TILING PROBLEM FOR LITTLEWOOD'S CONJECTURE

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by
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San Francisco, California
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CERTIFICATION OF APPROVAL

I certify that I have read *Tiling Problem for Littlewood's Conjecture* by Lok Shum Lui, and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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In mathematics, the Littlewood's conjecture is an open problem in Diophantine approximation, proposed by John Edensor Littlewood around 1930. It states that for any two real numbers α and β,

$$\inf_{n>0} n\|n\alpha\|\|n\beta\| = 0,$$

where $\|\|$ is the distance to the nearest integer. The goal of this project is to formulate a tiling problem (of the plane) that is equivalent to the Littlewood's Conjecture for $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. A tile in the tiling defined here may be thought of as a generalization of the continued fraction expansion of a single real number. In this project, we show that $\mathbb{R}^2$ is covered by the non-overlapping tiles associated to the pivots which play the role convergents of the continued fraction and show that the diameter of a tile is comparable to $-\log n\|n\alpha\|\|n\beta\|$ where n is the denominator of the associated convergent of the pair (α, β).

I certify that the Abstract is a correct representation of the content of this thesis.

____________________________________ ________________

Chair, Thesis Committee Date

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1 Introduction

In mathematics, the Littlewood's conjecture is an open problem in Diophantine approximation, proposed by John Edensor Littlewood around 1930 ([4],[5]). It states that

For any two real numbers α and β,

$$\liminf_{n\to\infty} \quad n\|n\alpha\|\|n\beta\| = 0,$$

where $\|\cdot\|$ is the distance to the nearest integer.

Note that for any positive integer n, we have the inequality $n\|n\alpha\|\|n\beta\| \geq 0$. Using this observation, Littlewood's conjecture can be reduced to:

For any two real numbers α and β,

$$\inf_{n\in\mathbb{Z}^+} \quad n\|n\alpha\|\|n\beta\| = 0.$$

The goal of this project is to formulate a tiling problem (of the Cartesian plane) that is equivalent to Littlewood's conjecture. From the latter version above, it is obvious that Littlewood's conjecture holds in the case when either α or β is rational. Thus, we focus on irrational pairs, i.e., $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$, in this project. Nevertheless, the irrationality of the pair is not neccessary for many results we found. Thoughout this paper, we will explicitly state the condition whenever it is pertinent.

The process of our formulation is divided into five parts: definition of a tile, introduction of the notion of pivots of a lattice, demonstration of the covering and non-overlapping properties of tiles associated with pivots, showing the diameters of those tiles are comparable to the quantity $-\log n\|n\alpha\|\|n\beta\|$, and finally the conclusion.

The definition of a tile is given in Section 2. In this section, we begin with introducing the lattice $\Lambda_{\alpha,\beta}$ and certain piecewise linear functions f_u associated to the vector $u \in \mathbb{Z}^3 \setminus \{0\}$ that will be used to define the tiles of the tiling. Then we continue to develop some basic properties of f_u. In addition, we introduce the notion of the vector u being (α, β)-good which is satisfied by any best approximation of the pair (α, β). Although this notion may appear a bit arbitrary at this point, the mystery will be unraveled as we proceed to the next section. Again, while some of the results we found in this project requires u to be (α, β)-good, some do not. The assumption will be stated explicitly for the maximal flexibility of our findings. Afterwards, we give the definiton of a tile toward the end of the section and examine a couple of its properties that are crucial to the further parts of the formulation.

In Section 3, we introduce the notion of pivots for a pair (α, β), which generalize the idea of convergents of the continued fraction of a single real number. Given a real number r, there are two ways to define a best Diophantine approximation of r:

Definition 1.1. *Best Diophantine Approximation of the First Kind*
Let r be a real number. The rational number $\frac{p}{q}$ is said to be a *best Diophantine approximation of the fisrt kind* of r if

$$\left|r - \frac{p}{q}\right| \leq \left|r - \frac{p'}{q'}\right|$$

for every rational number $\frac{p'}{q'}$ different from $\frac{p}{q}$ such that $0 < q' \leq q$ with strict inequality when $q' < q$.

Definition 1.2. *Best Diophantine Approximation of the Second Kind*
Let r be a real number. The rational number $\frac{p}{q}$ is said to be a *best Diophantine approximation of the second kind* of r if

$$|qr - p| \leq |q'r - p'|$$

for every rational number $\frac{p'}{q'}$ different from $\frac{p}{q}$ such that $0 < q' \leq q$ with strict inequality when $q' < q$.

Note that a best approximation of the second kind is also a best approximation of the first kind, but the converse is false. Furthermore, each best approximation of the second kind, as a classical result of Lagrange's Theorem on Number Theory [3], is a convergent of r's expression as a regular continued fraction. Given real numbers α and β, Littlewood's conjecture calls for the distances from $n\alpha$ and $n\beta$ to the nearest integers for each positive integer n. Thus, essentially we are dealing with the best approximations of α and β of the second kind. Pivots, after all, are elements in $\Lambda_{\alpha,\beta}$, the lattice associated with (α, β) defined by (2.1) below. There are two types of pivots: degenerate and nondegenerate. The beauty of them lies within their intrinsic property, the pivot denominator, which allows us to find a subsequence in Littlewood's conjecture; i.e.,

Given $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$, then

$$\inf_{q \in \mathbb{Z}_+} q\|q\alpha\|\|q\beta\| = \inf_{n \in \pi(\alpha,\beta)} n\|n\alpha\|\|n\beta\|$$

where $\pi(\alpha, \beta)$ is the set of pivot denominators of (α, β).

This claim will be proven in Theorem 3.2. Also, we show in Theorem 3.7 that the only degenerate pivots are those that lie in the coordinate plane given $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$.

Immediately after the introduction of pivots, in Section 4 we show that the tiles associated to inequivalent pivots of $\Lambda_{\alpha,\beta}$ cover $\mathbb{R}^2$ and that they do not overlap. These tiles are indexed by the pivots of $\Lambda_{\alpha,\beta}$. The tiling we attempt to establish in this project can then be thought of as a simultaneous generalization of the continued fraction of the numbers α and β. Thus, we have formulated a tiling problem that is equivalent to Littlewood's conjecture here. Nonetheless, the tiling makes no statement about the truth of the conjecture up to this point. The next section will shed some light on the connection thereof.

In Section 5, we show the diameters of the tiles associated with the inequivalent pivots are comparable to the quantity $-\log n\|n\alpha\|\|n\beta\|$ where n is the denominator of the associated convergent of the pair (α, β). In turn, we show a pair of irrational numbers (α, β) is a counterexample to Littlewood's conjecture [2] if and only if the diameters of the tiles of nondegenerate pivots are uniformly bounded. Finally, we end the project in Section 6 with the discussion of some further problems following our result.

2 Piecewise Linear Functions

The definition of tiles involves the function

$$W_{\alpha,\beta}(t,s) := \inf_{u\in\mathbb{Z}^3\setminus\{0\}} f_u(t,s)$$

where $f_u(t,s)$ are some piecewise linear functions. In this section, we introduce these piecewise linear functions and develop some of their properties.

Given $\alpha, \beta \in \mathbb{R}$, first we define the lattice

$$\Lambda_{\alpha,\beta} := h_{\alpha,\beta}\mathbb{Z}^3 \tag{2.1}$$

where $h_{\alpha,\beta}$ is the shear transformation

$$h_{\alpha,\beta} := \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

Also, for $(t,s) \in \mathbb{R}^2$, we define the scaling matrix $g_{t,s}$ by

$$g_{t,s} := \begin{pmatrix} e^{t+s} & 0 & o \\ 0 & e^{t-s} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}. \tag{2.3}$$

Now we formally define for all (t,s) in $\mathbb{R}^2$

$$W_{\alpha,\beta}(t,s) := \inf_{u\in\mathbb{Z}^3\setminus\{0\}} f_u(t,s) \tag{2.4}$$

where $f_u(t,s)$ are piecewise linear functions defined for all u in $\mathbb{Z}^3 \setminus \{0\}$ by

$$f_u(t,s) := \log \|g_{t,s}h_{\alpha,\beta}u\|_\infty. \tag{2.5}$$

The norm $\|\cdot\|_\infty$ in the above definition is the standard sup norm, i.e., for any $u = (x,y,z) \in \mathbf{R}^3$, $\|u\|_\infty := \max(|x|,|y|,|z|)$. In the following, we begin our development of some properties of f_u. Clearly, the function $W_{\alpha,\beta}$ is finite only if $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$.

