Convexity of Domains of Best Approximation

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Mathematics

by

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CERTIFICATION OF APPROVAL

I certify that I have read *Convexity of Domains of Best Approximation* by Bita Nosratieh and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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Convexity of Domains of Best Approximation

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The notion of best approximation defines a relation between rational points in $\mathbb{Q}^n$ and points in $\mathbb{R}^n$. For $n = 1$, the problem is well-understood but in higher dimensions much less is known. We explore the case where n=2. Given a rational point v, then the set of points $x \in \mathbb{R}^2$ that have v as a best approximant defines the domain of best approximation for v, denoted by $\Delta(v)$. It is known that $\Delta(v)$ is represented as an intersection of sets $\Delta_u(v)$, where $\Delta_u(v)$ is constructed by comparing the distance between v and x with the distance between any rational u and x. For the Euclidean norm each $\Delta_u(v)$ is an Apollonian circle; in particular convex. Thus $\Delta(v)$ is convex. This raises the question of convexity for all norms. This research gives a negative answer for the maximum norm. This is shown by first identifying the criterion for the convexity of $\Delta_u(v)$. Then we determine the sufficient condition that allows for simplifying $\Delta(v)$ to be defined by only two sets of $\Delta_u(v)$. Finally we show that if at least one $\Delta_u(v)$ is not convex then $\Delta(v)$ will not be convex.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee   Date

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0.1 Introduction

The study of best approximation typically involves choosing a real number and finding the set of all rational numbers that are closest to that real number. In this study we take a different point of view. To determine the domains of best approximation, we fix a rational point and construct the set of irrationals that this point is a best approximant for.

Each rational point in $\mathbb{Q}^2$ has a unique representation as $(\frac{p_1}{q}, \frac{p_2}{q})$ where

$$v = (p_1, p_2, q) \in \mathbb{Z}^2 \times \mathbb{Z}_{>0} \text{ and } \gcd(v) = 1 \tag{1}$$

Notation. For any v satisfying (1), we let

$$\dot{v} = (\frac{p_1}{q}, \frac{p_2}{q}) \in \mathbb{Q}^2 \text{ and } |v| := q$$

$\dot{v}$ is said to be a *best approximant* to $(x, y) \in \mathbb{R}^2$ relative to some given norm $\|\cdot\|$ if

$$\|q(x,y) - (p_1,p_2)\| = \min\{\|n(x,y) - (m_1,m_2)\| : (\frac{m_1}{n}, \frac{m_2}{n}) \in \mathbb{Q}^2, 1 \le n \le q\}.$$

In the case where $d = 1, \dot{v} = \frac{p}{q} \in \mathbb{Q}$ is a convergent of the continued fraction of $x \in \mathbb{R}$ if and only if it is a best approximant of x.

Let $\Delta(v)$ denote the domain of best approximation for v, then we define

$$\Delta(v) := \{\mathbf{x} \in \mathbb{R}^2 : \dot{v} \text{ is a a best approximant to } \mathbf{x}\}.$$

In [1] it was observed that $\Delta(v)$ is represented as

$$\Delta(v) = \left(\bigcap_{|u|<|v|} \Delta_u(v) \right) \bigcap \left(\bigcap_{|u|=|v|} \overline{\Delta_u(v)} \right)$$

for any u satisfying (1),

$$\Delta_u(v) = \{\mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, \dot{v}) < \frac{|u|}{|v|} d(\mathbf{x}, \dot{u})\} \qquad (2)$$

where $d(\ , \ )$ is the metric induced by the norm $\| \cdot \|$.

The study of best approximants has been around for many years. However the domains of best approximation have not yet been explored. Since the domains of best approximation are determined by computing the distance between a fixed rational point and the irrationals, the choice of norm becomes an important factor.

Let the Euclidean norm be denoted by

$$\|\cdot\|_2$$

where for any $\mathbf{x} = (x, y), \mathbf{x'} = (x', y') \in \mathbb{R}^2$

$$\|(\mathbf{x}, \mathbf{x'})\|_2 = \sqrt{(x - x')^2 + (y - y')^2}.$$

In the case of the Euclidean norm each $\Delta_u(v)$ is a circle and as a result $\Delta(v)$ which is the intersection of finitely many circles is convex. This raises the question of convexity for general norms; in particular the maximum norm. Let

$$\|\cdot\|_\infty$$

denote the maximum norm. So for all $\mathbf{x} = (x, y), \mathbf{x'} = (x', y') \in \mathbb{R}^2$

$$\|(\mathbf{x}, \mathbf{x'})\|_\infty = \max\{|x - x'|, |y - y'|\}.$$

This research reveals that $\Delta(v)$ is not always a convex set with respect to the maximum norm. We show this by first identifying the criterion for the convexity of $\Delta_u(v)$ with respect to the maximum norm. Then we determine a sufficient condition that allows for $\Delta(v)$ to be represented by the intersection of only two sets of $\Delta_u(v)$. Finally we show that if at least one $\Delta_u(v)$ is not convex then $\Delta(v)$ will not be convex. In the end we present an explicit example of a non convex $\Delta(v)$.

Chapter 1

Two-dimensional Sublattices

Lemma 1.1. *Let* $\Lambda \subset \mathbb{R}^2$ *be a two-dimensional lattice. Suppose* $u, v \in \Lambda$ *where* u, v *are linearly independent. Then*

$$\|u\|_2 \cdot \|v\|_2 \geq \text{area}(\Lambda).$$

Proof. Let θ denote the angle between u and v. Since u, v are linearly independent, then $\theta \neq 0$. The area of a parallelogram formed by u, v with diagonal $u + v$ is given by $\|u\|_2 \cdot \|v\|_2 \sin\theta$. The area of Λ equals the area of the parallelogram formed by two basis vectors in Λ. It follows that $\text{area}(\Lambda) \leq \|u\|_2 \cdot \|v\|_2 \sin\theta$. Since $0 < \sin\theta \leq 1$ then we have $\text{area}(\Lambda) \leq \|u\|_2 \cdot \|v\|_2$ as desired.

□

Lemma 1.2. *Let $\Lambda \subset \mathbb{R}^2$ be a two dimensional lattice. Let $v \in \Lambda$ be the a nonzero primitive vector such that*

$$\|v\|_\infty \leq \frac{\sqrt{\text{area}(\Lambda)}}{\sqrt{2}},$$

then $\|v\|_\infty$ is minimal with respect to the maximum norm among all nonzero vectors in Λ.

