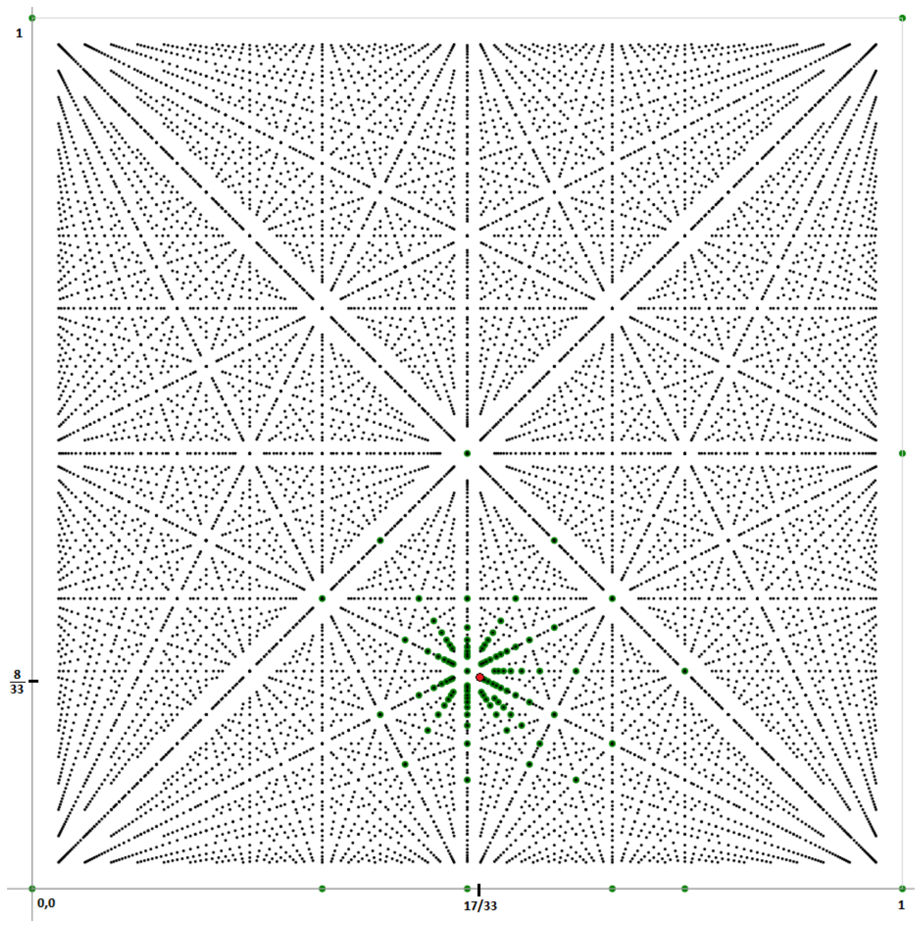

Figure 6: Graph of points $p = (\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2, q \leq 33, 0 < \frac{p_1}{q} < 1, 0 < \frac{p_2}{q} < 1$. From definition of best approximant, any of these could contribute to the shape of $\triangle(v)$. Green points are actual pruned subset which are the candidates for the final $\triangle(v)$ shape. The red point is the example point $(\frac{17}{33}, \frac{8}{33})$ provided in [1].