Definition 2.1. Let $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$. We say that u is (α,β)-*good* if m_1 is the nearest integer to $n\alpha$ and m_2 is the nearest integer to $n\beta$.

Remark. There are obvious generalizations to the case where u is not (α,β)-good, but in our applications, this condition always holds. See Proposition 3.1.

Recall that we denote $\|n\alpha\|$ by the distance from $n\alpha$ to the nearest integer and similarly for $\|n\beta\|$. In tsw-coordinates we define the following functions :

$$P_x : w = t + s + \log\|n\alpha\|, \tag{2.6}$$

$$P_y : w = t - s + \log\|n\beta\|, \text{ and} \tag{2.7}$$

$$P_z : w = -2t + \log n \tag{2.8}$$

For simplicity, we refer to the graphs of the functions P_x, P_y and P_z with the same symbols when the context is clear.

Proposition 2.1. *Let $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$. If u is (α, β)-good, then the graph of f_u is contained in $P_x \cup P_y \cup P_z$.*

Proof. This is shown by examining each of the following cases.

Case 1: f_u is realized by the t-coordinate. Then $w = \log(e^{t+s}\|n\alpha\|) = t + s + \log\|n\alpha\|$ and the associated plane is $\mathcal{P}_x : t + s - w = -\log\|n\alpha\|$ with normal $n_x = (1, 1, -1)$.

Case 2: f_u is realized by the s-coordinate. Then $w = \log(e^{t-s}\|n\beta\|) = t - s + \log\|n\beta\|$ and the associated plane is $\mathcal{P}_y : t - s - w = -\log\|n\beta\|$ with normal $n_y = (1, -1, -1)$.

Case 3: f_u is realized by the r-coordinate. Then $w = \log(e^{-2t}n) = -2t + \log n$ and the associated plane is $\mathcal{P}_z : 2t + w = \log n$ with normal $n_z = (-2, 0, -1)$.

Having exhausted all possible cases, we see that the graph of f_u is indeed contained in the union of P_x, P_y and P_z as it is claimed. $\square$

With some simple calculations, it is easy to see that

$$P_x = P_y \quad \text{if and only if} \quad s = \frac{1}{2}\log\frac{\|n\beta\|}{\|n\alpha\|},$$

$$P_x = P_z \quad \text{if and only if} \quad s = -3t - \log\frac{\|n\alpha\|}{n}, \quad \text{and}$$

$$P_y = P_z \quad \text{if and only if} \quad s = 3t + \log\frac{\|n\beta\|}{n}.$$

In ts-coordinates, we define the lines

$$\ell_{xy} \ : \ s = \frac{1}{2} \log \frac{\|n\beta\|}{\|n\alpha\|}, \tag{2.9}$$

$$\ell_{xz} \ : \ s = -3t - \log \frac{\|n\alpha\|}{n}, \text{ and} \tag{2.10}$$

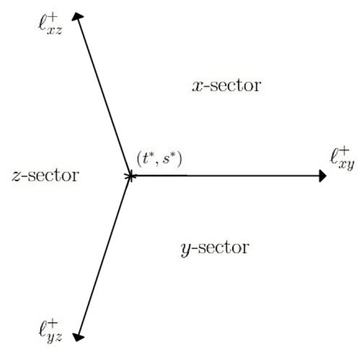
$$\ell_{yz} \ : \ s = 3t + \log \frac{\|n\beta\|}{n}. \tag{2.11}$$

Observe that the intersection of ℓ_{xy} and ℓ_{xz} lies on ℓ_{yz}. Indeed, the solution to the system is

$$(t^*, s^*) = \left(-\frac{1}{6} \log \frac{\|n\alpha\| \|n\beta\|}{n^2}, \ \frac{1}{2} \log \frac{\|n\beta\|}{\|n\alpha\|} \right). \tag{2.12}$$

Create the ray ℓ_{xy}^+ by appending the endpoint (t^*, s^*) to line ℓ_{xy} such that for all (t, s) within the ray, $P_x(t, s) > P_x(t^*, s^*)$. Similarly, creat the rays ℓ_{xz}^+, ℓ_{yz}^+ such that for all (t, s) within the rays, $P_z(t, s) > P_z(t^*, s^*)$ and $P_y(t, s) > P_y(t^*, s^*)$, respectively. The rays split the ts-plane producing three open sectors. With a little algebraic manipulation, we see that each of the functions P_x, P_y and P_z dominates the others in a sector. We call these sectors the x-sector, the y-sector, and the z-sector, respectively. For example, $P_x > P_y$ and $P_x > P_z$ in the x-sector. The partition by the rays is depicted in Figure 1.

Figure 1: Partition of the ts-plane by ℓ^+_{xy}, ℓ^+_{xz} and ℓ^+_{yz}.



Proposition 2.2. *Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$, and $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$. If u is (α, β)-good, then f_u has the global minimum occuring at (t^*, s^*).*

Proof. Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$, and $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$. Assume that u is (α, β)-good. We show that $f_u(t, s) > f_u(t^*, s^*)$ for all $(t, s) \in \mathbb{R}^2 \setminus \{(t^*, s^*)\}$. To aid the visualization of the following argument, the readers are invited to refer to Figure 1.

Case 1: (t, s) lies in a ray emanating from (t^*, s^*). Assume (t, s) lies in ℓ^+_{xy}. Then $f_u(t, s) = P_x(t, s)$ and $f_u(t^*, s^*) = P_x(t^*, s^*)$. Thus $f_u(t, s) > f_u(t^*, s^*)$ by the construction of the ray. Similarly, we see the result holds for the cases where (t, s) lies in ℓ^+_{xz} and ℓ^+_{yz}.

Case 2: (t, s) lies in an open sector. Note that for any (t, s) lying in an open sector, it can be connected to (t^*, s^*) by either a single line segment parallel to the gradient of the dominating function of the sector or first a line segment parallel to the gradient of the dominating function of the sector to the point (t', s') in a ray and then the line segment joining (t', s') and (t^*, s^*). We already showed that $f_u(t', s') > f_u(t^*, s^*)$ above. Thus, it remains to show $f_u(t, s) > f_u(t', s')$ for the two line segments case or $f_u(t, s) > f_u(t^*, s^*)$ for the other.

Case i: (t,s) lies in the open x-sector. Then the gradient is $\nabla P_x = \langle 1,1\rangle$. Thus, we have $t > t'$ and $s > s'$ or $t > t^*$ and $s > s^*$ depending on the number of connecting line segments. Evaluating f_u at the point, respectively we get $f_u(t,s) = P_x(t,s) > f_u(t',s')$ and $f_u(t,s) = P_x(t,s) > f_u(t^*,s^*)$.

Case ii: (t,s) lies in the open y-sector. Then the gradient is $\nabla P_y = \langle 1,-1\rangle$. Thus, we have $t > t'$ and $s < s'$ or $t > t^*$ and $s < s^*$. Evaluating f_u at the point, respectively we get $f_u(t,s) = P_y(t,s) > f_u(t',s')$ and $f_u(t,s) = P_y(t,s) > f_u(t^*,s^*)$.

Case iii: (t,s) lies in the open z-sector. Then the gradient is $\nabla P_z = \langle -2,0\rangle$. Thus, we have $t < t'$ and $s = s'$ or $t < t^*$ and $s = s^*$. Evaluating f_u at the point, respectively we get $f_u(t,s) = P_z(t,s) > f_u(t',s')$ and $f_u(t,s) = P_z(t,s) > f_u(t^*,s^*)$.

