AN OVERLAP CRITERION FOR THE TILING PROBLEM OF THE LITTLEWOOD CONJECTURE

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *AN OVERLAP CRITERION FOR THE TILING PROBLEM OF THE LITTLEWOOD CONJECTURE* by Lucy Hanh Odom and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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The Littlewood Conjecture is an open problem in number theory, particularly in Diophantine approximation. In this paper, we look at the continued fractions of a pair of real numbers (α, β) that is analogous to the tiling of the plane. We offer an extension to the work already done in Samantha Lui's thesis [3], where we will do analysis in the case where $\alpha, \beta \in \mathbb{Q}$. We will address some issues that arise in this case, and extend the definition of a tiling in [3] to work in the rational case.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Introduction

The Littlewood Conjecture [5] is an open problem in Diophantine approximation and was first proposed by John Edensor Littlewood in the 1930s. Take $x \in \mathbb{R}$, and let $\|x\|$ denote the distance from x to the nearest integer. Then for any $\alpha \in \mathbb{R}$ there exists integers p, q with $1 \leq q \leq Q$ such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{qQ},$$

i.e. $\|q\alpha\| \leq \frac{1}{Q}$. The behavior of $\|q\alpha\|$, as q varies through integers, embodies *approximation of α by rational numbers.* The Littlewood Conjecture handles the simultaneous approximation of two real numbers α, β. It asserts that for any two real numbers α and β,

$$\liminf_{n\to\infty} n\|n\alpha\|\|n\beta\| = 0, \tag{1.1}$$

where $\|\cdot\|$ is the distance to the nearest integer. It also states that α and β may be approximated, pretty well, by rational numbers with the same denominator.

It is clear that the conjecture holds under the assumptions that $\alpha, \beta \in \mathbb{Q}$. The conjecture also holds if the continued fraction of α (or β) has unbounded partial quotients, that is, if the integers $a_0, a_1, \ldots$ with $a_1, a_2, \ldots > 0$ such that

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

satisfy $\sup a_k = \infty$. As a shorthand notation we write the set of partial quotients as $\alpha = [a_1; a_2, a_3, \ldots]$. Perhaps the most important result on the Littlewood Conjecture also happens to be the most recent. In 2006, Einsiedler et. al. [1] proved that

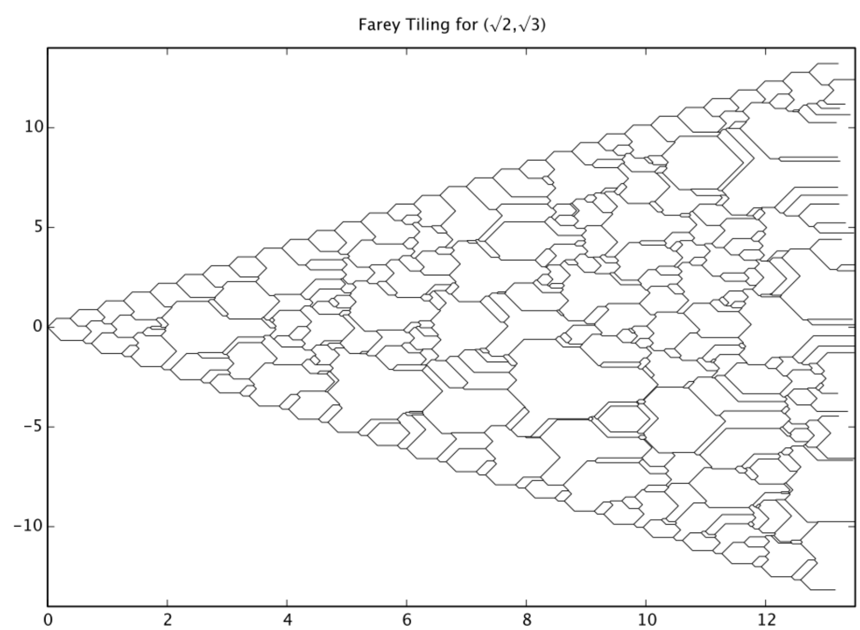
Theorem 1.1. *The set of (α, β) for which (1.1) fails has Hausdorff dimension 0.*

In [3], Lui showed that a pair of real numbers (α, β) corresponds to a Farey tiling of the xy-plane, with the assumption that $\alpha, \beta \in \mathbb{R} \backslash \mathbb{Q}$. One of the main results in [3] were

$$\inf_{n>0} n\|n\alpha\|\|n\beta\| > 0$$

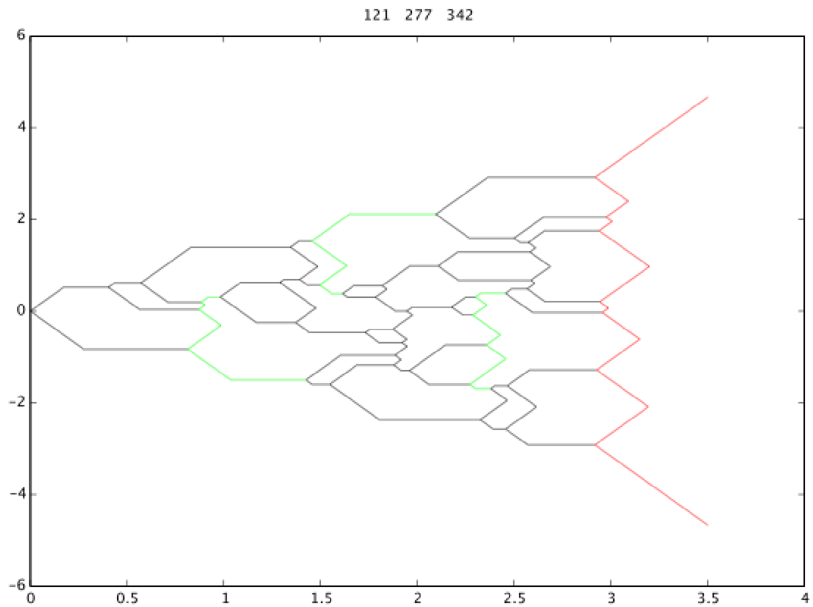
if and only if the diameters of each of the tiles are uniformly bounded. We will continue this geometric approach to the Littlewood Conjecture when $\alpha, \beta \in \mathbb{Q}$.

Below is a Farey tiling of the plane for the pair $(\alpha, \beta) = (\sqrt{2}, \sqrt{3})$. These tiles come from the partial quotients of $\sqrt{2}$ and $\sqrt{3}$ respectively. As $\sqrt{2} = [1; 2, 2, 2, \ldots]$, and $\sqrt{3} = [1; 1, 2, 1, 2, 1, \ldots]$, we see that the slopes of the upper boundary represent the partial quotients of $\sqrt{2}$, and the alternating slopes of the lower boundary represent the partial quotients of $\sqrt{3}$. Because of the fact that the continued fractions of irrational numbers do not terminate, then naturally there are infinitely many corresponding tiles.



We now wish to understand the tilings for the pair (α, β) in the rational case, and develop an analog of the continued fraction when $(\alpha, \beta) \in \mathbb{Q}^2$. Due to the finite number of partial quotients in the rational case, an implication of $(\alpha, \beta) \in \mathbb{Q}^2$ is that the number of tiles is finite, as can be seen by the next figure.