Proof. For any $\mathbf{x} \in \mathbb{R}^{\mathbf{2}}$ we have

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{2}\|\mathbf{x}\|_\infty \qquad (3)$$

Let $u \in \Lambda \backslash \{\mathbb{R}v\}$, then from Lemma 1.1 and equation (3) we get

$$\|u\|_2 \geq \frac{\text{area}(\Lambda)}{\|v\|_2} \geq \frac{\text{area}(\Lambda)}{\sqrt{2}\|v\|_\infty}.$$

Since $\sqrt{2}\|v\|_\infty \leq \sqrt{\text{area}(\Lambda)}$, then $\|u\|_2 \geq \sqrt{\text{area}(\Lambda)}$. By (3), $\sqrt{2}\|u\|_\infty \geq \sqrt{\text{area}(\Lambda)}$. Thus $\|u\|_\infty \geq \frac{\sqrt{\text{area}(\Lambda)}}{\sqrt{2}} \geq \|v\|_\infty$ as desired.

□

Definition 1.1. Let L denote a two-dimensional sublattice of $\mathbb{Z}^3$ such that $L = \mathbb{Z}u + \mathbb{Z}v$. Then L is said to be *primitive* if all integer vectors in the real span are integer linear combination of u and v, i.e. $\mathbb{Z}^3 \cap (\mathbb{R}u + \mathbb{R}v) = \mathbb{Z}u + \mathbb{Z}v$.

Given any v satisfying (1), i.e., $v \in \mathbb{Z}^2 \times \mathbb{Z}_{>0}$ and $\gcd(v) = 1$, we define

$$\mathcal{L}_{\mathbb{R}}(v)$$

a subset of $\bigwedge^2 \mathbb{R}^3$, as the kernel of the operation $\wedge v : \bigwedge^2 \mathbb{R}^3 \to \bigwedge^3 \mathbb{R}^3$ or equivalently as the image of $\wedge v : \bigwedge^1 \mathbb{R}^3 \to \bigwedge^2 \mathbb{R}^3$. Similarly, we have $\bigwedge^1 \mathbb{Z}^3 \to \bigwedge^2 \mathbb{Z}^3 \to \bigwedge^3 \mathbb{Z}^3$ where the image of the first map

$$\mathcal{L}_{\mathbb{Z}}(v)$$

defines a two-dimensional lattice in $\mathcal{L}_{\mathbb{R}}(v)$. An oriented two-dimensional lattice $L \subset \mathbb{Z}^3$ containing v determines an element of $\mathcal{L}_{\mathbb{Z}}(v)$ as follows. We choose a primitive $u = (a, b, c) \in L$ such that $L = \mathbb{Z}u + \mathbb{Z}v$ and (u, v) is consistent with the orientation of L. Then the corresponding element of $\mathcal{L}_{\mathbb{Z}}(v)$ is $u \wedge v$, an oriented, primitive, two-dimensional sublattice of $\mathbb{Z}^3$ and is independent of the choice of u.

Intuitively we can think of elements of $\mathcal{L}_{\mathbb{R}}(v)$ as cosets of $\mathbb{R}v$ in $\mathbb{R}^3$, i.e. lines in $\mathbb{R}^3$ that are parallel to $\mathbb{R}v$. Similarly we think of elements of $\mathcal{L}_{\mathbb{Z}}(v)$ as those lines parallel to $\mathbb{R}v$ that pass through points in $\mathbb{Z}^3$. It is convenient to represent $\mathcal{L}_{\mathbb{R}}(v)$ and $\mathcal{L}_{\mathbb{Z}}(v)$ as subsets of $\mathbb{R}^2$ in the following way. Each element in $\mathcal{L}_{\mathbb{R}}(v)$ can be associated to a point in $\mathbb{R}^2$ by taking the lines parallel to $\mathbb{R}v$ and intersecting with $\mathbb{R}^2 \times \{0\}$. In this way each unique intersection point in $\mathbb{R}^2$ represents an element in $\mathcal{L}_{\mathbb{R}}(v)$. This defines a bijection between $\mathcal{L}_{\mathbb{R}}(v)$ and $\mathbb{R}^2$. Let the image of $\mathcal{L}_{\mathbb{Z}}(v)$

under this bijection be denoted by

$$\mathcal{L}(v).$$

To define this set, consider $L \in \mathcal{L}_{\mathbb{Z}}(v)$ represented by $u \wedge v$, where $u = (a, b, c)$ and $v = (p_1, p_2, q)$. Let ℓ be the line parallel to $\mathbb{R}v$ passing through u. Then ℓ intersects $\mathbb{R}^2 \times \{0\}$ in the point

$$(a - \frac{cp_1}{q}, b - \frac{cp_2}{q}, 0)$$

as shown in figure 1.1.

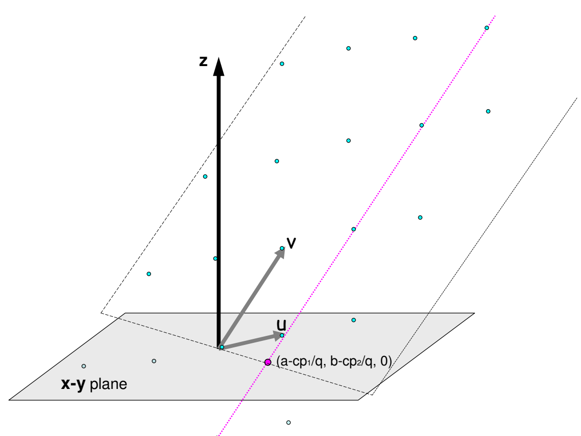

Figure 1.1: $L = u \wedge v$

So,

$$\mathcal{L}(v) := \{(a - \frac{cp_1}{q}, b - \frac{cp_2}{q}) : a, b, c \in \mathbb{Z}\}.$$