Therefoe, f_u has the global minimum occuring at (t^*,s^*). □

Corollary. *Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$, and $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$. If u is (α,β)-good and $n\|n\alpha\|\|n\beta\| < 1$, then the global minimum of f_u is less than 0.*

Proof. Since (t^*,s^*) is the intersection of the three closed sectors, the global minimum of $f_u = f_u(t^*,s^*) = P_x(t^*,s^*) = P_y(t^*,s^*) = P_z(t^*,s^*)$. Given $n\|n\alpha\|\|n\beta\| < 1$, we have $f_u(t^*,s^*) = P_z(t^*,s^*) = -2t^* + \log n = -2(-\frac{1}{6}\log\frac{\|n\alpha\|\|n\beta\|}{n^2}) + \log n = \frac{1}{3}\log n\|n\alpha\|\|n\beta\| < 0$. □

Definition 2.2. Let $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$. We define

$$\Delta(u) := \{(t,s) \ : \ f_u(t,s) \leq 0\}.$$

Let $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$. Assume u is (α, β)-good and $n\|n\alpha\|\|n\beta\| < 1$, then by solving for $P_x \leq 0$, $P_y \leq 0$ and $P_z \leq 0$ in the respective sectors, we see that $\Delta(u)$ is a right triangle in the ts-plane bounded by the lines $P_x = 0$, $P_y = 0$, and $P_x = 0$, and having diameter

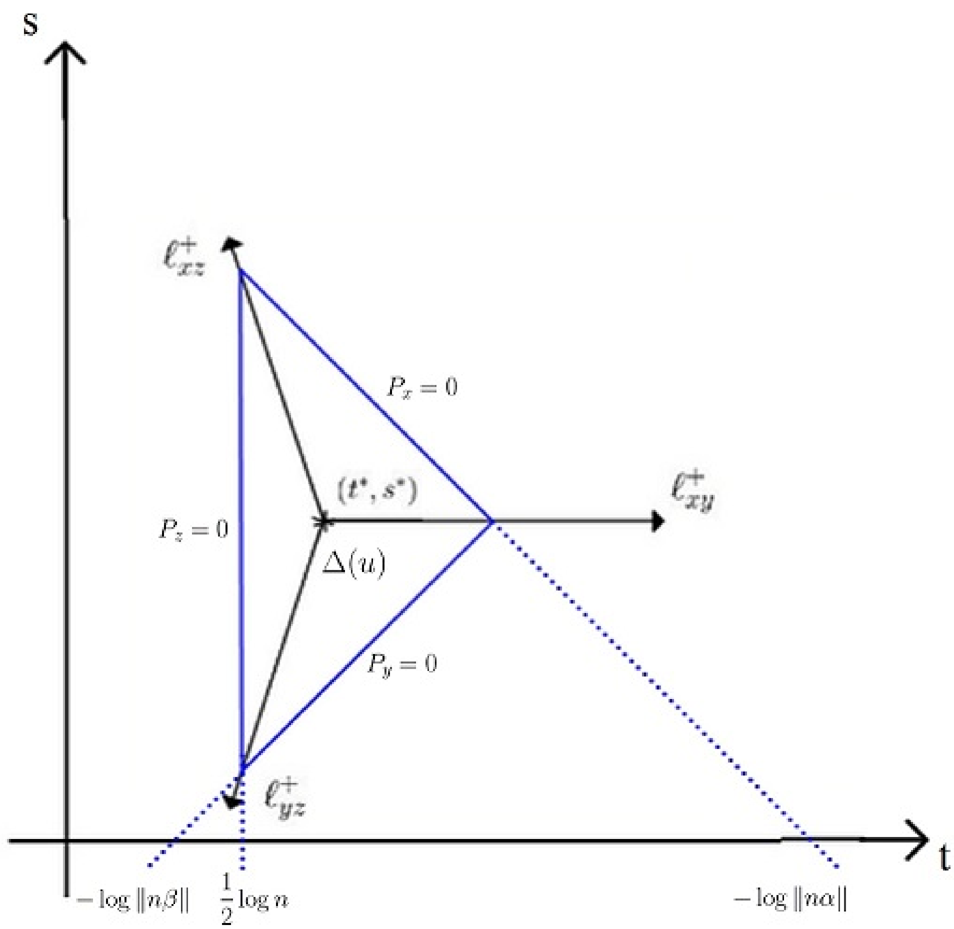
$$\operatorname{diam}(\Delta(u)) = -\log(n\|n\alpha\|\|n\beta\|) \tag{2.13}$$

and centroid (t^*, s^*) given by

$$(t^*, s^*) = \left(-\frac{1}{6}\log\frac{\|n\alpha\|\|n\beta\|}{n^2},\ \frac{1}{2}\log\frac{\|n\beta\|}{\|n\alpha\|}\right). \tag{2.14}$$

Remark. Given $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$, and $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$ where u is (α, β)-good, the formula given in (2.13) suggests that the conditions $\Delta(u)$ being nonempty and $n\|n\alpha\|\|n\beta\| \leq 1$ are equivalent.

Figure 2: The set $\Delta(u)$ in the ts-plane given $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$ and $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$ where u is (α, β)-good and $n\|n\alpha\|\|n\beta\| < 1$. We assume $\|n\beta\| > \|n\alpha\|$ without loss of generality,.



After the introduction of the piecewise linear functions f_u and the development of some of their properties, we are ready to define a tile.

Definition 2.3. A *tile* for the tiling of the pair (α, β) associated with $u \in \mathbb{Z}^3 \setminus \{0\}$ is a set

$$\tau(u) := \{(t, s) : W_{\alpha,\beta}(t, s) = f_u(t, s)\}. \tag{2.15}$$

We have already seen that the diameter of $\Delta(u)$ is given by $-\log n\|n\alpha\|\|n\beta\|$. To properly formulate the tiling problem that is equivalent to Littlewood's conjecture, loosely speaking, we must have the set $\tau(u)$ "no bigger than" $\Delta(u)$. We show in the following the aforementioned in a more precise manner.

Theorem 2.3. *(Minkowski Convex Body Theorem [1])*
Let Λ be a lattice in $\mathbb{R}^d$, and let $C \subset \mathbb{R}^d$ be a symmetric convex set with $\mathrm{vol}(C) > 2^d \det \Lambda$. Then C contains a point of Λ different from zero.

Illustrated by the example of the open unit cube in the standard integer lattice, we see that the strict inequality in the Minkowski Convex Body Theorem cannot be relaxed, however we have:

Corollary. *Let Λ be a lattice in $\mathbb{R}^d$, and let $C \subset \mathbb{R}^d$ be a symmetric convex set with $\mathrm{vol}(C) = 2^d \det \Lambda$. Then $\overline{C}$, the closure of C, contains a point of Λ different from zero.*

Lemma 2.4. *If $(t, s) \in \mathbb{R}^2$, then $f_u(t, s) \leq 0$ for some $u \in \mathbb{Z}^3 \setminus \{0\}$.*

Proof. Since $\Lambda_{\alpha,\beta}$ is the image of $\mathbb{Z}^3$ under the shear transformation $h_{\alpha,\beta}$, its fundamental region has volume 1. Also, $g_{t,s}$ has determinant 1. Thus, the volume of $g_{t,s}\Lambda_{\alpha,\beta}$ equals $|g_{t,s}|\,\mathrm{vol}(\Lambda_{\alpha,\beta}) = 1$. Then by the corollary to Theorem 2.3, there is a nonzero element in the intersection of the lattice and the unit cube $[-1, 1]^3$. Hence, there is some vector $g_{t,s}h_{\alpha,\beta}u \in g_{t,s}\Lambda_{\alpha,\beta}$ with $\|g_{t,s}h_{\alpha,\beta}u\|_\infty \leq 1$. Thus, $f_u(t, s) \leq 0$ for some $u \in \mathbb{Z}^3 \setminus \{0\}$. □

Theorem 2.5. *Given $u \in \mathbb{Z}^3 \setminus \{0\}$, the set $\tau(u)$ is contained in $\Delta(u)$.*

Proof. Let $u \in \mathbb{Z}^3 \setminus \{0\}$. Assume $(t, s) \in \tau(u) \setminus \Delta(u)$ for some u. Then $W_{\alpha,\beta}(t, s) = f_u(t, s) > 0$. It follows that $f_{u'}(t, s) > 0$ for all $u' \in \mathbb{Z}^3 \setminus \{0\}$. This is a contradiction by Lemma 2.4. Therefore, $\tau(u) \subset \Delta(u)$. □

Theorem 2.6. *Let $u \in \mathbb{Z}^3 \setminus \{0\}$. If $\tau(u)$ is nonempty, it contains the centroid of $\Delta(u)$.*

Proof. Suppose the centroid $(t^*, s^*) \notin \tau(u)$. By hypothesis there is $(t, s) \in \tau(u)$ such that $W_{\alpha,\beta}(t, s) = f_u(t, s) < f_u(t^*, s^*)$. This is a contradiction to Proposition 2.2. Hence, $\tau(u)$ contains the centroid of $\Delta(u)$. □

3 Pivots

In this section, we define pivots of $\Lambda_{\alpha,\beta}$ and develop examine some properties of them.