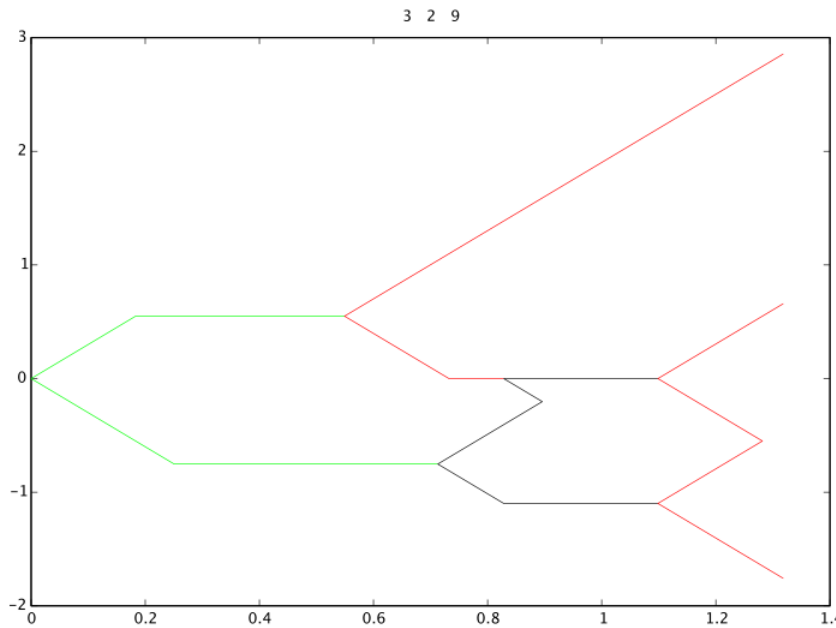
Figure 1.1: Tiling for $p_1 = 121, p_2 = 277, q = 342$.



In the following sections, we assume that the pair of rational numbers $(\frac{p_1}{q}, \frac{p_2}{q})$ satisfies $\gcd(p_1, q) = 1$ and $\gcd(p_2, q) = 1$, i.e., the pair $(\frac{p_1}{q}, \frac{p_2}{q})$ is *totally reduced*. Then we will show that the same definition for $\tau(u)$ in Lui's thesis defines a tiling.

However, we will see that if $(\frac{p_1}{q}, \frac{p_2}{q})$ is not totally reduced, then Lui's definition for the irrational case will break down. In particular, the tiles may overlap. Below is an example where $\gcd(p_1, q) \neq 1$ and $\gcd(p_2, q) \neq 1$.

Figure 1.2: Tiling for $p_1 = 3, p_2 = 2, q = 9$.



The above figure shows that the pair $(\frac{1}{3}, \frac{2}{9})$ contains two non-degenerate tiles, which are the red colored ones. The overlap takes place in a portion of the green with another portion of the black one. The portion of the green tile lies directly beneath the sliver of the upper "sector" of the the black tile. In the next sections, we will define the sectors of a tile.

The goal of this project is to investigate the cause of these tiles overlapping, and establish an overlap criterion for the case of rational pairs (α, β) .

Chapter 2

Encoding of A-orbits

To each lattice $\Lambda \subset \mathbb{R}^n$, we will start by defining a covering by "tiles" of the positive diagonal subgroup $A \subset \mathrm{SL}(n, \mathbb{R})$ given by

$$A = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} : \forall i, a_i > 0, \prod_{i=1}^{n} a_i = 1 \right\}.$$

These tiles come from the encoding of the A-orbit of Λ through the shortest nonzero vector in $a\Lambda$ with respect to the sup norm $\|\cdot\|_\infty$ as a varies over A.

A special case of this construction can be referred to in Appendix A, where $n = 3$ and $a_1 = e^{t+s}, a_2 = e^{t-s}, a_3 = e^{-2t}$, and the mapping $(t, s) \mapsto A$ is a homeomorphism.

2.1 Λ-boxes

Let $\Lambda \subset \mathbb{R}^n$ be a lattice, by which we mean the discrete subgroup of $\mathbb{R}^n$ spanned by n linearly independent vectors.

Definition 2.1. We define a *box* to be a subset of $\mathbb{R}^n$ of the form

$$B(u) = \{x \in \mathbb{R}^n : |x_i| \leq u_i \text{ for } i = 1, \ldots, n\}$$

where $u_1, \ldots, u_n$ are nonnegative real numbers, any number of which are permitted to vanish. When this happens, the box is said to be *degenerate*. The *dimension* of $B(u)$ refers to the dimension of the smallest subspace $W \subset \mathbb{R}^n$ such that $B \subset W$. The *interior* and *boundary* of a box B are taken relative to the subspace W and are denoted by $\mathrm{int}B$ and ∂B, respectively.

Remark. Note that this definition allows for *degenerate* boxes, i.e. a box for which $\prod_{i=1}^n u_i = 0$. We say that a box $B \subset \mathbb{R}^n$ is *degenerate* if and only if its dimension is $< n$.

Definition 2.2. By a Λ-*box*, we mean a box B such that

$$\mathrm{int}B \cap \Lambda = \{0\} \quad \text{and} \quad \partial B \cap \Lambda \neq \varnothing.$$

We partially order the set of all Λ-boxes by inclusion, with respect to which the notions of *maximal* and *minimal* Λ-boxes are then defined.

Lemma 2.1. *For any non-degenerate box $B \subset \mathbb{R}^n$ there is a unique $a \in A$ such that $Q = aB$ is a cube.*

Proof. For $a \in A$, we have

$$a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

such that $a_i > 0$ and $\prod_{i=1}^{n} a_i = 1$. If aB is a cube, then $r = a_1 u_1 = \cdots = a_n u_n$, where r is the common value of the length of the sides of the cube. It follows that

$$r^n = (a_1 u_1) \cdots (a_n u_n) = (a_1 \cdots a_n)(u_1 \cdots u_n) = \mathrm{vol}(B),$$

and so $r = \sqrt[n]{\mathrm{vol}(B)}$. Hence, the unique $a \in A$ is the diagonal matrix with elements

$$a_1 = \frac{\sqrt[n]{\mathrm{vol}(B)}}{u_1}, \quad \cdots, \quad a_n = \frac{\sqrt[n]{\mathrm{vol}(B)}}{u_n}$$

that make $Q = aB$ a cube. □

Notation. Let $a \in A$ and $\Lambda \subset \mathbb{R}^3$. Then the set of all Λ-boxes is parametrized by A via the map sending

$$a \in A \quad \mapsto \quad B_a = a^{-1} Q(a\Lambda)$$

where $Q(\cdot)$ denotes the largest cube containing no nonzero lattice points in its interior. As a shorthand notation, we shall write $B = a^{-1}Q$.

Then one observation we can make is that

$$B \text{ is a } \Lambda\text{-box} \Leftrightarrow aB \text{ is an } (a\Lambda)\text{-cube.}$$

If we work with boxes in $\mathbb{R}^3$, the box will have a *multi-face* of codimension 1. In general, a multi-face of codimension k has 2^k components.

Definition 2.3. Given a non-degenerate box B, there is a unique $a \in A$ such that $Q = aB$ is a cube. The *boundary* ∂B is partitioned into the *multi-faces*

$$\partial_I B = a^{-1}\left(\partial Q \cap V_I\right), \quad \varnothing \neq I \subset \{1, \cdots, n\},$$

where $V_I = \{x \in \mathbb{R}^n : \|x\|_\infty = |x_i| \text{ if and only if } i \in I\}$. We refer to any of the $2^{\#I}$ connected components of $\partial_I B$ as a *face* of B of "type" I.

Remark. For a box in $\mathbb{R}^3$, V_I in the above definition can be geometrically thought of as a cone shaped like a pyramid with two components extending in either the x, y, or z direction, which can sometimes be labeled as 1, 2, or 3 respectively.

Figure 2.1: An example of $V_{\{3\}}$ in $\mathbb{R}^3$.

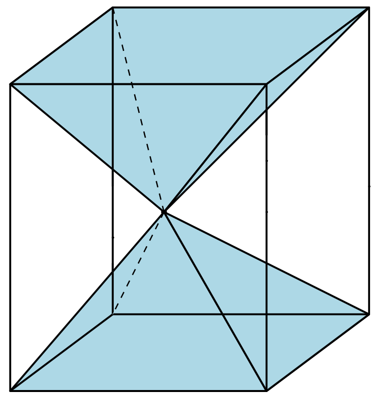
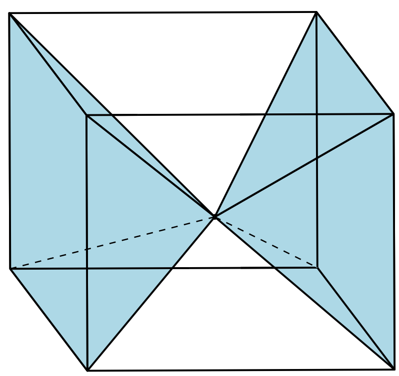


Figure 2.2: An example of $V_{\{1\}}$ in $\mathbb{R}^3$.