To determine the area of $\mathcal{L}(v)$, consider $u, w \in \mathbb{Z}^3$ such that $\{v, u, w\}$ are basis vectors for $\mathbb{Z}^3$, i.e. $|\det(v, u, w)| = 1$. Let P denote the parallelogram formed by u and w. We associate each $x \in P$ with a point $y \in \mathbb{R}^2$ in the following way: $y \in \mathbb{R}^2$ is the unique point of intersection of the line through x that is parallel to $\mathbb{R}v$ with $\mathbb{R}^2 \times \{0\}$. Let Q be the set of all such y points. Since $\{u, w\}$ projects to a basis for $\mathcal{L}(v)$, then Q is a parallelogram in $\mathbb{R}^2$ and $\text{area}(Q) = \text{area}(\mathcal{L}(v))$. The parallelopiped formed by $Q \times [0, |v|]$ is a fundamental domain for $h_v\mathbb{Z}^3$, where $h_v = \begin{pmatrix} 1 & 0 & -p_1/q \\ 0 & 1 & -p_2/q \\ 0 & 0 & 1 \end{pmatrix}$. Its volume is equal to $\text{area}(Q) \cdot |v| = 1$. Thus,

$$\text{area}(\mathcal{L}(v)) = \frac{1}{|v|}.$$

Now we wish to set the notation for the norm of elements in $\mathcal{L}(v)$ and the notation for the norm of the corresponding elements in $\mathcal{L}_{\mathbb{Z}}(v)$. If $v = (p_1, p_2, q)$ and $u = (a, b, c)$ then $L \in \mathcal{L}(v)$ corresponds to $u \wedge v \in \mathcal{L}_{\mathbb{Z}}(v)$. Let the norm of L as an element in $\mathcal{L}(v)$ be denoted by

$$\| \cdot \|_{\mathcal{L}(v)}$$

and the norm of L as an element in $\mathcal{L}_{\mathbb{Z}}(v)$ be denote by

$$|L|.$$

Then we define

$$\|L\|_{\mathcal{L}(v)} = \frac{|L|}{|v|} \qquad (4)$$

where the norm $|\cdot| : \mathcal{L}_{\mathbb{Z}}(v) \to \mathbb{R}_{\geq 0}$ is given by

$$|L| = \|\langle aq - p_1 c, , bq - p_2 c\rangle\|.$$

Notation. In the case where the norm on $\mathbb{R}^2$ is the maximum norm, we will denote $\|L\|_{\mathcal{L}(v)}$ and $|L|$ by $\|L\|_\infty$ and $|L|_\infty$, respectively.

Let $d(\ ,\ )$ denote the metric on $\mathbb{R}^2$ induced by the norm $\|\cdot\|$. Then $\|u \wedge v\| = |u| d(\dot{u}, \dot{v})$ so we have

$$d(\dot{u}, \dot{v}) = \frac{|u \wedge v|}{|u||v|} \qquad (5).$$

For further details on two-dimensional sublattices see section 2.1 in [1].

Chapter 2

Convexity of $\Delta_u(v)$ Relative to the Maximum Norm

To determine the convexity of $\Delta_u(v)$ with respect to the maximum norm we take a geometric approach. Consider u, v satisfying (1) where $0 < |u| < |v|$. Recall from (2) that $\Delta_u(v) = \{\mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, \dot{v}) < \frac{|u|}{|v|} d(\mathbf{x}, \dot{u})\}$. Let s denote the slope of the line connecting $\dot{v}$ and $\dot{u}$. We shall show that the shape of $\Delta_u(v)$ depends on the following parameters:

$$\lambda = \frac{|u|}{|v|}, \quad \sigma = \min\left(|s|, \frac{1}{|s|}\right), \quad r_u = \frac{|u \wedge v|}{|v|^2}.$$

In particular, we shall show that $\Delta_u(v)$ is either a convex quadrilateral (trapezoid or kite) or a non-convex hexagon.

Let us shift $\dot{v}$ to the origin and denote it by $\dot{\mathbf{v}}$. Accordingly we replace $\dot{u}$ with $\dot{u}-\dot{v}$ and denote it by $\dot{\mathbf{u}}$. To determine $d(\mathbf{x},\dot{\mathbf{v}})$ and $d(\mathbf{x},\dot{\mathbf{u}})$ with respect to the maximum norm, it is convenient to divide $\mathbb{R}^2$ into nine regions defined by the following four lines: $y = x, \quad y = -x, \quad y = -x + \frac{r_u}{\lambda}(\sigma+1), \quad y = x + \frac{r_u}{\lambda}(\sigma-1)$. See Figure 2.1.

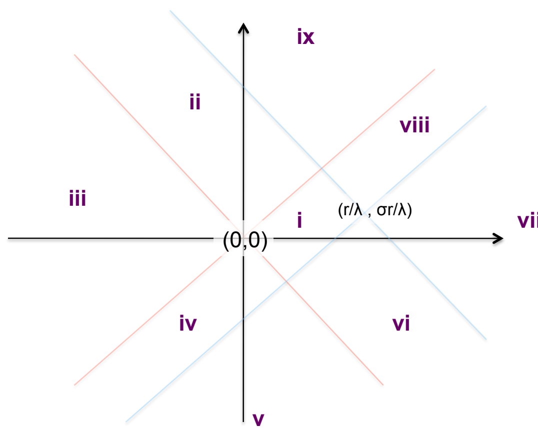

Figure 2.1: $\mathbb{R}^2$ divided into nine regions

Then for any $\mathbf{x} = (x, y) \in \mathbb{R}^2$ we have

$$d(\mathbf{x},\dot{\mathbf{v}}) = \begin{cases} x & \mathbf{x} \in \mathbf{i}, \mathbf{vi}, \mathbf{vii}, \mathbf{viii} \\ y & \mathbf{x} \in \mathbf{ii}, \mathbf{ix} \\ -x & \mathbf{x} \in \mathbf{iii} \\ -y & \mathbf{x} \in \mathbf{iv}, \mathbf{v} \end{cases} \qquad \mathbf{d}(\mathbf{x},\dot{\mathbf{u}}) = \begin{cases} -x + \frac{r_u}{\lambda} & \mathbf{x} \in \mathbf{i}, \mathbf{ii}, \mathbf{iii}, \mathbf{iv} \\ -y + \frac{\sigma r_u}{\lambda} & \mathbf{x} \in \mathbf{v}, \mathbf{vi} \\ x - \frac{r_u}{\lambda} & \mathbf{x} \in \mathbf{vii} \\ y - \frac{\sigma r_u}{\lambda} & \mathbf{x} \in \mathbf{viii}, \mathbf{ix}. \end{cases}$$

To obtain the equation of the boundary lines of $\Delta_u(v)$ in each region, we set $d(\mathbf{x}, \dot{\mathbf{v}}) = \lambda \mathbf{d}(\mathbf{x}, \dot{\mathbf{u}})$. Let l_j denote the boundary line of $\Delta_u(v)$ in region j.