Given $u = (a, b, c) \in \mathbb{R}^3$, we let

$$R(u) := [-|a|, |a|] \times [-|b|, |b|] \times [-|c|, |c|].$$

Given a rectangle R of the form $R(\textit{eccentricity})$ as above, we define the following subsets of the boundary of R, ∂R:

- the corners of R, $\partial'' R = \{\pm|a|\} \times \{\pm|b|\} \times \{\pm|c|\}$,
- the open x-face of R, $\partial_x R = \{-|a|, |a|\} \times (-|b|, |b|) \times (-|c|, |c|)$,
- the open x-edge of R, $\partial'_x R = (-|a|, |a|) \times \{-|b|, |b|\} \times \{-|c|, |c|\}$.

$\partial_y R, \partial_z R, \partial'_y R$ and $\partial'_z R$ are similarly defined. Then

$$\partial R = \partial_x R \cup \partial_y R \cup \partial_z R \cup \partial'_x R \cup \partial'_y R \cup \partial'_z R \cup \partial'' R$$

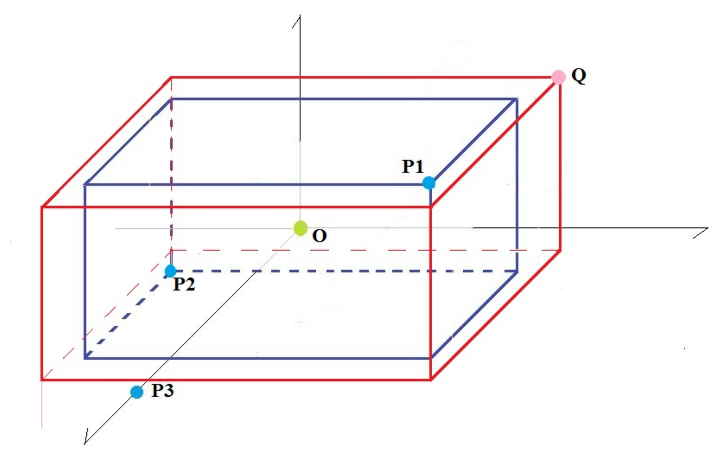
.

Remark. We choose the term ‘corners’ as opposed to ‘vertices’ to enhance the visualization of the box R. The two terms possess the same meaning by our context otherwise.

Definition 3.1. Let $u \in \Lambda$. Then u is said to be a *pivot* of Λ if $R(u) \cap (\Lambda \setminus \{0\}) \subset \partial'' R$.

Example 3.1. Figure 3 below shows a section of a lattice about the origin and all of its neighboring lattice points where Q has coordinates whose absolute values are strictly greater than those of P_1. In the figure, P_1, P_2 and P_3 are pivots of the lattice while Q is not because the intersection of $R(Q)$ with the lattice contains the non-zero lattice points P_1 and P_2 which are not the vertices of $R(Q)$. Moreover, the pivot P_1 and P_2 are equivalent and P_3 is a degenerate pivot whose definitions will be introduced shortly.

Figure 3: Examples of pivot.



Proposition 3.1. *Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$ and $u \in \mathbb{Z}^3 \setminus \{0\}$. If $h_{\alpha,\beta}u$ is a pivot of $\Lambda_{\alpha,\beta}$, then u is (α, β)-good.*

Proof. Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$ and $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$. Assume $h_{\alpha,\beta}u$ is a pivot of $\Lambda_{\alpha,\beta}$. We show u is (α, β)-good. Let p be the nearest integer to $n\alpha$. Suppose $m_1 \neq p$, then $\partial R(h_{\alpha,\beta}u)$ contains $h_{\alpha,\beta}v$ where $v = (p, m_2, n)$; this is a contracdiction. Therefore, u is (α, β)-good. □

Definition 3.2. Let $u = (m_1, m_2, n) \in \mathbb{Z}^3$ for some positive n. We say u is a *convergent* of (α, β) if $h_{\alpha,\beta}u$ is a pivot of $\Lambda_{\alpha,\beta}$.

Before we go further with the notion of pivots, we see in the following the prominent role they play on the contemplation of the Littlewood Conjecture.

Definition 3.3. Let $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$ for some positive n. We say that the integer n is a *pivot denominator* of (α, β) if $h_{\alpha,\beta}u$ is a pivot of (α, β). We denote the pivot denominator by $|u|$.

Theorem 3.2. *Let $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Then*

$$\inf_{q \in \mathbb{Z}_+} q\|q\alpha\|\|q\beta\| = \inf_{n \in \pi(\alpha,\beta)} n\|n\alpha\|\|n\beta\| \tag{3.1}$$

where $\pi(\alpha, \beta)$ is the set of pivot denominators of (α, β).

Proof. Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$ and q be a positive integer. Suppose q is not a pivot denominator of (α, β). Let $v = (p_1, p_2, q)$ where p_1 is the nearest integer to $q\alpha$ and p_2 is the nearest integer to $q\beta$. Then the box $R(h_{\alpha,\beta}v)$ contains a point of $\Lambda_{\alpha,\beta} \setminus \{0\}$ that is not a corner. Since $|p_1 - q\alpha| = \|q\alpha\| > 0$ and $|p_2 - q\beta| = \|q\beta\| > 0$, the point must lie in the interior of R. Thus, $R(h_{\alpha,\beta}v)$ contains a pivot $h_{\alpha,\beta}u$, for some $u \in \mathbb{Z}^3 \setminus \{0\}$ with $|u| = n < q$, and hence $R(h_{\alpha,\beta}u) \subset R(h_{\alpha,\beta}v)$. Therefore, we have $n\|n\alpha\|\|n\beta\| < q\|q\alpha\|\|q\beta\|$ completing the proof. $\square$

Next, we examine some properties of pivots.

By the construction of $\Lambda_{\alpha,\beta}$, it is easy to see that the scaling by $g_{t,s}$ preserves pivots.

Proposition 3.3. *Let $(t,s) \in \mathbb{R}^2$ and $u \in \Lambda_{\alpha,\beta}$. If u is a pivot of $\Lambda_{\alpha,\beta}$, then $g_{t,s}u$ is a pivot of $g_{t,s}\Lambda_{\alpha,\beta}$.*

Proposition 3.4. *Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$ and $u \in \mathbb{Z}^3 \setminus \{0\}$ be a convergent with $|u| = n$. Then $n\|n\alpha\|\|n\beta\| < 1$.*

Proof. The box $R(h_{\alpha,\beta}u)$ is a symmetric convex subset of $\mathbb{R}^3$ having volume $8n\|n\alpha\|\|n\beta\|$. Since the interior of $R(h_{\alpha,\beta}u)$ contains no element of $\Lambda_{\alpha,\beta}$ different from zero, the Minkowski Convex Body Theorem implies $n\|n\alpha\|\|n\beta\| < 1$. $\square$

Definition 3.4. Let u and v be pivots of Λ. We say that the two pivots are *equivalent* if $R(u) = R(v)$.