The next definition extends the previous one to include degenerate boxes.

Definition 2.4. Given a box B, there is a unique $a \in A$ such that $Q = aB$ is a cube of the same dimension as B and the linear map defined by a is the identity on the orthogonal complement of B. Then ∂B is partitioned in multi-faces $\partial_I B$ defined by the same formula as above, but with I restricted to nonempty subsets of the set of supporting indices.

Note that the codimension of a face type I is the cardinality of I.

Notation. We shall write $\partial_i B$ as a shorthand for $\partial_{\{i\}} B$, and we allow i to range over the set $\{1, \ldots, n\}$. We use $\bullet$ to denote the set of supporting indices so that $\partial_\bullet B$ refers to the set of *corners* of B.

Lemma 2.2. *Let B be a Λ-box. Then B is minimal if and only if the only nonzero lattice points in B lie in the corners of B, i.e. $\partial B \cap \Lambda \subset \partial_\bullet B$.*

Proof. If ∂B contains a lattice point not in the corners, then B cannot be minimal. Conversely, if all nonzero lattice points in B lie in the corners, then any smaller box is not a Λ-box. $\square$

Definition 2.5. We say a Λ-box is *relatively maximal* if it is maximal with respect to the set of all Λ-boxes of $\dim B$.

Note that a non-degenerate Λ-box is maximal if and only if it is relatively maximal, whereas a degenerate Λ-box might be relatively maximal, but never maximal.

Lemma 2.3. *Let B be a Λ-box. Then B is relatively maximal if and only if it contains a lattice point in each codimension one face, i.e. $\partial_i B \cap \Lambda \neq \varnothing$ for each i.*

Proof. If $\partial_i B \cap \Lambda = \varnothing$, then B can be extended in the i^{th} direction to a larger Λ-box of the same dimension, and is hence not maximal. Conversely, if every codimension one face contains a lattice point, then B is relatively maximal because any box B'

containing B and having the same dimension will have one of these lattice points in its interior and is thus not a Λ-box. $\square$

2.2 Pivots and Tiles

Definition 2.6. We say that $u \in \Lambda$ is a *pivot* of Λ if $B(u)$ is a minimal Λ-box. The pivot u is *degenerate* if the box $B(u)$ is degenerate; or equivalently, if any coordinate of u vanishes. We say $u \in \Lambda$ is a *trivial pivot* if all but one component vanishes.

Definition 2.7. The set of *pivot classes* of the lattice Λ is defined by

$$\Pi(\Lambda) = \{u \in \Lambda : B(u) \text{ is a minimal } \Lambda\text{-box }\}/\sim$$

where $u \sim v$ if $B(u) = B(v)$.

Definition 2.8. The *tile* associated with a pivot $u \in \Pi(\Lambda)$ is denoted by

$$\tau(u) = \{a \in A : \|au\|_\infty \leq \|aw\|_\infty \text{ for all } 0 \neq w \in \Lambda\}.$$

Note that this is well-defined since $u \sim v$ implies $\|au\|_\infty = \|av\|_\infty$ for all $a \in A$, so that $\tau(u) = \tau(v)$.

The next theorem can be attributed to [3] in a special case.

Theorem 2.4. *For any lattice $\Lambda \in \mathbb{R}^3$, we have $A = \bigcup_{u \in \Pi(\Lambda)} \tau(u)$.*

Proof. Given $a \in A$ there is a $u \in \Lambda$ such that

$$\|au\|_\infty = \min\{\|av\|_\infty : 0 \neq v \in \Lambda\},$$

because $a\Lambda \setminus \{0\}$ is a closed subset of $\mathbb{R}^3 \setminus \{0\}$. If u is a pivot, then $a \in \tau(u)$ and we are done. If not, let $Q = [-r, r]^n$ where $r = \|au\|_\infty$. Since $a\Lambda$ contains no nonzero lattice points in the interior of Q, then $B(au)$ is an $(a\Lambda)$-box, but it is not minimal, since au is not a pivot of $a\Lambda$. Let B be a minimal $(a\Lambda)$-box contained in $B(au)$. By Lemma 2.2, $B = B(au_0)$ for some pivot u_0 of Λ. Since $B(au_0) \subset B(au) \subset [-r, r]^3$, we have $\|au_0\|_\infty = r$. Therefore, $a \in \tau(u_0)$ and we are done. $\square$

2.3 Connection to Littlewood Conjecture

Let $\Lambda_{\alpha,\beta}$ denote the lattice

$$\Lambda_{\alpha,\beta} := \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^3.$$

The *height* of a pivot refers to the absolute value of its z-coordinate, which only depends on its pivot class.

Definition 2.9. Given $\alpha, \beta \in \mathbb{R}$, let

$$\pi(\alpha, \beta) := \{n \in \mathbb{Z}_{>0} : \Lambda_{\alpha,\beta} \text{ has a pivot of height } n\}.$$

Remark. There are exactly two pivot classes of height zero. These are $(1, 0, 0)$ and $(0, 1, 0)$. The tiles corresponding to these pivots are unbounded. If α, β are both irrational, then these are the only unbounded tiles.

Remark. For $\Lambda_{\alpha,\beta}$ any pivot has integer height. Moreover, a pivot of nonzero height is uniquely determined by its height up to equivalence.

Theorem 2.5 ([3])**.** *For any pair of real numbers α and β,*

$$\inf_{n>0} n\|n\alpha\|\|n\beta\| = \inf_{n\in\pi(\alpha,\beta)} n\|n\alpha\|\|n\beta\|$$

where $\pi(\alpha, \beta)$ is the set of positive pivot heights in $\Lambda_{\alpha,\beta}$.

Remark. Theorem 2.5 reduces the Littlewood Conjecture while considering $n \in \pi(\alpha, \beta)$.

Notation. Although we are abusing notation a bit here, we shall write $\tau(n)$ for $\tau(u)$, where $n > 0$ uniquely determines the pivot.

The Euclidean metric on $\mathbb{R}^2$ is transferred onto A via the isomorphism

$$(t,s) \to \begin{pmatrix} e^{t+s} & 0 & 0 \\ 0 & e^{t-s} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}.$$

which allows us to measure the size of tiles. The following gives a geometric interpretation of a counterexample to Littlewood's Conjecture. The proof of this result can be found in [3].

Theorem 2.6 ([3])**.** *For each $n \in \pi(\alpha, \beta)$, the following inequalities hold:*

$$\frac{1}{9}\left(-\log n\|n\alpha\|\|n\beta\| - \log 6\right) \leq \mathit{diam}\ \tau(u_n) \leq -\log n\|n\alpha\|\|n\beta\|.$$

As a consequence,

$$\inf_{n>0} n\|n\alpha\|\|n\beta\| > 0 \quad \textit{if and only if} \quad \sup_{n\in\pi(\alpha,\beta)} \mathit{diam}\ \tau(u_n) < \infty.$$

Since the sup norm is not strictly convex, it is possible for distinct tiles to overlap. When both α and β are irrational, this does not happen.

Theorem 2.7 ([3])**.** *Assume α and β are irrational. Let u and v be pivots of $\Lambda_{\alpha,\beta}$ such that $B(u) \neq B(v)$. Then $\tau(u) \cap \tau(v)$ has empty interior.*

We wish to extend the argument in [3] to include cases where $(\alpha, \beta) \in \mathbb{Q}^2$.