Let us first show that $\Delta_u(v) \cap (\mathbf{vii} \cup \mathbf{ix} \cup \mathbf{viii}) = \emptyset$. For any $\mathbf{x} \in (\mathbf{vii} \cup \mathbf{ix})$, we have $d(\mathbf{x}, \dot{\mathbf{v}}) > \mathbf{d}(\mathbf{x}, \dot{\mathbf{u}})$. Since $0 < \lambda < 1$, then $\Delta_u(v) \cap (\mathbf{vii} \cup \mathbf{ix}) = \emptyset$. Region **viii** is the intersection of three half planes defined by the $y = x$, $y = -x + \frac{r_u}{\lambda}(\sigma + 1)$ and $y = x + \frac{r_u}{\lambda}(\sigma - 1)$ lines. Notice for any $\mathbf{x} \in \mathbf{viii}$ we have $x \geq y$. However if $\mathbf{x} \in \Delta_u(v)$, then $d(\mathbf{x}, \dot{\mathbf{v}}) < \lambda d(\mathbf{x}, \dot{\mathbf{u}}) \implies x < \lambda y - \sigma r_u < y$. Thus, $\Delta_u(v) \cap \mathbf{viii} = \emptyset$.

The following are the equations of the boundary lines in the remaining six regions:

Table 2.1: Boundary lines of $\Delta_u(v)$

$l_1 :$	$x = \frac{r_u}{1+\lambda}$
$l_2 :$	$y = -\lambda x + r_u$
$l_3 :$	$x = \frac{-r_u}{1-\lambda}$
$l_4 :$	$y = \lambda x - r_u$
$l_5 :$	$y = \frac{-\sigma r_u}{1-\lambda}$
$l_6 :$	$y = -\frac{1}{\lambda}x + \frac{\sigma r_u}{\lambda}$

Let us define $T(\lambda, \sigma, r_u)$, a trapezoid with sides given by l_1, l_2, l_3 and l_4 and the four vertices denoted by P_j as follows:

Table 2.2: $T(\lambda, \sigma, r_u)$

Lines Intersecting	Vertex
l_1, l_2	$P_1 = \left(\frac{r_u}{1+\lambda}, \frac{r_u}{1+\lambda}\right)$
l_2, l_3	$P_2 = \left(\frac{-r_u}{1-\lambda}, \frac{r_u}{1-\lambda}\right)$
l_3, l_4	$P_3 = \left(\frac{-r_u}{1-\lambda}, \frac{-r_u}{1-\lambda}\right)$
l_4, l_1	$P_4 = \left(\frac{r_u}{1+\lambda}, \frac{r_u}{1+\lambda}\right)$

Similarly we define $H(\lambda, \sigma, r_u)$ as a hexagon with sides given by l_1, l_2, l_3, l_4, l_5 and l_6 with following vertices:

Table 2.3: $H(\lambda, \sigma, r_u)$

Lines Intersecting	Vertex
l_1, l_2	$P_1 = \left(\frac{r_u}{1+\lambda}, \frac{r_u}{1+\lambda}\right)$
l_2, l_3	$P_2 = \left(\frac{-r_u}{1-\lambda}, \frac{r_u}{1-\lambda}\right)$
l_3, l_4	$P_3 = \left(\frac{-r_u}{1-\lambda}, \frac{-r_u}{1-\lambda}\right)$
l_4, l_5	$\hat{P}_4 = \left(\frac{r_u}{\lambda}(1 - \frac{\sigma}{1-\lambda}), \frac{-\sigma r_u}{1-\lambda}\right)$
l_5, l_6	$P_5 = \left(\frac{\sigma r_u}{1-\lambda}, \frac{-\sigma r_u}{1-\lambda}\right)$
l_6, l_1	$P_6 = \left(\frac{r_u}{1+\lambda}, \frac{r_u}{\lambda}(\sigma - \frac{1}{1+\lambda})\right)$

See Figure 2.2.

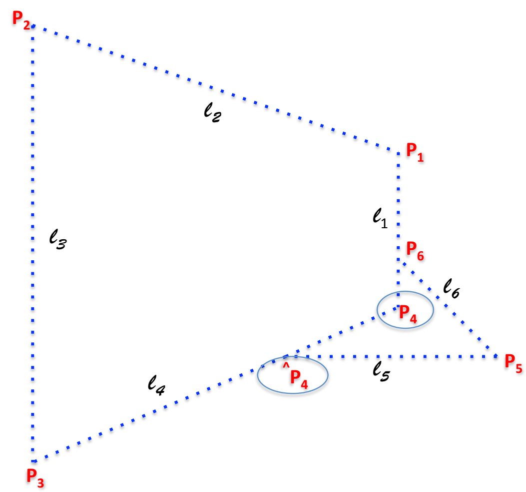

Figure 2.2: $\Delta(\lambda, \sigma, r_u)$

Theorem 2.1. *Given the maximum norm on* $\mathbb{R}^2$*, let* $u, v \in \mathbb{Z}^3$ *be two primitive vectors such that* $0 < |u| < |v|$*. Let* ρ *be the unique isometry of the square* $[1, -1] \times [1, -1]$ *that sends* $\dot{u} - \dot{v}$ *to the vector* $(\frac{r_u}{\lambda}, \frac{\sigma r_u}{\lambda})$*. Then*

$$\Delta_u(v) = \begin{cases} \rho T(\lambda, \sigma, r_u) + \dot{v} & \text{if} \quad \lambda \leq \frac{1-\sigma}{1+\sigma} \\ \rho H(\lambda, \sigma, r_u) + \dot{v} & \text{otherwise.} \end{cases}$$

Proof. Notice that p_1, p_2 and p_3 are independent of σ, so they will always be three vertices of $\Delta_u(v)$. Let $\lambda \leq \frac{1-\sigma}{1+\sigma}$, we determine if l_4 intersects l_1 or if l_4 intersects l_5 by comparing the x-coordinate of p_4 with the x-coordinate of $\hat{p}_4$. Since $\lambda \leq \frac{1-\sigma}{1+\sigma} \implies$