Theorem 3.5. *Given $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. If u and v are convergents of (α,β) with $|u| = |v| = n$ for some positive integer n, then $h_{\alpha,\beta}u$ and $h_{\alpha,\beta}v$ are equivalent.*

Proof. Let $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Suppose $u = (m_1, m_2, n)$ and $v = (p_1, p_2, n)$ are convergents of (α, β), then $|m_1 - n\alpha| = |p_1 - n\alpha| = \|n\alpha\| > 0$ and $|m_2 - n\beta| = |p_2 - n\beta| = \|n\beta\| > 0$. Thus, $R(h_{\alpha,\beta}u) = R(h_{\alpha,\beta}v)$. $\square$

Proposition 3.6. *The standard basis vectors e_1 and e_2 in $\mathbb{R}^3$ are pivots of $\Lambda_{\alpha,\beta}$.*

Proof. Recall that $\Lambda_{\alpha,\beta}$ is the image of $\mathbb{Z}^3$ by the transformation $h_{\alpha,\beta}$ which is a shear parallel to the xy-plane. Then $R(e_1) \cap (\Lambda_{\alpha,\beta} \setminus \{0\}) \subset \partial'' R(e_1) = (\pm 1, 0, 0)$ and $R(e_2) \cap (\Lambda_{\alpha,\beta} \setminus \{0\}) \subset \partial'' R(e_2) = (0, \pm 1, 0)$. Hence, we see that e_1 and e_2 are pivots of $\Lambda_{\alpha,\beta}$. $\square$

Definition 3.5. A pivot of Λ is said to be *degenerate* if one or more of its coordinates vanishes.

Remark. Let $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. If u is a convergent of (α, β), then $h_{\alpha,\beta}u$ is a nondegenerate pivot of $\Lambda_{\alpha,\beta}$.

Theorem 3.7. *If $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$, then the only degenerate pivots of $\Lambda_{\alpha,\beta}$ are trivial, i.e., the vectors $\pm e_1$ and $\pm e_2$.*

Proof. Let $u = (m_1, m_2, n) \in \mathbb{Z}^3 \setminus \{0\}$ such that $h_{\alpha,\beta}u = (m_1 - n\alpha, m_2 - n\beta, n)$ is a degenerate pivot of $\Lambda_{\alpha,\beta}$. Then $(m_1 - n\alpha)(m_2 - n\beta)n = 0$. Since $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$, n must be zero. Thus, $u = (m_1, m_2, 0)$. Then $R(u) \cap (\Lambda_{\alpha,\beta} \setminus \{0\}) \subset \partial'' R = \{\pm|m_1|, \pm|m_2|, 0\}$. Hence by the construction of $\Lambda_{\alpha,\beta}$, $u = (\pm 1, 0, 0) = \pm e_1$ and $(0, \pm 1, 0) = \pm e_2$. $\square$

Definition 3.6. Let $\Lambda \subset \mathbb{R}^3$ be a nonempty lattice. The *systole* of Λ is defined to be

$$\operatorname{sys}(\Lambda) := \{u \in \Lambda : \|u\|_\infty = \|\Lambda\|\}$$

where $\|\Lambda\| := \inf\{\|u\|_\infty : u \in \Lambda, u \neq 0\}$ is the length of a shortest vector in Λ and $\|\cdot\|_\infty$ is the sup norm as previously defined. We refer to $\overline{B}(0, \|\Lambda\|) := [-\|\Lambda\|, \|\Lambda\|]^3$ as the *systole cube.*

Remark. The systole of Λ, $sys(\Lambda)$, is nonempty because $\Lambda \setminus \{0\}$ is closed.

Proposition 3.8. *Let $\Lambda \subset \mathbb{R}^3$ be a lattice. $sys(\Lambda)$ contains a pivot.*

Proof. We begin by noting that a vector in the systole has length $\|\Lambda\|$ with respect to the sup norm. Hence, it occupies at least one close face of the systole cube. Let $u \in \operatorname{sys}(\Lambda)$. Assume u occupies an open z-face. If u is the only occupant (up to equivalence), then $R(u) \cap (\Lambda \setminus \{0\}) \subseteq \partial'' R$ and we are done. So assume otherwise. Pick an occupant of an open z-face having the minimal magnitude of x-coordinate. If there is only one of such vector (up to equivalence), then again we are done.

So again, assume otherwise. Then pick the occupant with the minimal magnitude of y-coordinate. This vector is unique (up to equivalence) and is a pivot.

By a similar argument, we can track down a pivot starting with a vector in the systole occupying an open edge. And finally, if a vector u in the systole occupies a corner of the systole cube, then $R(u) = \bar{B}(0, \|\Lambda\|)$ and clearly it is a pivot. Therefore, $\mathrm{sys}(\Lambda)$ contains a pivot. □

Theorem 3.9. *Let $u \in \mathbb{Z}^3 \backslash \{0\}$. Then u is a convergent of (α, β) if and only if there is a neighborhood U around the centroid (t^*, s^*) of $\Delta(u)$ on which $W_{\alpha,\beta}(t,s) = f_u(t,s)$.*

Proof. Let $u \in \mathbb{Z}^3 \setminus \{0\}$. Suppose u is a convergent of (α, β). Then $g_{t^*,s^*} h_{\alpha,\beta} u$ is a nondegenerate pivot of $g_{t^*,s^*} \Lambda_{\alpha,\beta}$ by Proposition 3.3. By Proposition 2.2, $g_{t^*,s^*} h_{\alpha,\beta} u$ occupies an x-, a y-, and a z-face. Hence, the corners of the systole cube of $g_{t^*,s^*} \Lambda_{\alpha,\beta}$ and $\partial'' R(g_{t^*,s^*} h_{\alpha,\beta} u)$ coincide. Thus, the systole cube contains only one vector – $g_{t^*,s^*} h_{\alpha,\beta} u$ – up to equivalence. So, the discreteness of the lattice implies that there is $\epsilon_0 > 0$ such that whenever $\epsilon_0 \geq \epsilon > 0$, there exists $\delta_0 > 0$ such that $\bar{B}(0, \epsilon)$ contains only one vector $g_{t+\delta, s+\delta} h_{\alpha,\beta} u$ up to equivalence. Hence, there is a neighborhood U about (t^*, s^*) such that for all (t,s) in $U, W_{\alpha,\beta}(t,s) = f_u(t,s)$.

Conversely, suppose there is a neighborhood U about the centroid (t^*, s^*) of $\Delta(u)$ such that for all (t,s) in $U, f_u(t,s) = W_{\alpha,\beta}(t,s)$. The numbers $f_u(t,s)$ are bounded for (t,s) are ranging over the supposed neighborhood U. Thus, we see that $h_{\alpha,\beta} u$ cannot be a degenerate pivot. Now, we show that $h_{\alpha,\beta} u$ is a pivot of $\Lambda_{\alpha,\beta}$. Let $(t,s) \in U$. Then $f_u(t,s) = W_{\alpha,\beta}(t,s)$. Thus, $(t,s) \in \tau(u)$. So, by Theorem 2.6, $(t^*, s^*) \in \tau(u)$ and hence $f_u(t^*, s^*) = W_{\alpha,\beta}(t^*, s^*)$. Therefore, $g_{t^*,s^*} h_{\alpha,\beta} u$ is a pivot of $g_{t^*,s^*} \Lambda_{\alpha,\beta}$ implying u is a convergent of (α, β) by Proposition 3.3. □

4 Tilings of $\mathbb{R}^2$

In this section, we show that the tiles associated with two inequivalent pivots cover $\mathbb{R}^2$ and do not overlap. Thoughout this section, we assume $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$.

Define

$$\Pi(\alpha, \beta) := \{u \in \mathbb{Z}^3 \setminus \{0\} : h_{\alpha,\beta} u \text{ is a pivot of } \Lambda_{\alpha,\beta}\}.$$

Theorem 4.1. *If $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$, then $\bigcup_{\Pi(\alpha,\beta)} \tau(u) = \mathbb{R}^2$.*

Proof. Note that $\tau(u) \subset \mathbb{R}^2$ implies $\bigcup_{\Pi(\alpha,\beta)} \tau(u) \subset \mathbb{R}^2$. Now we show the containment in the other direction. Let $(t, s) \in \mathbb{R}^2$. By the Proposition 3.8, $sys(g_{t,s}\Lambda_{\alpha,\beta})$ contains a pivot of $g_{t,s}\Lambda_{\alpha,\beta}$ of the form $g_{t,s}h_{\alpha,\beta}u$ for some $u \in \mathbb{Z}^3 \setminus \{0\}$. Since $g_{t,s}h_{\alpha,\beta}u \in sys(g_{t,s}\Lambda_{\alpha,\beta})$, we have $f_u(t, s) = \log \|g_{t,s}h_{\alpha,\beta}u\|_\infty = \log \|g_{t,s}\Lambda_{\alpha,\beta}\| = W_{\alpha,\beta}(t, s)$. Hence, we see that $(t, s) \in \tau(u)$ and thus the result $\bigcup_{\Pi(\alpha,\beta)} \tau(u) \supset \mathbb{R}^2$. □

We have just shown that given $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$, the tiles associated to the inequivalent pivots of $\Lambda_{\alpha,\beta}$ cover the plane. Now, we show that any pair of such tiles do not overlap.