Chapter 3

Overlap Criterion

Due to the fact that the sup norm is not strictly convex, it is possible for tiles in $\{\tau(u)\}_{u\in\Pi(\Lambda)}$ to overlap. In [3] it was shown that this does not happen for the lattice $\Lambda = \Lambda_{\alpha,\beta}$ if both α and β are irrational. The issue of overlap is more delicate in the case $(\alpha, \beta) = (\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2$ and depends in some nontrivial way on the arithmetic properties of the pair.

Definition 3.1. We say that $\tau(u)$ and $\tau(v)$ *overlap* if $\mathrm{int}(\tau(u) \cap \tau(v)) \neq \varnothing$.

Remark. In other words, if $\tau(u)$ and $\tau(v)$ overlap, then there exists an $a \in \tau(u)\cap\tau(v)$ that has a neighborhood also contained in $\tau(u) \cap \tau(v)$. Note that

$$\mathrm{int}(\tau(u) \cap \tau(v)) = \mathrm{int}(\tau(u)) \cap \mathrm{int}(\tau(v)).$$

Definition 3.2. We say $(p_1, p_2, q) \in \mathbb{Z}^3$ is a "*primitive triple representing a pair* $(\alpha, \beta) \in \mathbb{Q}^2$ *in lowest terms*" if $\gcd(p_1, p_2, q) = 1$ and $q > 0$.

Remark. Here we consider the pair $(\alpha, \beta) = (\frac{p_1}{q}, \frac{p_2}{q})$ to be represented by $(p_1, p_2, q) \in \mathbb{Z}^3$.

Definition 3.3. We say $\theta \in \mathbb{Q}^d$ has *uniform height* if each rational coordinate has the same height. In the lowest terms of representation, $\theta = \frac{p}{q}$, where $(p, q) \in \mathbb{Z}^d \times \mathbb{Z}_{>0}$ and $\gcd(p_1, \ldots, p_d, q) = 1$. Moreover, θ has *uniform height* if and only if $\gcd(p_i, q) = 1$ for each $i = 1, \ldots, d$.

One of the main goals of this section is to establish the following theorem.

Theorem 3.1. *For any $\theta \in \mathbb{R}^d$, let Λ_θ denote the sheared integer lattice*

$$\Lambda_\theta := \begin{pmatrix} 1 & & & -\theta_1 \\ & \ddots & & \vdots \\ & & 1 & -\theta_d \\ & & & 1 \end{pmatrix} \mathbb{Z}^{d+1}. \tag{3.1}$$

Then no two tiles in $\{\tau(u) : u \in \Pi(\Lambda_\theta)\}$ overlap provided each coordinate of θ is irrational, i.e. $\theta \in (\mathbb{R} \setminus \mathbb{Q})^d$, or if $\theta \in \mathbb{Q}^d$ and has uniform height.

The starting point for the proof of this theorem is Lemma 3.2, but we first need to introduce the sectors of tiles $\tau(u)$.

3.1 Sector of Tiles

Consider $u \in \Pi(\Lambda)$, where Λ is fixed.

Definition 3.4. For each $u \in \Pi(\Lambda)$, the *open sectors* of the tile $\tau(u)$ are given by

$$\tau_i(u) = \{a \in \mathrm{int}\tau(u) : au \in V_{\{i\}}\}, \quad \varnothing \neq i \in \{1, \ldots, n\}.$$

where $V_I = \{x \in \mathbb{R}^n : \|x\|_\infty = |x_i| \text{ if and only if } i \in I\}$.

We next make an important observation which bridges the connection between an element $a \in A$ belonging to a sector of a tile of $\tau(u)$ with a pivot of a lattice u lying on an open face of a Λ-box B.

$$a \in \tau_i(u) \text{ if and only if } u \in \partial_i B_a.$$

In the three-dimensional setting, the above observation says that an element $a \in A$ belonging to the x-sector of $\tau(u)$ corresponds to a pivot u on the x-face of Λ-box. Similar for y, z cases.

Figure 3.1: An element $a \in A$ lying in $\tau_x(u)$.

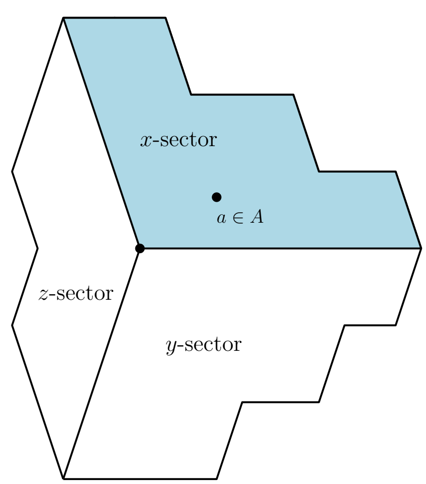
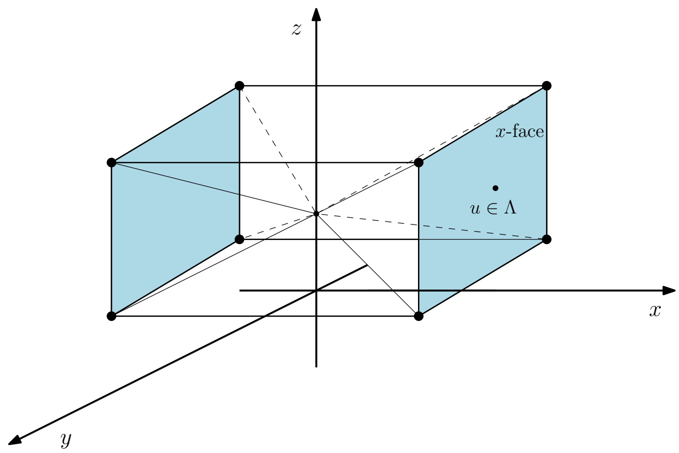


Figure 3.2: A pivot $u \in \Lambda$ on the boundary $\partial_x B_a$ of the open x-face of some Λ-box B.



Lemma 3.2. *Let u and v be inequivalent pivots of Λ. If $\tau_i(u) \cap \tau_j(v) \neq \varnothing$, then $i = j$; moreover, u and v belong to the multi-face $\partial_i B$ of some Λ-box B.*

Proof. Choose any $a \in \tau_i(u) \cap \tau_j(v)$ and let $B = a^{-1}Q$ where Q is the largest cube in $a\Lambda$ that contains no nonzero lattice points. Since $a \in \tau_i(u)$ we have $au \in V_{\{i\}}$ so that $u \in \partial_i B$. Similarly, $a \in \tau_j(v)$ implies $v \in \partial_j B$. Note that B is a Λ-box so that, to finish the proof, it only remains to show $i = j$.

Suppose not. Let $a_\delta = a \cdot \exp X_\delta(i,j)$ where $X_\delta(i,j)$ is the matrix whose (i,i)-entry and (j,j)-entry are $-\delta$ and δ, respectively, and such that all other entries vanish. Then for any $\delta > 0$ we have $\|a_\delta u\|_\infty < \|a_\delta v\|_\infty$ so that $a_\delta \notin \tau(v)$. Since $a_\delta \to a$ as $\delta \to 0$, this contradicts the fact that $a \in \mathrm{int}\tau(v)$. $\square$

Remark. This theorem implies that overlap between two tiles $\tau(u)$ and $\tau(v)$ must occur in sectors of the same "type". We do not know whether a tile can overlap with other tiles in multiple sectors.

If we study the analysis on how the tiles may overlap, we will find that there are endless possibilities in which they can overlap, some of which can lead to very complex situations. Fortunately, by Lemma 3.2, an important consequence is that tiles can only overlap in the same sectors.

Remark. Referring to Figure 1.2 in the Introduction section, the example $(3, 2, 9)$ has two non-degenerate tiles and they overlap in their common x-sector. A portion of the x-sector of the green one lies below the x-sector of the black one.