$\sigma \leq \frac{1-\lambda}{1+\lambda}$ we have

$$-\frac{\sigma}{1-\lambda} \geq -\frac{1}{1+\lambda} \implies 1 - \frac{\sigma}{1-\lambda} \geq \frac{\lambda}{1+\lambda} \implies \frac{r_u}{\lambda}(1 - \frac{\sigma}{1-\lambda}) \geq \frac{r_u}{1+\lambda}.$$

So l_4 intersects l_1 and does not connect with l_5. To check whether l_6 intersects l_1, we compare the y-coordinate of p_6 with the y-coordinate of p_4. Since $\sigma \leq \frac{1-\lambda}{1+\lambda}$ we have

$$\sigma - \frac{1}{1+\lambda} \leq -\frac{\lambda}{1+\lambda} \implies \frac{1}{\lambda}(\sigma - \frac{1}{1+\lambda}) \leq -\frac{1}{1+\lambda} \implies \frac{r_u}{\lambda}(\sigma - \frac{1}{1+\lambda}) \leq -\frac{r_u}{1+\lambda}.$$

So l_1 does not intersect l_6. Thus $\Delta_u(v) = \rho T(\lambda, \sigma, r_u) + \dot{v}$.

Conversely if $\lambda > \frac{1-\sigma}{1+\sigma}$ then $\hat{p_4}$, p_5 and p_6 are the vertices of $\Delta_u(v)$ and we have $\Delta_u(v) = \rho H(\lambda, \sigma, r_u) + \dot{v}$.

□

Corollary 2.2. $\Delta_u(v)$ *is convex with respect to the maximum norm if and only if* $\lambda \leq \frac{1-\sigma}{1+\sigma}$ *or* $\sigma = 1$.

Proof. Let $\lambda \leq \frac{1-\sigma}{1+\sigma}$, by Theorem 2.1 we have a trapezoid which is convex. Notice that if $\sigma = 1$ we get $\hat{p_4} = p_3$ and $p_6 = p_1$. Then $\Delta_u(v)$ is convex as in figure 2.3

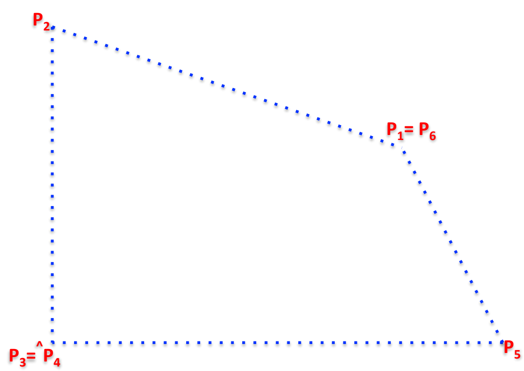

Figure 2.3: $\Delta_u(v)$

Conversely if $\lambda > \frac{1-\sigma}{1+\sigma}$ we have $\Delta_u(v) = \rho H(\lambda, \sigma, r_u) + \dot{v}$ and therefore not convex. $\square$

Figure 2.4 illustrates the relationship between λ, σ and convexity of $\Delta_u(v)$. For example given $\sigma = \frac{1}{3}$, the critical value for λ that determines convexity of $\Delta_u(v)$ is $\frac{1}{2}$.

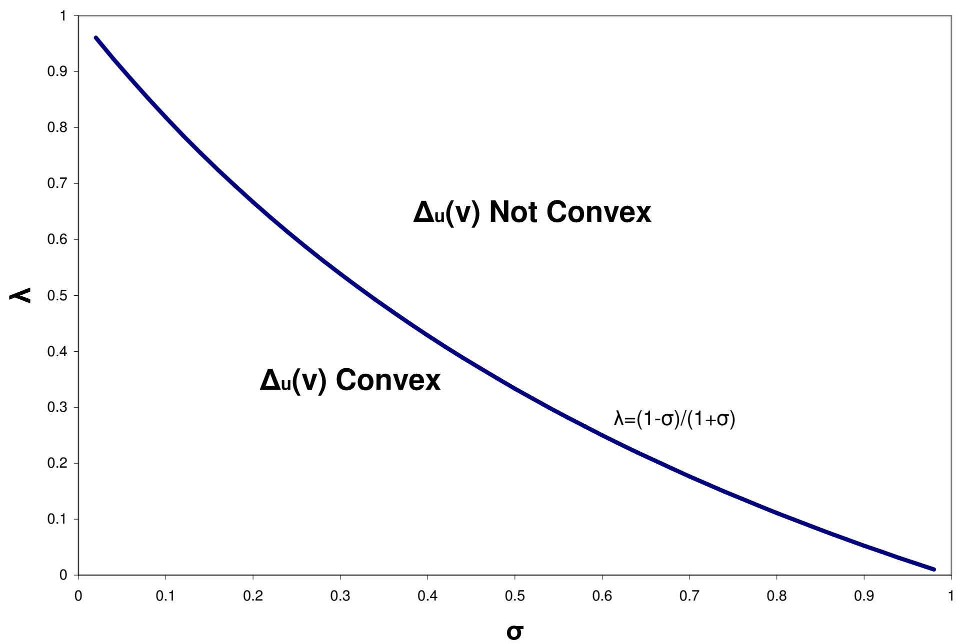

Figure 2.4: σ vs λ

Chapter 3

Simplifying $\Delta(v)$

Since $\Delta(v)$ is represented by the intersection of finitely many $\Delta_u(v)$, our goal is to simplify it so that it is defined by the intersection of only two sets. Let the smallest nonzero element of $\mathcal{L}(v)$ relative to a given norm be denoted by

$$L(v).$$

Given $L(v)$ we define u_+ and u_- as the unique vectors satisfying (1) such that $L = u_\pm \wedge v$ and $|u_\pm| \leq |v|$ (*Lemma 2.10 in [1]*).

Notation: Let Δ_+ be the shorthand for $\Delta_{u_+}(v)$ and similarly Δ_- for $\Delta_{u_-}(v)$.

The goal of this section is to prove the following.

Theorem 3.1. *Suppose* $L \in \mathcal{L}(v)$ *such that*

$$|L|_\infty < \frac{|v|^{1/2}}{2\sqrt{2}}.$$

Then $\Delta(v) = \Delta_+ \cap \Delta_-$.