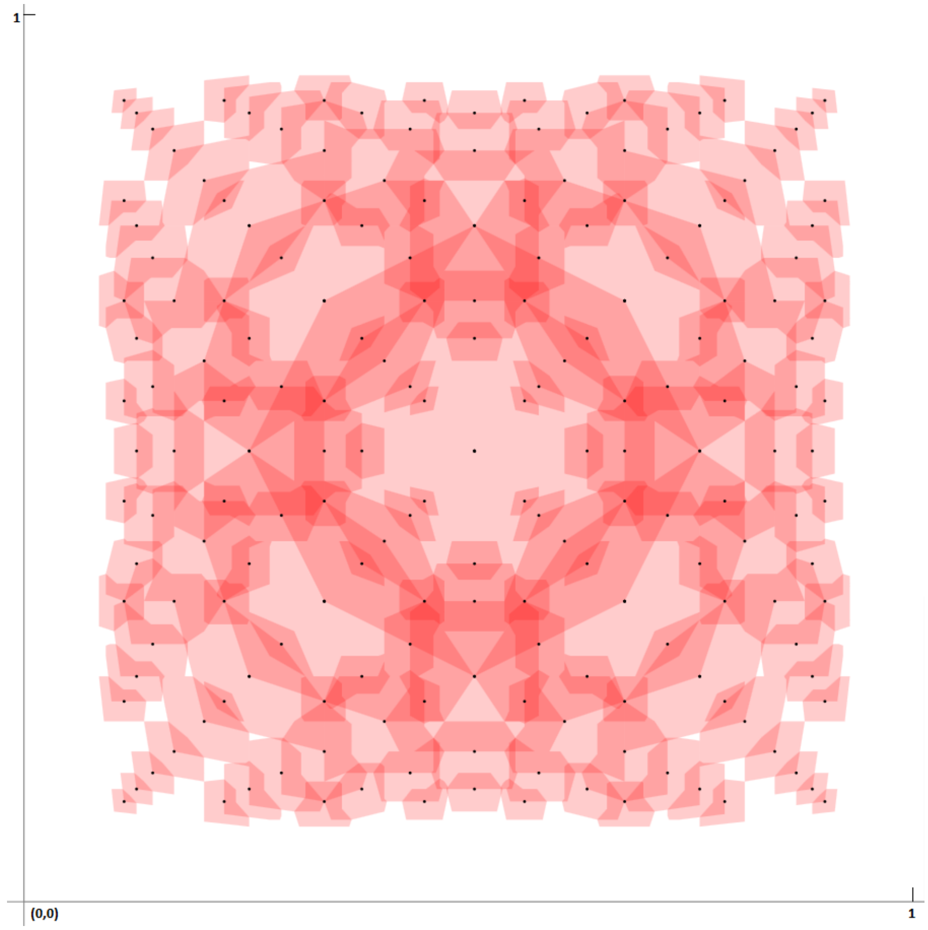

Figure 7: Example of $\triangle_u(v)$'s for rational points inside a unit square with denominators of 9 or less.