Consider the i^{th} coordinate projection map $\pi_i : (\mathbb{Z}/q\mathbb{Z})^n \longrightarrow \mathbb{Z}/q\mathbb{Z}$. It is not hard to see that any Λ_θ-box contains at most one pivot class in $\partial_{d+1}B$ due to the fact that a coset of $\ker \pi_{d+1}$ can only meet Λ in a shifted rectangular lattice. In the following section, we shall abstract the relevant properties of these sets, which will enable us to show that the tiles of any lattice with "rectangular cross sections" do not overlap.

Chapter 4

Subsets Closed Under $\neg$

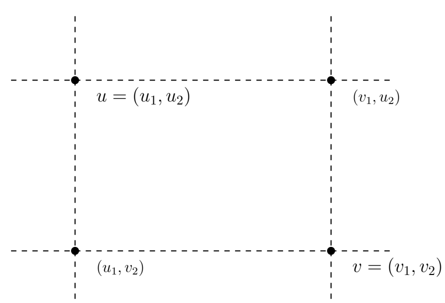
Definition 4.1. Given $u, v \in \mathbb{R}^n$, and $i \in \{1, \ldots, n\}$ we denote by

$$u \neg_i v$$

the vector obtained from u by replacing its i^{th} component with that of v.

Definition 4.2. Let Σ be a subset of $\mathbb{R}^n$. We say that Σ is *closed under* $\neg$, or simply $\neg$-closed, if $u, v \in \Sigma$ implies that $u \neg_i v \in \Sigma$ for each $i = 1, \ldots, n$.

Figure 4.1: A property of Σ in $\mathbb{R}^2$.



Example 4.1. In $\mathbb{R}^2$, this definition says that given any two points in Σ, the intersection of the horizontal line through one point, say (u_1, u_2) and the vertical line through the other point (v_1, v_2) is again in Σ (and vice versa).

Definition 4.3. Let Γ be a discrete subgroup of $\mathbb{R}^n$. We say Γ is *rectangular* if it is generated by its elements on the coordinate axes.

Lemma 4.1. *Cosets of rectangular discrete subgroups if* $\mathbb{R}^n$ *are* $\neg$*-closed.*

Proof. Note that $\neg$-closed sets are preserved under translations as well as the action of A. Also, a product of a $\neg$-closed set in $\mathbb{R}^k$ and a $\neg$-closed set in $\mathbb{R}^l$ is a $\neg$-closed set in $\mathbb{R}^{k+l}$. Finally, $\mathbb{Z}^k \times \{0\}$ as a subset of $\mathbb{R}^l$ where $k < l$ is $\neg$-closed. $\square$

Remark. The intersection of an arbitrary number of rectangular discrete subgroups is again rectangular.

Definition 4.4. Let Σ be a subset of $\mathbb{R}^n$. By a *minimal vector* of Σ we mean $v \in \Sigma$ such that $u \in \Sigma \cap B(v)$ implies $u \sim v$, i.e. $B(u) = B(v)$.

For the following result, we prove that if Σ is a $\neg$-closed set, then there exists a unique minimal vector up to equivalence.

Lemma 4.2. *Any* $\neg$*-closed set* Σ *has at most one minimal vector up to* $\sim$.

Proof. For any $u, v \in \Sigma$, there exists $w \in \Sigma$ such that $B(w) = B(u) \cap B(v)$. If u and v are minimal vectors, then $B(u) = B(v) = B(w)$ so that $u \sim v$. $\square$

We will next attempt to make the connection between subsets closed under $\neg$ and Λ-boxes in order to prove a result about the case when the pair of rational numbers $(\alpha, \beta) = (\frac{p_1}{q}, \frac{p_2}{q})$ is totally reduced.

4.1 Lattices with Rectangular Cross Sections

Recall that a pivot of Λ is said to be *trivial* if it lies on a coordinate axis and *degenerate* if it belongs to some coordinate hyperplane $\ker \pi_i$.

Definition 4.5. Let Λ be a lattice in $\mathbb{R}^n$. The i^{th} *cross section* of Λ refers to the discrete subgroup $\Gamma_i := \Lambda \cap \ker \pi_i$, where π_i is the i^{th} coordinate projection. We say Γ_i is *rectangular* if each vector in Γ_i can be written as an integer combination of the trivial pivots of Λ. We say Λ has *rectangular cross sections* if Γ_i is rectangular for each $i = 1, \ldots, n$.

Theorem 4.3. *Suppose Λ is a lattice in $\mathbb{R}^n$ with rectangular cross sections. Then no two tiles in $\{\tau(u)\}_{u \in \Pi(\Lambda)}$ overlap.*

Proof. If $\operatorname{int}(\tau(u) \cap \tau(v)) \neq \varnothing$, then there exists i, j such that $\tau_i(u) \cap \tau_j(v) \neq \varnothing$, hence by Lemma 3.2, it suffices to show that every Λ-box B contains at most one pivot class in each of its multi-faces. Let $\Sigma_i = \Lambda \cap H_i$, where H_i is the coset of $\ker \pi_i$ containing one of the components of $\partial_i B$. Each pivot in $\partial_i B$ determines a minimal vector in Σ_i, and distinct pivot classes determine inequivalent minimal vectors. By

Lemma 4.1, Σ_i is closed under $\neg$ because it is a coset of Γ_i, which is rectangular. Lemma 4.2 now implies that $\partial_i B$ can contain at most one pivot class. $\square$

Remark. Lemma 4.2 was key to prove the above theorem, as it uses the fact that $\neg$-closed sets have at most one minimal vector up to equivalence, which implies the multi-faces of a Λ-box can never contain two pivots. From this we can deduce that two tiles $\tau(u)$ and $\tau(v)$ do not overlap in the same i^{th} sector if Λ has rectangular cross sections.

Theorem 3.1 is a corollary of Theorem 4.3, by the next lemma.

Lemma 4.4. *The lattice $\Lambda_\theta \subset \mathbb{R}^{d+1}$ has rectangular cross sections if and only if each coordinate of $\theta \in \mathbb{R}^d$ is irrational or $\theta \in \mathbb{Q}^d$ has uniform height.*

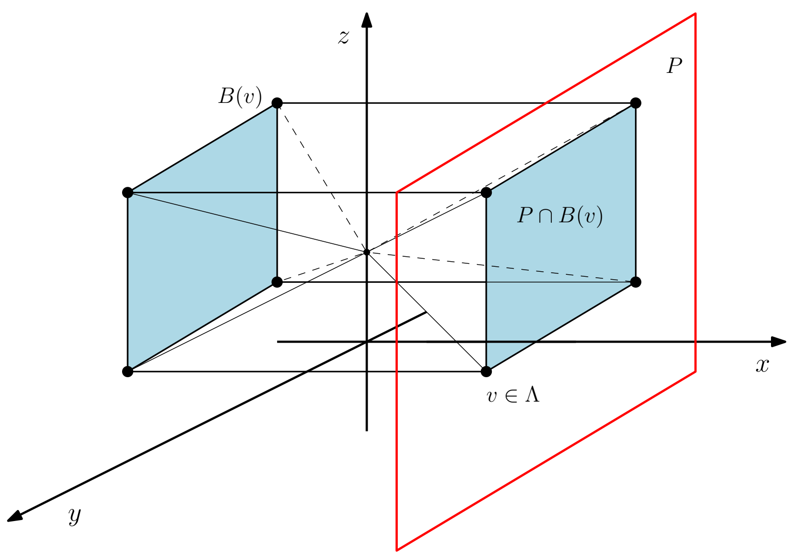
Proof. Note that $\Gamma_{d+1} = \mathbb{Z}^d \times \{0\}$ is rectangular, regardless of θ. Note also that $\Gamma_i \supset Z_i := (\mathbb{Z}^d \times \{0\}) \cap \ker \pi_i$ and equality holds if and only if θ_i is irrational, in which case Γ_i is rectangular. Thus, Λ_θ has rectangular cross sections if each θ_i is irrational.