Lemma 3.2. *Suppose* $L \in \mathcal{L}(v)$ *such that*

$$|L|_\infty < \frac{|v|^{1/2}}{2\sqrt{2}}$$

then $L = L(v)$, *the smallest element in* $\mathcal{L}(v)$ *and for any* $L' \neq L(v), |L'|_\infty \geq 4|L(v)|_\infty$.

Proof. Let $L' \in \mathcal{L}(v)$ such that $L' \neq L$. From equation (3), we have $\|L\|_2 \cdot \|L'\|_2 \leq 2\|L\|_\infty \cdot \|L'\|_\infty$ and by lemma 1.1 we know $\|L\|_2 \cdot \|L'\|_2 \geq \frac{1}{|v|}$. Combining these results and using equation (4) we have

$$\begin{aligned}
\|L\|_\infty \cdot \|L'\|_\infty &\geq \frac{1}{2|v|} \\
|L|_\infty \cdot |L'|_\infty &\geq \frac{|v|}{2} \\
|L'|_\infty &\geq \frac{|v|}{2|L|_\infty}.
\end{aligned}$$

Since $|L|_\infty < \frac{|v|^{1/2}}{2\sqrt{2}}$, then $|L'|_\infty \geq 4|L|_\infty$. Thus L is the smallest element in $\mathcal{L}(v)$, and we have $L = L(v)$. □

Let $u \notin L(v)$, to show that $\Delta_u(v)$ is redundant in $\Delta(v)$, we use the result of Theorem 2.11 in [1]: Let $B(\mathbf{x}, r) \subset \mathbb{R}^2$ denote the open ball at $\mathbf{x}$ with radius r. Let $v = (p_1, p_2, q) \in \mathbb{Z}^2 \times \mathbb{Z}_{>0}$ such that $\gcd(v) = 1$, then for any $u \in \mathbb{Z}^2 \times \mathbb{Z}_{>0}$ and $u \neq \mathbb{R}v$, we have

$$(i)\Delta_u(v) \supset B(\dot{v}, \frac{r_u}{2}) \qquad \text{where} \qquad r_u = \frac{|u \wedge v|}{|v|^2}$$

$$(ii)\Delta_+ \cap \Delta_- \subset B(\dot{v}, 2r) \qquad \text{where} \qquad r = \frac{|L(v)|}{|v|^2}.$$

Lemma 3.3. *Let $L \in \mathcal{L}(v)$ such that $|L|_\infty < \frac{|v|^{1/2}}{2\sqrt{2}}$, then*

$$\Delta_+ \cap \Delta_- \subset \Delta_u(v) \qquad \textit{for all} \quad u \in \mathbb{Z}^3 \backslash L.$$

Proof. Let $u \notin L$, then $u \wedge v = L'$. By lemma 3.2, $|L'|_\infty \geq 4|L|_\infty$ and we have $r_u = \frac{|u \wedge v|}{|v|^2} \geq 4\frac{|L(v)|}{|v|^2} = 4r$. Thus by Theorem 2.11 in [1], $\Delta_+ \cap \Delta_- \subset \Delta_u(v)$. □

It follows that for any $u \notin L(v), \Delta_u(v)$ is redundant and can be dropped from the intersection in the definition of $\Delta(v)$. Let $v = (p_1, p_2, q)$ as in (1) and $u = (a, b, c) \in L(v)$ such that $u \neq \{v, u_\pm\}$. Since $\{u_+, v\}$ is a basis for $L(v)$, we have $u = \alpha u_+ + \beta v$ for some uniquely determined $\alpha, \beta \in \mathbb{Z}$. Notice that $u \wedge v = \alpha(u_+ \wedge v) = -\alpha(u_- \wedge v)$. So if $|\alpha| \geq 4$, then $r_u \geq 4r$ and $\Delta_u(v)$ is redundant. Let Δ_α be shorthand for $\Delta_{u_\alpha}(v)$. It remains to show for $\alpha = 2, 3, -2$ and -3, $\Delta_\alpha \supset \Delta_+ \cap \Delta_-$.

Lemma 3.4. *Let v be as in* (1) *and $|u_-| \leq |u_+|$. Suppose $u_\alpha \in L(v)$ such that $u_\alpha \neq \{v, u_\pm\}$. Then*

$$\begin{aligned} \Delta_\alpha \supset \Delta_+ \quad & \textit{if} \;\; \alpha = 2, 3 \\ \Delta_\alpha \supset \Delta_- \quad & \textit{if} \;\; \alpha = -2, -3 \end{aligned}$$

.

Proof. Since $|u_-| \leq |u_+|$, it follows that $0 < \lambda_- \leq \frac{1}{2}$ and $\frac{1}{2} \leq \lambda_+ < 1$. Consider $\theta = (x, y, 1) \in \mathbb{R}^3$, then $\dot{\theta} = (x, y) \in \mathbb{R}^2$.

Suppose $\dot{\theta} \in \Delta_+$. Then $d(\dot{v}, \dot{\theta}) < \lambda_+ d(\dot{u_+}, \dot{\theta})$ and by (5) this is equivalent to $|v \wedge \theta| < |u_+ \wedge \theta|$.

Let $\alpha = 2$, then $u_2 = 2u_+ - v$. It follows that $|u_2 \wedge \theta| \geq 2|u_+ \wedge \theta| - |v \wedge \theta|$. Since $|u_+ \wedge \theta| > |v \wedge \theta|$, then $2|u_+ \wedge \theta| - |v \wedge \theta| > |u_+ \wedge \theta| > |v \wedge \theta|$. So $|u_2 \wedge \theta| > |v \wedge \theta|$

which is equivalent to $\dot{\theta} \in \Delta_2$.

Let $\alpha = 3$, if $\frac{1}{2} \leq \lambda_+ \leq \frac{2}{3}$. Then $u_3 = 3u_+ - v$. So we have $|u_3 \wedge \theta| \geq 3|u_+ \wedge \theta| - |v \wedge \theta| > |u_+ \wedge \theta| > |v \wedge \theta|$. If $\frac{2}{3} \leq \lambda_+ < 1$, then $u_3 = 3u_+ - 2v$ and it follows that $|u_3 \wedge \theta| \geq 3|u_+ \wedge \theta| - 2|v \wedge \theta| > |u_+ \wedge \theta| > |v \wedge \theta|$. Thus $\dot{\theta} \in \Delta_3$.