Let $u, v \in \mathbf{Z}^3 \setminus \{0\}$ such that $h_{\alpha,\beta}u$ and $h_{\alpha,\beta}v$ are nondegenerate pivots of $\Lambda_{\alpha,\beta}$. Define

$$f_{u,v}(t,s) := \min\{f_u(t,s), f_v(t,s)\} \tag{4.1}$$

and let

$$\Omega_u := \{(t,s) : f_{u,v}(t,s) = f_u(t,s)\}. \tag{4.2}$$

Ω_v is similarly defined.

Recall that if $\alpha, \beta \notin \mathbb{Q}$ and u is convergents of (α, β) with $|u| = n$, the centroid of $\Delta(u)$ is

$$(t^*, s^*) = \left(-\frac{1}{6}\log\frac{\|n\alpha\|\|n\beta\|}{n^2},\ \frac{1}{2}\log\frac{\|n\beta\|}{\|n\alpha\|}\right).$$

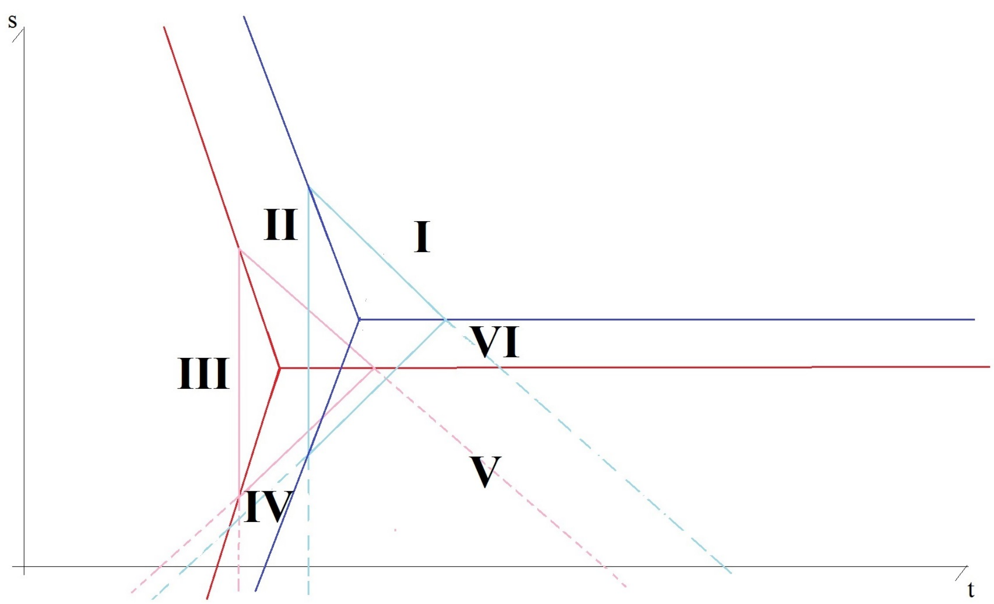
Lemma 4.2. *Given $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Let u, v be two convergents of (α, β). If $|u| < |v|$, then $\Delta(v) \nsubseteq \Delta(u)$.*

Proof. Assume $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Let u, v be two convergents of (α, β) with $n = |u| < |v| = q$. Then $h_{\alpha,\beta}u$ and $h_{\alpha,\beta}v$ are inequivalent. Thus $R(h_{\alpha,\beta}u) \neq R(h_{\alpha,\beta}v)$. Clearly, $h_{\alpha,\beta}u \notin R(h_{\alpha,\beta}v)$. Then $\alpha, \beta \notin \mathbb{Q}$ implies that either $\|n\alpha\| > \|q\alpha\|$ or $\|n\beta\| > \|q\beta\|$. By the construction of $\Delta(u)$ (see Figure 2), one easily sees that $\Delta(v) \nsubseteq \Delta(u)$. □

Lemma 4.3. *Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$, and u, v be two convergents of (α, β) with $|u| = n$ and $|v| = q$. If $n < q$, $\|n\alpha\| > \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$, then Ω_u and Ω_v do not overlap.*

Proof. Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$, and u, v be two convergents of (α, β) with $|u| = n$ and $|v| = q$. Referring to Figure 2, Figure 4 below demonstrates the partition of the ts-plane into six regions by the rays associated with $\Delta(u)$ and $\Delta(v)$ given $n < q$, $\|n\alpha\| > \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$. By region, we mean the set of points bounded together with its boundaries. In the following, we show that if $f_u = f_v$ occurs in a region, then such subset in the region is 1-dimensional.

Figure 4: The partition of the ts-plane into six regions by the rays associated with $\Delta(u)$ and $\Delta(v)$ given $n < q$, $\|n\alpha\| > \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$.



Region I : Region I is the intersection of the x-section of u and x-section of v. Let (t, s) be a point of the region, then we have $f_u(t, s) = P_x(t, s)$ (for u) $= t + s + \log \|n\alpha\|$ and $f_v(t, s) = P_x(t, s)$ (for v) $= t + s + \log \|q\alpha\|$. Since $0 < \|q\alpha\| < \|n\alpha\| < \frac{1}{2}$, we get $f_v(t, s) < f_u(t, s)$. Thus, Region I is a subset of $\Omega_v \setminus \Omega_u$.

Region III : Region III is the intersection of the z-section of u and z-section of v. Let (t, s) be a point of the region, then we have $f_u(t, s) = P_z(t, s) = -2t + \log n$ and $f_v(t, s) = P_z(t, s) = -2t + \log q$. Since $1 \leq n < q$, we get $f_u(t, s) < f_v(t, s)$. Thus, Region III is a subset of $\Omega_u \setminus \Omega_v$.

Region V : Region V is the intersection of the y-section of u and y-section of v. By a similar argument to Region I, it can be seen that $f_v(t, s) < f_u(t, s)$ for any point (t, s) in the region. Thus, Region V is a subset of $\Omega_v \setminus \Omega_u$.

Region VI : Region VI is the intersection of the x-section of u and y-section of v. Let (t, s) be a point of the region, then we have $f_u(t, s) = P_x(t, s) = t + s + \log \|n\alpha\|$ and $f_v(t, s) = P_y(t, s) = -t - s + \log \|q\beta\|$. Note that Region VI is a region bounded between the rays ℓ^+_{xy} of both u and v. Thus, we see that $\frac{1}{2} \log \frac{\|n\beta\|}{\|n\alpha\|} \leq s \leq \frac{1}{2} \log \frac{\|q\beta\|}{\|q\alpha\|}$ resulting $0 < \log \frac{\|n\beta\|}{\|q\beta\|} \leq f_u(t, s) - f_v(t, s) \leq \log \frac{\|n\alpha\|}{\|q\alpha\|}$. Hence, Region VI is a subset of $\Omega_v \setminus \Omega_u$.

Region II : Region II is the intersection of the x-section of u and z-section of v. Let (t, s) be a point of the region, then we have $f_u(t, s) = P_x(t, s) = t + s + \log \|n\alpha\|$ and $f_v(t, s) = P_z(t, s) = -2t + \log q$ leading to $f_u(t, s) - f_v(t, s) = 3t + s + \log \frac{\|n\alpha\|}{q}$. Hence, we see that Region II is divided into two parts by the line $s = -3t - \log \frac{\|n\alpha\|}{q}$ in which is a subset of $\Omega_u \setminus \Omega_v$ if $s < -3t - \log \frac{\|n\alpha\|}{q}$ and is a subset of $\Omega_v \setminus \Omega_u$ if $s > -3t - \log \frac{\|n\alpha\|}{q}$.