If $\theta \in \mathbb{Q}^d$ has uniform height q, then $v = (0, \ldots, 0, q)$ is a trivial pivot of Λ_θ and $\Gamma_i = Z_i + Zv$ is rectangular for each $i = 1, \ldots, d$. Hence Λ_θ has rectangular cross sections if $\theta \in \mathbb{Q}^d$ has uniform height.

It remains to consider $\theta \in \mathbb{R}^d$ that have both a rational coordinate θ_i and an irrational coordinate θ_j. The existence of the latter implies that any trivial pivot of Λ_θ has vanishing last component. Hence, no vector in $\Gamma_i \setminus Z_i$ can be expressed as a sum of trivial pivots, so that the cross section Γ_i is not rectangular. $\square$

Example 4.2. Given a three-dimensional pivot $v \in \Lambda$ (of $\Lambda_{\frac{p_1}{q},\frac{p_2}{q}}$), recall that $B(v)$ is a minimal Λ-box. Let P be a plane containing the x-face of the Λ-box $B(v)$. As $v \in \Lambda$ was a minimal vector, the intersection of $B(v)$ with P gives rise to a rectangular subset. Hence Λ meets the plane P in a rectangular cross section. We see that $u \in P \cap B(v)$ implies $B(u) = B(v)$. By Theorem 4.3, the tiles of $\tau(u)$ and $\tau(v)$ do not overlap.

Figure 4.2: The pivot $v \in \Lambda$ is a minimal vector in $\Lambda \cap P$.



Example 4.3. Observe the case of our interest, the three-dimensional lattice

$$\Lambda_{\frac{p_1}{q},\frac{p_2}{q}} = \begin{pmatrix} 1 & 0 & -\frac{p_1}{q} \\ 0 & 1 & -\frac{p_2}{q} \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^3.$$

We will use the results thus far to develop the overlap criterion for such a lattice, which will be discussed in the following section.

Chapter 5

Results and Discussion

Recall that the pair of rational numbers $(\alpha, \beta) = (\frac{p_1}{q}, \frac{p_2}{q})$ is *totally reduced* if

$$\gcd(p_1, q) = 1 = \gcd(p_2, q).$$

We previously defined the pair $(\alpha, \beta) = (\frac{p_1}{q}, \frac{p_2}{q})$ having uniform height if the above condition is satisfied in the multi-dimensional case.

Lemma 5.1. *If* $\gcd(p_1, q) = 1$, *then the lattice* $\Lambda_{\frac{p_1}{q}, \frac{p_2}{q}}$ *meets the* $x = 0$ *plane in a rectangular cross section. Similarly, if* $\gcd(p_2, q) = 1$, *then the lattice meets the* $y = 0$ *plane in a rectangular cross section.*

Proof. A vector in $\Lambda_{\frac{p_1}{q}, \frac{p_2}{q}}$ is of the form $(a - \frac{p_1}{q} \cdot c, b - \frac{p_2}{q} \cdot c, c)$ for some $(a, b, c) \in \mathbb{Z}^3$. If $a = \frac{p_1}{q} \cdot c$, then $\gcd(p_1, q) = 1$ implies $c = kq$, where $k = \frac{a}{p_1}$. The vector we obtain is then $(0, b - p_2 k, kq)$, which is the $\mathbb{Z}$-span of $(0, 1, 0)$ and $(0, 0, q)$, i.e. the integer combination of the trivial pivots of $\Lambda_{\frac{p_1}{q}, \frac{p_2}{q}} \cap \{x = 0\}$. Conversely, any vector in

$\mathbb{Z}(0,1,0)+\mathbb{Z}(0,0,q)$ is realized: given $m(0,1,0)+n(0,0,q)$, let $c = nq$, $b = p_2 n + m$, and $a = p_1 n$. Similarly for $\gcd(p_2, q) = 1$. $\square$

Remark. Lemma 5.1 asserts that the affine subspace containing the x-face of the cube $Q = aB$ intersects $\Lambda_{\frac{p_1}{2},\frac{p_2}{q}}$ in a rectangular lattice. There is a unique pivot of that face by Lemma 4.2. This means there is no overlap in the x-face of the cube $Q = aB$. This yields similar results for $\Lambda_{\frac{p_1}{q},\frac{p_2}{q}} \cap \{y = 0\}$.

5.1 Conclusion

To conclude our work, we were able to achieve the goal of developing an overlap criterion for the tiling problem of the Littlewood Conjecture. In the process, we were able to reformulate the original definition of tiling in [3] to extend into higher dimensions. The final product of this work gives us the following theorem.

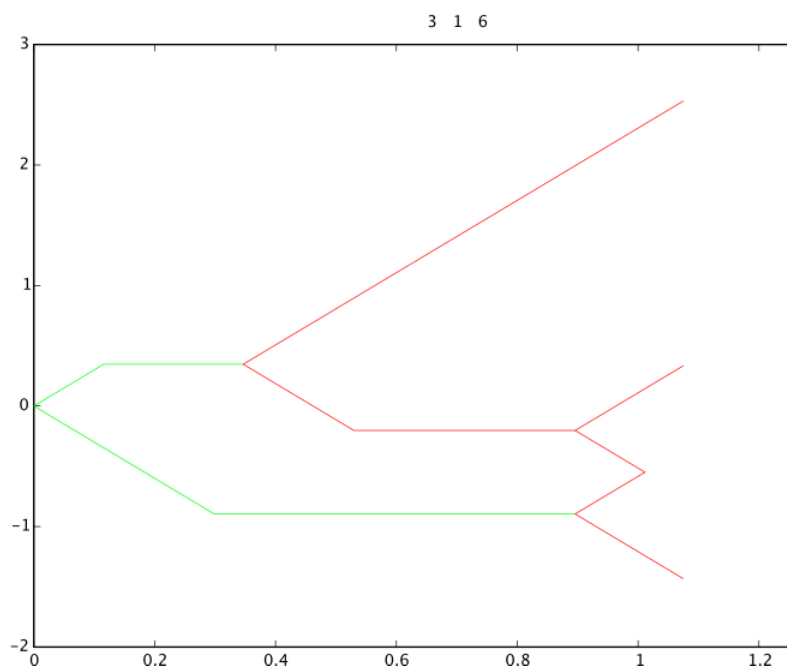
Theorem 5.2 (Overlap Criterion)**.** *For any $\theta \in \mathbb{R}^d$, let Λ_θ denote the sheared integer lattice*

$$\Lambda_\theta := \begin{pmatrix} 1 & & & -\theta_1 \\ & \ddots & & \vdots \\ & & 1 & -\theta_d \\ & & & 1 \end{pmatrix} \mathbb{Z}^{d+1}. \tag{5.1}$$

Then no two tiles in $\{\tau(u) : u \in \Pi(\Lambda_\theta)\}$ overlap provided each coordinate of θ is irrational, i.e. $\theta \in (\mathbb{R} \setminus \mathbb{Q})^d$, or if $\theta \in \mathbb{Q}^d$ and has uniform height, i.e, $\gcd(p_i, q) = 1$, *and where* $\theta_i = \frac{p_i}{q}$.

Let us transition to the case of our interest, when $\theta \in \mathbb{R}^2$. Strictly speaking, the first part of the theorem was proven in [3], so that there is no overlap of tiles when $\theta = (\alpha, \beta)$ are irrational. The second part of the theorem states that if the rational pair $(\alpha, \beta) = (\frac{p_1}{q}, \frac{p_2}{q})$ has uniform height (is totally reduced), then there is no overlap of tiles. The converse is not necessarily true, that is, the nonoverlapping of a rational pair of tiles does not mean that it is totally reduced. As an example of pair that is *not totally reduced*, but its tiles do not overlap, consider $(p_1, p_2, q) = (3, 1, 6)$.

Figure 5.1: Tiling for $p_1 = 3, p_2 = 1, q = 6$.



As in the Figure 1.2 from the Introduction section, we obtain two nondegenerate tiles, which are the red colored ones. However, overlap does not occur here because the green tile is a "full" tile, in which the red tiles do not overlap in either the x- or y-sector.