Suppose $\dot{\theta} \in \Delta_-$. This implies $|v \wedge \theta| < |u_- \wedge \theta|$. If $\alpha = -2$, then $u_2 = 2u_-$, and $u_2 \wedge \theta = -2(u_- \wedge \theta)$. So $\dot{\theta} \in \Delta_2$. Suppose $\alpha = -3$ and let $0 < \lambda_- < \frac{1}{3}$, then since $u_3 = 3u_-$, it follows that $\dot{\theta} \in \Delta_-$. For the case where $\frac{1}{3} \leq \lambda_- \leq \frac{1}{2}$, we have $u_3 = 3u_- - v$. So $|u_3 \wedge \theta| \geq 3|u_- \wedge \theta| - |v \wedge \theta| > |v \wedge \theta|$. Hence $\dot{\theta} \in \Delta_3$. $\square$

This completes the proof of Theorem 3.1.

Chapter 4

Convexity of $\Delta(v)$ Relative to the Maximum Norm.

Theorem 4.1. *Suppose* $\Delta(v) = \Delta_+ \cap \Delta_-$ *and* $\lambda_+ > \frac{1-\sigma}{1+\sigma} > 0$. *Then* $\Delta(v)$ *is not convex.*

Proof. Let $\lambda_+ > \frac{1-\sigma}{1+\sigma} > 0$, then by Theorem 2.1, $\Delta_+ = \rho H(\lambda_+, \sigma, r) + \dot{v}$. It is enough to show $p_{6+} = \left(\frac{r}{1+\lambda_+}, \frac{r}{\lambda_+}(\sigma - \frac{1}{1+\lambda_+})\right) \in \Delta_-$.

The boundary lines of Δ_- are listed in the following table.

Table 4.1: Boundary lines for Δ_-

l_{1-} :	$x = \frac{r}{1-\lambda_-}$
l_{2-} :	$y = -\lambda_- x - r$
l_{3-} :	$x = \frac{-r}{1+\lambda_-}$
l_{4-} :	$y = \lambda_- x + r$
l_{5-} :	$y = \frac{\sigma}{1-\lambda_-}$
l_{6-} :	$y = y = -\frac{1}{\lambda_-} x - \frac{\sigma r}{\lambda_-}$

See figure 4.1 for the diagram of Δ_-.

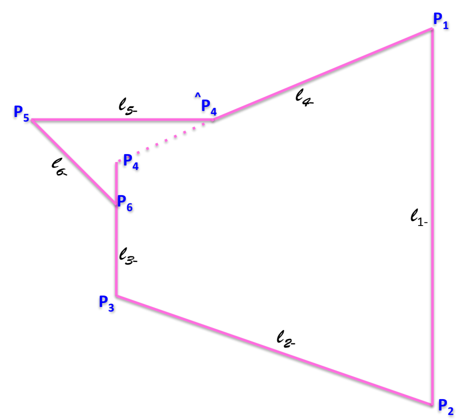

Figure 4.1: Δ_-

Since $\lambda_- = 1 - \lambda_+$ then $l_{1-} : x = \frac{r}{\lambda_+}$. So the x-coordinate of $p_{6+} = \frac{r}{1+\lambda_+}$ is to the left of l_{1-}. It remains to check if l_{4-} and l_{2-} bound p_{6+}. Notice that p_{6+} is bounded by the $y = x$ and $y = -x$ lines. At $x = \frac{r}{1+\lambda_+}$ we evaluate l_{4-} and l_{2-}

$$l_{4-} : y = \frac{2r}{1+\lambda_+} \qquad\qquad l_{2-} : y = -\frac{2r}{1+\lambda_+}.$$

So the $y = x$ line lies below l_{4-} and the $y = -x$ line lies above l_{2-}. It follows that $p_{6+} \in \Delta_-$. Thus $\Delta(v)$ is not convex.

□

The graph below illustrates the impact that convexity of Δ_+ and Δ_- have on the convexity of $\Delta_+ \cap \Delta_-$ with respect to the maximum norm.

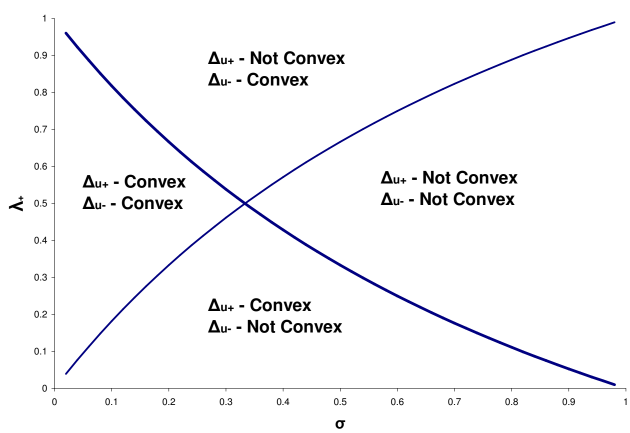

Figure 4.2: Convexity of Δ_+, Δ_- and $\Delta_+ \cap \Delta_-$

4.1 Example of a Non-Convex $\Delta(v)$

Consider the primitive vectors $v = (17, 8, 33)$ and $u = (15, 7, 29)$. Since

$$|(15, 7, 29) \wedge (17, 8, 33)| = 2 < \frac{\sqrt{33}}{2\sqrt{2}},$$

then by Lemma 3.2 $(15, 7, 29) \wedge (17, 8, 33) = L(v)$, the smallest element in $\mathcal{L}(v)$.

By Theorem 3.1, we have $\Delta(v) = \Delta_+ \cap \Delta_-$ with Δ_+ defined by $u_+ = (15, 7, 29), \lambda_+ = \frac{29}{33}, r = \frac{2}{(33)^2}$ and Δ_- defined by $u_- = (2, 1, 4), \lambda_- = \frac{4}{33}, r = \frac{2}{(33)^2}$.