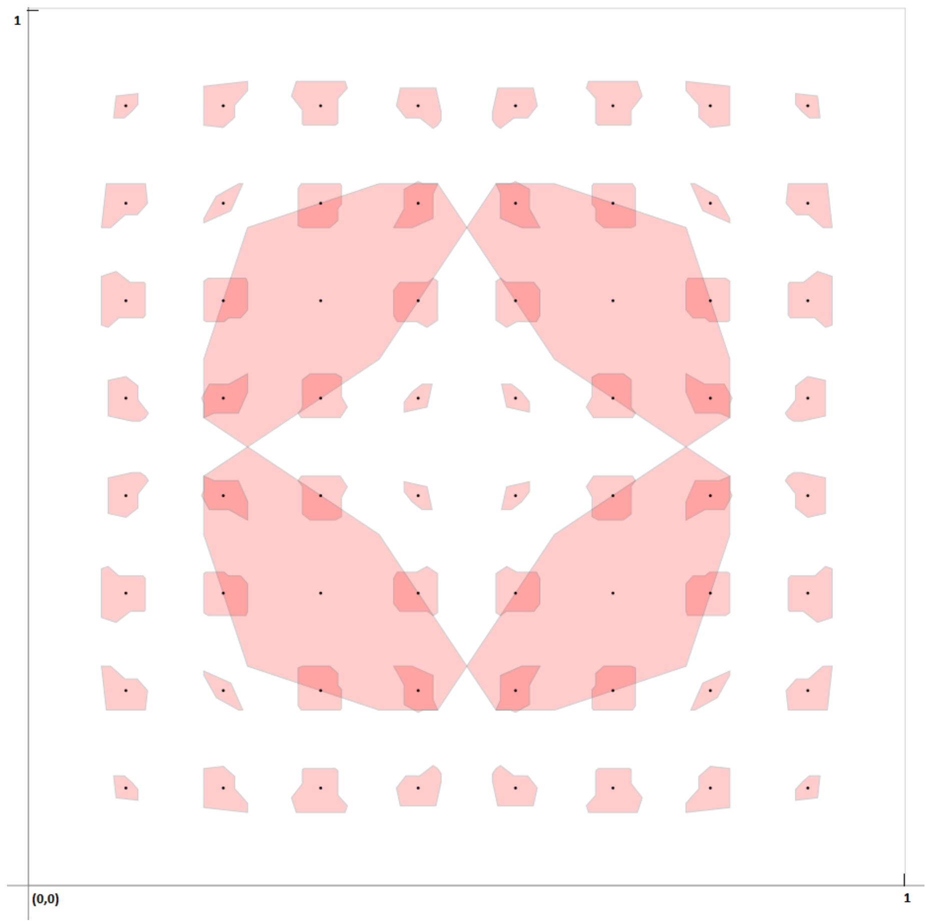

Figure 8: Example of $\triangle(v)$s for rational points inside a unit square with denominator precisely 9. The four points with large sets are those that have $\gcd(p_1, p_2, 9) \neq 1$, i.e. $(\frac{6}{9}, \frac{3}{9})$.

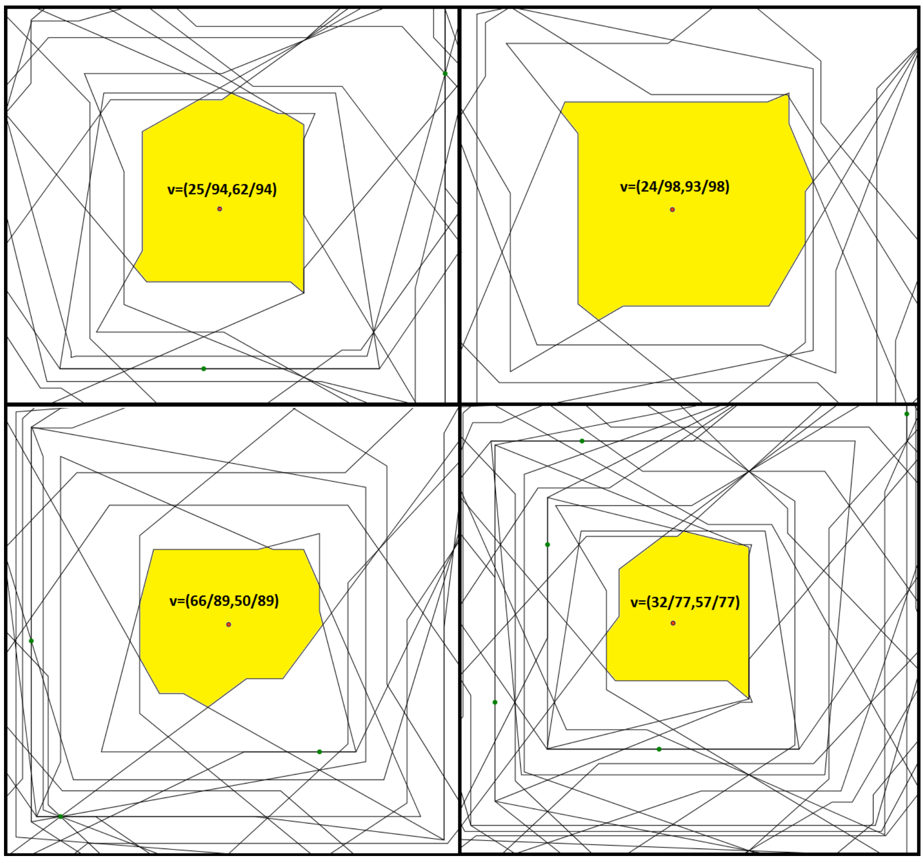

Figure 9: Example of complex $\triangle(v)$'s.

7 References

[1] Nosratieh, B., Thesis. San Francisco State University, Mathematics (2010).
[2] Cheung, Y., Hausdorff dimension of the set of singular pairs, Annals of Math, 173, (2011), 127-167
[3] Weiler, Kevin and Atherton, Peter. "Hidden Surface Removal using Polygon Area Sorting", Computer Graphics, 11(2):214-222, 1977.
[4] C.D. Olds, Continued Fractions, Random House, NY (1963).
[5] Weisstein, Eric W. "Lagrange's Continued Fraction Theorem." From MathWorld– A Wolfram Web Resource. http://mathworld.wolfram.com/LagrangesContinuedFractionTheorem.html