For the three-dimensional lattice $\Lambda_{\alpha,\beta}$, the important observation in Section 3.1 says that an element $a \in A \subset \mathrm{SL}(3, \mathbb{R})$ in an open i^{th} sector of a tile corresponds to a pivot on an open i^{th} face of a Λ-box. This leaves us with three things to consider. We look at the plane plane intersecting the three-dimensional Λ-box in the x-face, y-face, and the z-face.

We were able to immediately rule out the fact that $\Lambda_{\alpha,\beta} \cap \{z = 0\}$ is a rectangular cross section due to the nature of the sheared integer lattice

$$\Lambda_{\alpha,\beta} := \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^3.$$

More precisely, a vector in $\Lambda_{\alpha,\beta}$ takes the form $(m_1 - \alpha n, m_2 - \beta n, n)$ for some $(m_1, m_2, n) \in \mathbb{Z}^3$. If $n = 0$, then the vector obtained is the $\mathbb{Z}$-span of $(1, 0, 0)$ and $(0, 1, 0)$, which means that the three-dimensional lattice $\Lambda_{\alpha,\beta}$ meets the plane $z = 0$ in a rectangular cross section. As before, rectangular cross-sections imply there is a unique pivot. Hence, the z-face of a Λ-box can never contain two pivots. A geometric consequence is that for pivots u and v, their tiles $\tau(u)$ and $\tau(v)$ cannot overlap in the z-sector.

The treatment of the intersections of the $x = 0$ plane and $y = 0$ plane with the lattice $\Lambda_{\alpha,\beta}$ was more delicate in the sense that it required the extra totally reduced condition. This case has already been taken care of by Lemma 5.1. Therefore, the totally reduced condition ensures that there is no overlapping of tiles.

Appendix A: Piecewise Linear Functions

We provide a precise breakdown of the formulation of the original definitions of a lattice, tiling, and sectors of a tile in the three-dimensional context. Much of this work can be found in [3].

These definitions motivated the exploration of the overlap condition for the rational pair of real numbers (α, β), but once we were able to extend the definitions to higher dimensions, we found that these definitions were no longer needed. In particular, we include the computations of the different sectors of a tile. We do not use the following computations in the proofs of our results, and so we include it here as a reference.

Given $\alpha, \beta \in \mathbb{R}$, let $h_{\alpha,\beta}$ be the *shear transformation* defined by

$$h_{\alpha,\beta} := \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}.$$

For $(t, s) \in \mathbb{R}^2$, let the *scaling matrix* be defined by

$$g_{t,s} := \begin{pmatrix} e^{t+s} & 0 & 0 \\ 0 & e^{t-s} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}.$$

In addition, let

$$\Lambda_{\alpha,\beta} := h_{\alpha,\beta}\mathbb{Z}^3.$$

Then a *lattice* is defined by

$$\Lambda := g_{t,s}\Lambda_{\alpha,\beta}. \tag{5.2}$$

Fix $u \in \mathbb{Z}^3 \backslash \{0\}$, where $u = (m_1, m_2, n)$ and $n > 0$. Then

$$f_u(t, s) := \log \|g_{t,s}h_{\alpha,\beta}u\|_\infty, \tag{5.3}$$

where $\|\cdot\|_\infty$ denotes the usual sup norm. Unraveling the definition of the function from (5.3)

$$\begin{aligned} f_u(t, s) &= \log \|g_{t,s}h_{\alpha,\beta}u\| \\ &= \log \|(e^{t+s}(m_1 - n\alpha), e^{t-s}(m_2 - n\beta), e^{-2t}n)\| \\ &= \log \left[\max(e^{t+s}|m_1 - n\alpha|, e^{t-s}|m_2 - n\beta|, e^{-2t}|n|)\right] \end{aligned}$$

we have

$$f_u(t,s) = \begin{cases} t + s + \log|m_1 - n\alpha| \\ t - s + \log|m_2 - n\beta| \\ -2t + \log n. \end{cases}$$

Or equivalently,

$$f_u(t,s) = \begin{cases} t + s + \log\|n\alpha\| \\ t - s + \log\|n\beta\| \\ -2t + \log n, \end{cases}$$

where we assume m_1 and m_2 are nearest integers to $n\alpha$ and $n\beta$, respectively. That is

$$|m_1 - n\alpha| = \|n\alpha\|, \quad \text{and} \quad |m_2 - n\beta| = \|n\beta\|.$$

In (t,s) coordinates, we define the following planes

$$P_x := t + s + \log\|n\alpha\| \tag{5.4}$$

$$P_y := t - s + \log\|n\beta\| \tag{5.5}$$

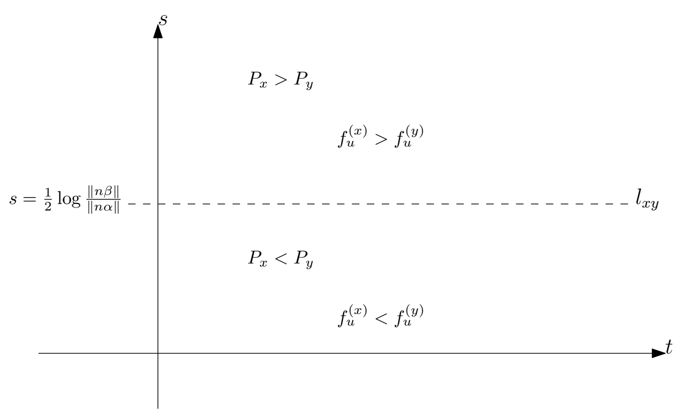
$$P_z := -2t + \log n \tag{5.6}$$

To see where these planes intersect in (t,s) coordinates, we exhaust all possible cases of lines by setting the planes equal to each other.

$P_x = P_y$: This is given by the horizontal line l_{xy}.

$$t + s + \log \|n\alpha\| = t - s + \log \|n\beta\| \tag{5.7}$$
$$s = \frac{1}{2} \log \frac{\|n\beta\|}{\|n\alpha\|} \tag{5.8}$$

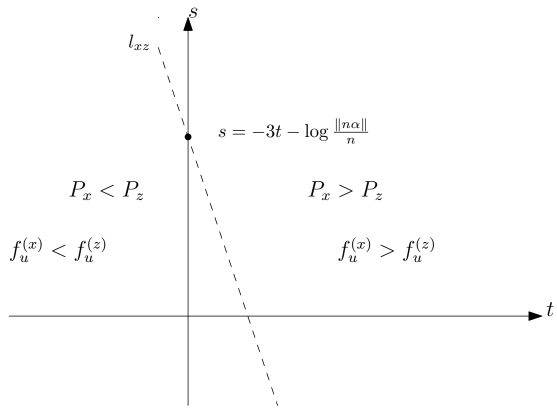
Figure 5.2: The line l_{xy} with regions where the planes P_x and P_y lie in relation to one another.



$P_x = P_z$: This is given by the line l_{xz} with slope -3.

$$t + s + \log \|n\alpha\| = -2t + \log n \tag{5.9}$$
$$s = -3t - \log \frac{\|n\alpha\|}{n} \tag{5.10}$$

Figure 5.3: The line l_{xz} with regions where the planes P_x and P_z lie in relation to one another.