To study the shape of $\Delta(v)$ we shift and normalize $\dot{v}, \dot{u_+}$ and $\dot{u_-}$ as described in chapter 2. So

$$\dot{\mathbf{v}} = (0, 0), \qquad \dot{\mathbf{u}}_+ = \left(\frac{2}{(29)(33)}, -\frac{1}{(29)(33)}\right), \qquad \dot{\mathbf{u}}_- = \left(-\frac{2}{(4)(33)}, \frac{1}{(4)(33)}\right)$$

with the slope of the line containing $\dot{\mathbf{v}}, \dot{\mathbf{u}}_+, \dot{\mathbf{u}}_-$ equal to $-\frac{1}{2}$ we have $\sigma = \frac{1}{2}$. So we reflect $\dot{\mathbf{u}}_+, \dot{\mathbf{u}}_-$ over the horizontal axis to obtain Δ_+ and Δ_-.

Note that $\frac{1-\sigma}{1+\sigma} = \frac{1}{3}$ so $\lambda_+ = \frac{29}{33} > \frac{1}{3}$ and $\lambda_- = \frac{4}{33} < \frac{1}{3}$. By Theorem 2.1, Δ_+ is not convex and Δ_- is convex. Finally by Theorem 4.1, we confirm that $\Delta(v)$ is not convex.

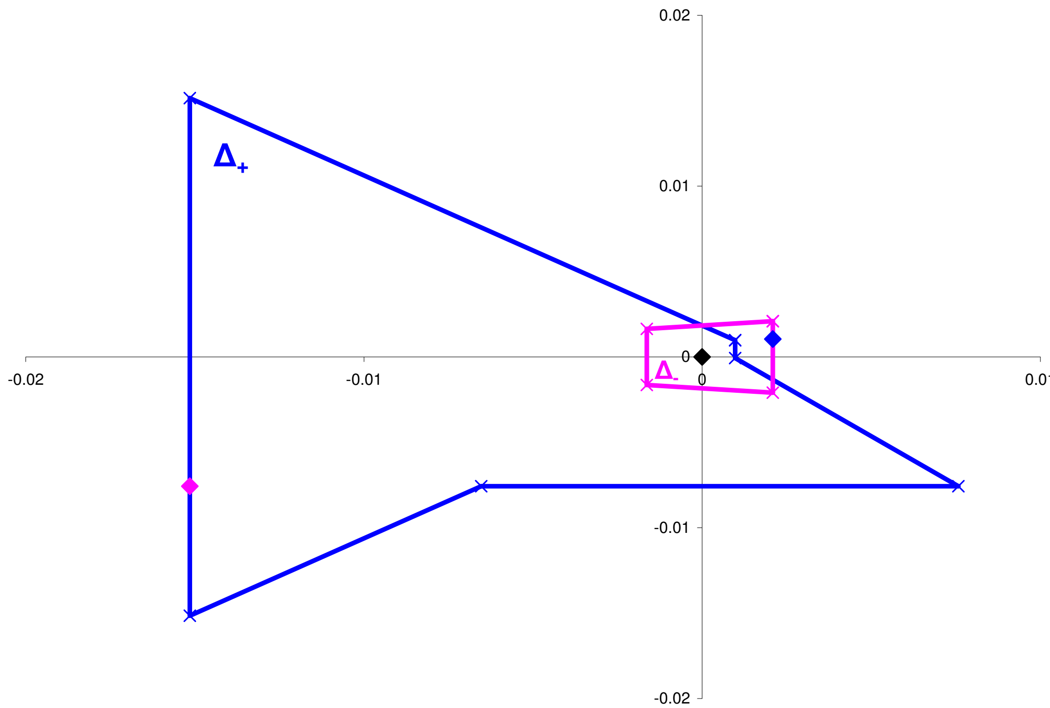

Figure 4.3: $\Delta(v)$ for $v = (17, 8, 33)$

Chapter 5

Conclusion

We have shown that the domains of best approximation with respect to the maximum norm are not necessarily convex. We accomplished this by reducing the number of sets that cover $\Delta(v)$ to two sets. One can explore the possibility of relaxing the sufficient condition that led us to this result (Theorem 3.1), and classify a broader set of rational numbers for which their domains of best approximation will not be convex. Furthermore one can study these sets with respect to other norms and determine for which other norms are these sets not convex?

REFERENCES

[1] Cheung, Y., *Hausdorff dimension of the set of singular pairs*, Annals of Math (accepted Nov. 5, 2009)

Appendix A

Lemma 5.1. *Let $\Lambda \subset \mathbb{R}^2$ be a two dimensional lattice, i.e., a discrete subgroup of $\mathbb{R}^2$ of full rank. Let $v \in \Lambda$ be the shortest nonzero vector with respect to the Euclidean norm. Then*

$$\|v\|_2 \leq \sqrt{\frac{2}{\sqrt{3}}}\sqrt{\operatorname{area}(\Lambda)}.$$

Proof. Let v be the shortest primitive vector in Λ. Let $u \notin \mathbb{Z}v$ be the next shortest vector in Λ and θ denote the angle between v and u. Then

$$\begin{aligned} \|u \pm v\|_2^2 &\geq \|u\|_2^2 \\ \|u\|_2^2 \pm 2(u \cdot v) + \|v\|_2^2 &\geq \|u\|_2^2 \\ \|v\|_2^2 &\geq \mp 2\|u\|_2\|v\|_2 \cos\theta \end{aligned}$$

So $\frac{1}{2} > \frac{\|v\|_2}{2\|u\|_2} \geq |\cos\theta|$. Thus $\frac{\pi}{3} < \theta < \frac{2\pi}{3} \implies \frac{\sqrt{3}}{2} < \sin\theta \leq 1$. Since

$$\begin{aligned} \text{area}(\Lambda) =& \|u\|_2 \|v\|_2 \sin\theta \\ \implies \|v\|_2 =& \frac{\text{area}(\Lambda)}{\|u\|_2 \sin\theta} < \frac{\text{area}(\Lambda)}{\|v\|_2 \sin\theta} \\ \implies \|v\|_2^2 <& \frac{\text{area}(\Lambda)}{\sin\theta} < \text{area}(\Lambda) \cdot \frac{2}{\sqrt{3}} \end{aligned}$$

Thus, $\|v\|_2 < \sqrt{\text{area}(\Lambda)}\sqrt{\frac{2}{\sqrt{3}}}$. □

Notice that if $\|v\|_2 \leq \sqrt{\text{area}(\Lambda)}$, then v is the shortest vector in Λ.