Region IV : By a similar argument for Region II, it can be seen that Region IV which is the intersection of the y-section of u and z-section of v is divided into two parts by the line $s = 3t + \log \frac{\|n\beta\|}{q}$ in which is a subset of $\Omega_u \setminus \Omega_v$ if $s > 3t + \log \frac{\|n\beta\|}{q}$ and is a subset of $\Omega_v \setminus \Omega_u$ if $s < 3t + \log \frac{\|n\beta\|}{q}$.

Note that the lines $s = -3t - \log \frac{\|n\alpha\|}{q}$ in Region II and $s = 3t + \log \frac{\|n\beta\|}{q}$ in Region IV intersect at the point on the common boundary of the two regions $s = \frac{1}{2} \log \frac{\|n\beta\|}{\|n\alpha\|}$, $(\frac{-1}{6} \log \frac{\|n\alpha\|\|q\beta\|}{q^2}, \frac{1}{2} \log \frac{\|n\beta\|}{\|n\alpha\|})$, which we shall call the *type*-0 *double point*. Concluding from the analysis above, we have

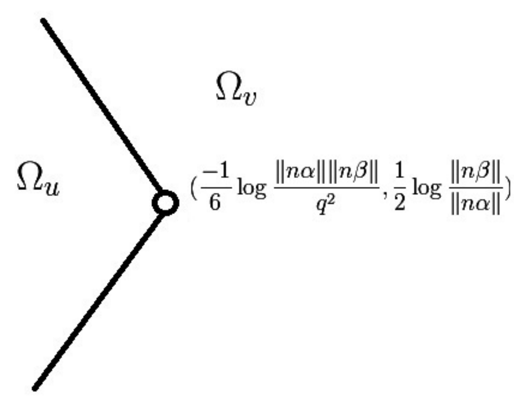
$$f_{u,v}(t,s) = f_u(t,s) \text{ if } \ 3t + \log\frac{\|n\beta\|}{q} < s < -3t - \log\frac{\|n\alpha\|}{q},$$

$$f_{u,v}(t,s) = f_u(t,s) = f_v(t,s) \text{ if } \ s = 3t + \log\frac{\|n\beta\|}{q} \text{ or } s < -3t - \log\frac{\|n\alpha\|}{q},$$

$$f_{u,v}(t,s) = f_v(t,s) \text{ otherwise},$$

completing the proof. □

Figure 5: The graph of the intersection of Ω_u and Ω_v given $n = |u| < |v| = q$, $\|n\alpha\| > \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$.



Lemma 4.4. *Let u, v be two convergents of (α, β) with $|u| = n$ and $|v| = q$. Assume $n < q$, $\|n\alpha| > \|q\alpha\|$ and $\|n\beta\| < \|q\beta\|$, then Ω_u and Ω_v do not overlap.*

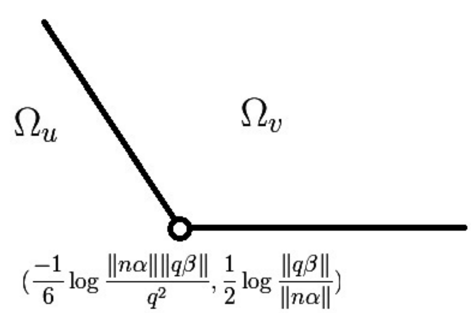
Proof. Let $\alpha \notin \mathbb{Q}$, $\beta \notin \mathbb{Q}$, and u, v be two convergents of (α, β) with $|u| = n$ and $|v| = q$. Suppose also $n < q$, $\|n\alpha\| > \|q\alpha\|$ and $\|n\beta\| < \|q\beta\|$, then by analyzing the regions corresponding to the hypotheses as we do in the previous proof, we find that the points in the intersection of Ω_u and Ω_v are precisely described by

$$s = -3t + \log q - \log\|n\alpha\| \text{ if } t \leq \frac{-1}{6}\log\frac{\|n\alpha\|\|q\beta\|}{q^2},$$

$$s = \frac{1}{2}\log\frac{\|q\beta\|}{\|n\alpha\|} \text{ otherwise,}$$

and the two lines intersect at the point $(\frac{-1}{6}\log\frac{\|n\alpha\|\|q\beta\|}{q^2}, \frac{1}{2}\log\frac{\|q\beta\|}{\|n\alpha\|})$, which we shall call the *type-+3 double point*. Moreover, we have the left side to the graph of the above piecewise linear functions on the ts-plane a subset of $\Omega_u \setminus \Omega_v$ and the right side to the graph of the above piecewise linear functions a subset of $\Omega_v \setminus \Omega_u$ and thus completing the proof. □

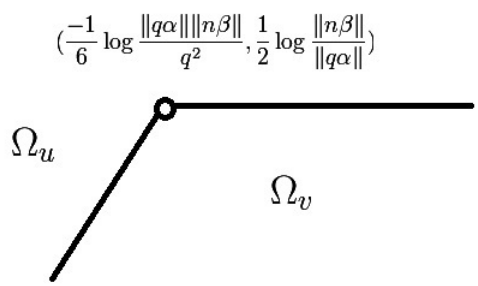
Figure 6: The graph of the intersection of Ω_u and Ω_v given $|u| < |v|$, $\|n\alpha\| > \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$.



By a very similar argument to Lemma 4.4, we have also:

Lemma 4.5. *Let u, v be two convergents of (α, β) with $|u| = n$ and $|v| = q$. Assume $n < q$, $\|n\alpha\| < \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$, then Ω_u and Ω_v do not overlap.*

Figure 7: The graph of the intersection of Ω_u and Ω_v given $|u| < |v|$, $\|n\alpha\| < \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$.



We shall call the intersection point of the two lines above the *type-* -3 *double point.*

Theorem 4.6. *Let u, v be two convergents of (α, β) with $|u| < |v|$. Then $\tau(u)$ and $\tau(v)$ do not overlap.*

Proof. Let u, v be two convergents of (α, β) with $|u| < |v|$. Then Lemma 4.2 implies that one of the three cases : case 1. $\|n\alpha\| > \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$, case 2. $\|n\alpha\| > \|q\alpha\|$ and $\|n\beta\| < \|q\beta\|$ and case 3. $\|n\alpha\| < \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$ must happen because $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Thus, Ω_u and Ω_v do not overlap by Lemma 4.3 , 4.4 and 4.5. Recalling the definition of τ, it is obvious that $\tau(u) \subset \Omega_u$ and $\tau(v) \subset \Omega_v$. Therefore, we have the result, $\tau(u)$ and $\tau(v)$ do not overlap, as desired. $\square$

5 Comparison of the Diameters of $\tau(u)$ and $\Delta(u)$

After we see that $\mathbb{R}^2$ is covered by non-overlapping tiles associated with the pivots of $\Lambda_{\alpha,\beta}$, in this section we show that the diameters of $\Delta(u)$ and $\tau(u)$ are comparable. Thoughout this section, we assume $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$.

Note that $\Delta(u)$ is nonempty for any convergent u of (α, β) with $n\|n\alpha\|\|n\beta\| < 1$. In the natural way we define the centroid of $\tau(u)$.

Definition 5.1. Let u be a convergent of (α, β). The *centroid* of $\tau(u)$ is defined to be the centroid of $\Delta(u)$.

Notation. In this section, we use the abbreviation $\hat{u}$ for $h_{\alpha,\beta}u$ for $u \in \Pi(\alpha, \beta)$ where

$$\Pi(\alpha, \beta) := \{u \in \mathbb{Z}^3 \setminus \{0\} : h_{\alpha,\beta}u \text{ is a pivot of } \Lambda_{\alpha,\beta}\}$$

is the set defined in the beginning of the previous section.