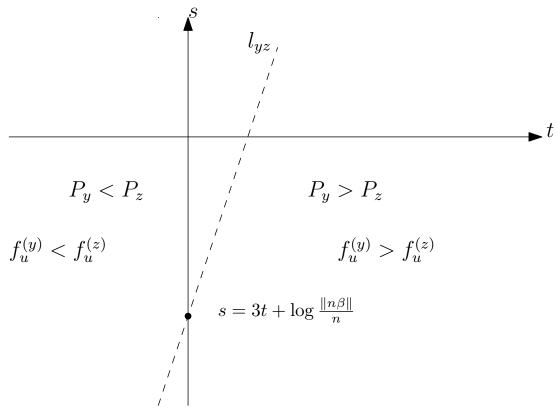


$P_y = P_z$: This is given by the line l_{yz} with slope +3.

$$t - s + \log \|n\beta\| = -2t + \log n \tag{5.11}$$

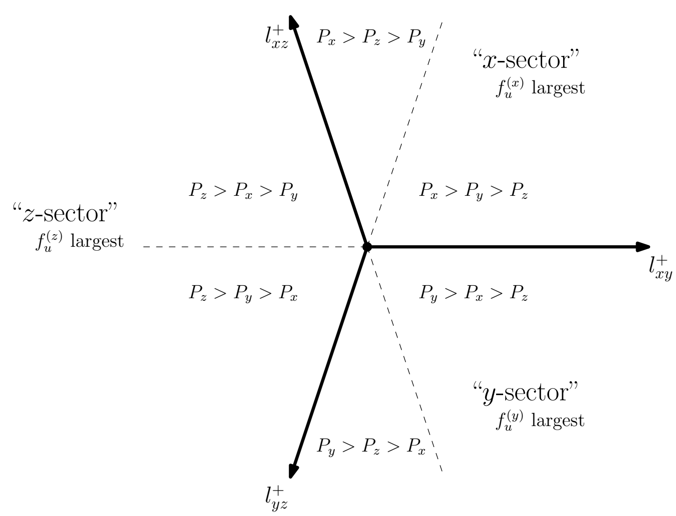
$$s = 3t + \log \frac{\|n\beta\|}{n} \tag{5.12}$$

Figure 5.4: The line l_{yz} with regions where the planes P_y and P_z lie in relation to one another.



Now putting these lines together, we obtain regions where the planes P_x, P_y, and P_z dominate all others. These regions also indicate where the function $f_u(t, s)$ is the largest in *sectors*, as seen by the following figure.

Figure 5.5: An illustration of the x, y, and z sectors.



So far, we have been working with the two-dimensional (t, s) plane. We now consider an additional axis, the w-axis and do analysis on the three-dimensional space. Define

$$\Gamma_u := \{(t, s, w) : w = f_u(t, s)\}. \tag{5.13}$$

From [3], it was proven that Γ_u is contained in the union of the three planes. To see this, we compute the intersection of Γ_u with the ts-plane, that is, when $w = 0$.

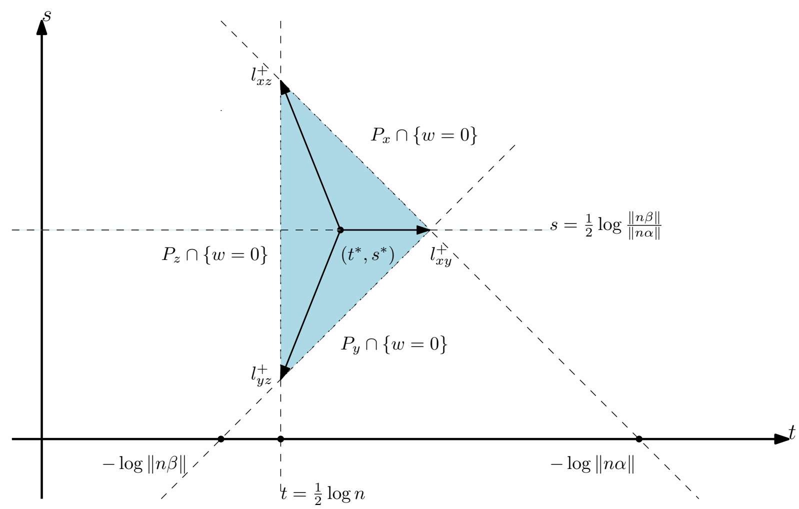
$$\begin{aligned}
P_x \cap \{w = 0\} : &\quad t + s = -\log \|n\alpha\| \\
P_y \cap \{w = 0\} : &\quad t - s = -\log \|n\beta\| \\
P_z \cap \{w = 0\} : &\quad t = \frac{1}{2} \log n
\end{aligned}$$

In addition, define

$$\Delta(u) := \{f_u(t, s) \leq 0\}. \tag{5.14}$$

The function $\Delta(u)$ gives us a triangular-shaped cone that lies below the plane $w = 0$. The set of lines $P_x \cap \{w = 0\}$ and $P_y \cap \{w = 0\}$ have slopes -1 and +1 respectively, so we can observe that $\Delta(u)$ is a right triangle bounded by the union of the three planes.

Figure 5.6: The set $\Delta(u)$ in the ts-plane.



Observe that Γ_u has a unique global minimum which occurs at the centroid (t^*, s^*). It is clear that $s^* = \frac{1}{2}\log\frac{\|n\beta\|}{\|n\alpha\|}$. The t^* coordinate is given by the intersection of the lines l_{xz} and l_{yz}.

$$\begin{aligned} -3t - \log\frac{\|n\alpha\|}{n} &= 3t + \log\frac{\|n\beta\|}{n} \\ t &= -\frac{1}{6}\log\frac{\|n\alpha\|\|n\beta\|}{n^2} \end{aligned}$$

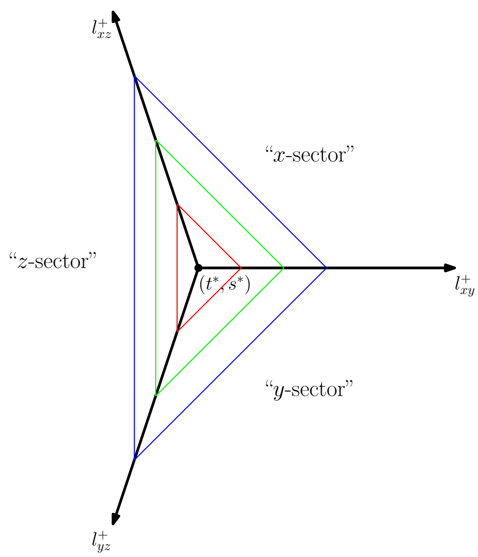
Thus, the centroid is given as

$$(t^*, s^*) = \left(-\frac{1}{6}\log\frac{\|n\alpha\|\|n\beta\|}{n^2}, \frac{1}{2}\log\frac{\|n\beta\|}{\|n\alpha\|}\right). \tag{5.15}$$

From [3], the length of the hypotenuse of the right triangle $\Delta(u)$ is the diameter of $\Delta(u)$ defined by

$$\text{diam } \Delta(u) := -\log n\|n\alpha\|\|n\beta\| \tag{5.16}$$

Figure 5.7: The contours/level sets of Γ_u are right angled triangles.



Now that we have established the piecewise linear functions f_u for all $u \in \mathbb{Z}^3\backslash\{0\}$, we proceed with the definition of tiling. Define the graph of the triangle as

$$W_{\alpha,\beta}(t,s) := \log \inf\{\|g_{t,s}h_{\alpha,\beta}u\| : u \in \mathbb{Z}^3\backslash\{0\}\},$$

which is equivalent to

$$W_{\alpha,\beta}(t,s) := \inf_{u\in\mathbb{Z}^3\backslash\{0\}} f_u(t,s). \tag{5.17}$$

Definition 5.1. Given a pair (α, β) associated with $u \in \mathbb{Z}^3\backslash\{0\}$, a **tile** is a set

$$\tau(u) := \{(t, s) : W_{\alpha,\beta}(t, s) = f_u(t, s)\}. \tag{5.18}$$

Fix $u \in \mathbb{Z}^3\backslash\{0\}$, where $u = (m_1, m_2, n)$ and $n > 0$. We will refer to u as a *pivot* of the lattice $\Lambda_{\alpha,\beta}$.

Geometrically, a tile can be thought of as "blocks" of polygons that partition the $\mathbb{R}^2$ plane. Given the continued fractions of the pair of real numbers (α, β), the tiles can be indexed by the collection of pivots of $\Lambda_{\alpha,\beta}$.

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