Definition 5.2. Let $\hat{u}$ and $\hat{v}$ be pivots of Λ. We say $\hat{v}$ is a *z-neighbor* of $\hat{u}$ if there is an origin-centered symmetric box B such that

1. $\Lambda \cap \text{int}(B) = \{0\}$,
2. $\hat{v}$ occupies an open z-face of B, and
3. $\hat{u}$ occupies an open z-edge of B.

The relations *x-neighbor* and *y-neighbor* are similarly defined.

Proposition 5.1. *Given $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$, every nondegenerate pivot has a neighbor of each kind.*

Proof. Let $\hat{u}$ be a nondegenerate pivot of Λ. Then $R(\hat{u})$ is not occupied in the open z-face. Extend $R(\hat{u})$ in the positive z-direction until a nonzero lattice point is in the open z-face. Call the result from the extension $R_z(\hat{u})$. By construction, the interior of $R_z(\hat{u})$ contains no nonzero lattice points. Note that $R_z(\hat{u})$ is finite because it has volume at most 8 by the Minkowski Convex Body Theorem. Let $\hat{v}$ be a nonzero lattice point in the open z-face with the minimal $|x|$- and $|y|$- coordinates. Then $\hat{v}$ is a pivot, $R_z(\hat{u})$ is a box such that $\hat{v}$ is in its open z-face, u is in its open z-edge, and $\text{int } B_z(\hat{u}) \cap \Lambda = \{0\}$. Therefore, $\hat{v}$ is a z-neighbor of $\hat{u}$. The argument for the existence of $\hat{u}$'s x- and a y- neighbors is similar. ☐

Proposition 5.2. *Let $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. If $\hat{u}$ is a pivot of $\Lambda_{\alpha,\beta}$, then the z-neighbor of $\hat{u}$ is unique up to equivalence.*

Proof. Let $\hat{u}$ be a pivot of $\Lambda_{\alpha,\beta}$ and $\hat{v}_1$ and $\hat{v}_2$ be two z-neighbors of $\hat{u}$ with $|v_1| > 0$ and $|v_2| > 0$. Then $|v_1| = |v_2|$ and thus $\hat{v}_1$ and $\hat{v}_2$ are equivalent by Theorem 3.5. ☐

Proposition 5.3. *Let $\hat{u}$ be a pivot of $\Lambda_{\alpha,\beta}$. If $\alpha \notin \mathbb{Q}$, then the x-neighbor of $\hat{u}$ is unique up to equivalence.*

Proof. Assume $\alpha \notin \mathbb{Q}$. Let $\hat{u}$ be a pivot of $\Lambda_{\alpha,\beta}$ and $\hat{v}_1$ and $\hat{v}_2$ with $|v_1| = q_1$ and $|v_2| = q_2$ be two x-neighbors of $\hat{u}$. Then, $\|q_1\alpha\| = \|q_2\alpha\|$. It follows $|q| = |q'|$, for otherwise $\|q\alpha\| = \|q'\alpha\|$ implies $\alpha \in \mathbb{Q}$ which is a contradiction. Therefore, $\hat{v}_1$ and $\hat{v}_2$ are equivalent by Theorem 3.5. ☐

Proposition 5.4. *Let $\hat{u}$ be a pivot of $\Lambda_{\alpha,\beta}$. If $\beta \notin \mathbb{Q}$, then the y-neighbor of $\hat{u}$ is unique up to equivalence.*

Let $\hat{u}$ be nondegenerate and $\hat{u_x}, \hat{u_y}$ and $\hat{v}$ be its x-, y- and z-neighbors, respectively. Assume $|u| = n$, $|v| = q$, $|u_x| = n_x$ and $|u_y| = n_y$ such that

$$\max(n_x, n_y) < n < q.$$

Note that the absolute x-coordinate of u_x is $\|n_x\alpha\|$ unless $n_x = 0$ in which case it is one. Similarly, the absolute y-coordinate of u_y is $\|n_y\alpha\|$ unless $n_y = 0$ in which case it is one.

Proposition 5.5. *The coordinates of the double point between $\tau(u)$ and $\tau(v)$ are given by*

$$\left(-\frac{1}{6}\log\frac{\|n\alpha\|\|n\beta\|}{q^2}, \frac{1}{2}\log\frac{\|n\beta\|}{\|n\alpha\|}\right). \tag{5.1}$$

Proof. From the proof for Proposition 5.1, we see that $n < q$, $\|n\alpha\| > \|q\alpha\|$ and $\|n\beta\| > \|q\beta\|$. Then referring to Lemma 4.3, we see that the double point between $\tau(u)$ and $\tau(v)$ is $(-\frac{1}{6}\log\frac{\|n\alpha\|\|n\beta\|}{q^2}, \frac{1}{2}\log\frac{\|n\beta\|}{\|n\alpha\|})$. □

Proposition 5.6. *The coordinates of the double point between $\tau(u)$ and $\tau(u_x)$ are given by*

$$\left(-\frac{1}{6}\log\frac{\|n_x\alpha\|\|n\beta\|}{n^2}, \frac{1}{2}\log\frac{\|n\beta\|}{\|n_x\alpha\|}\right) \tag{5.2}$$

unless $n_x = 0$ in which case the correct formula is obtained by replacing $\|n_x\alpha\|$ with 1.

Proof. Using a similar strategy in searching for the z-neighbor as in the proof for Proposition 5.1, we find the x-neighbor of $\hat{u}$ satisties the conditions : $n_x < n$, $\|n_x\alpha\| > \|n\alpha\|$ and $\|n_y\beta\| < \|n\beta\|$. Then referring to Lemma 4.4, we see that the double point between $\tau(u)$ and $\tau(u_x)$ is $(-\frac{1}{6}\log\frac{\|n_x\alpha\|\|n\beta\|}{n^2}, \frac{1}{2}\log\frac{\|n\beta\|}{\|n_x\alpha\|})$ unless $n_x = 0$ in which case the correct formula is obtained by replacing $\|n_x\alpha\|$ with 1. □

By a similar argument, we also find:

Proposition 5.7. *The coordinates of the double point between $\tau(u)$ and $\tau(u_y)$ are given by*

$$\left(-\frac{1}{6}\log\frac{\|n\alpha\|\|n_y\beta\|}{n^2}, \frac{1}{2}\log\frac{\|n_y\beta\|}{\|n\alpha\|}\right) \tag{5.3}$$

unless $n_y = 0$ in which case the correct formula is obtained by replacing $\|n_y\beta\|$ with 1.

Definition 5.3. The *spine* of $\tau(u)$, $\mathrm{spine}(u)$, is the union of the three line segments joining the centroid of $\tau(u)$ to the corresponding double points of its three neighbors.

Theorem 5.8. *Let $\alpha \notin \mathbb{Q}, \beta \notin \mathbb{Q}$ and u be a convergent of (α, β). Then $\mathrm{spine}(u)$ is a subset of $\tau(u)$.*

Proof. Let $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Without loss of generality, we assume $\hat{u}$ to be the nondegenerate pivot defined above. Let $\hat{w}$ be a neighbor of $\hat{u}$ and (t, s) be a point in the line segment joining the centroid (t^*, s^*) of $\tau(u)$ and the double point determined by u and w. We show that $(t, s) \in \tau(u)$. Suppose $\hat{w}$ is the z-neighbor of $\hat{u}$, then there is a box R containing $\hat{u}$ in an open z-edge and $\hat{w}$ in an open z-face. By Proposition 5.5, the point (t, s) lies in ray ℓ^+_{xy} (see (2.9) and figure 2). Thus, we see that $f_u(t, s) = f_w(t, s)$. Hence, the three coordinates of $g_{t,s}\hat{u}$ are equal. So, Proposition 3.3 implies that that $g_{t,s}\hat{u}$ is contained in the systole cube of $g_{t,s}\Lambda_{\alpha,\beta}$. Consequently, we have $(t, s) \in \tau(u)$. The results can easily be seen if we replace z-neighbor with x- and y-neighbors. Therefore, the set $\mathrm{spine}(u)$ is a subset of $\tau(u)$. □

Proposition 5.9. *The total length of* spine(u) *is at least*

$$\frac{1}{3}\log\frac{q\|n_x\alpha\|\|n_y\beta\|}{n\|n\alpha\|\|n\beta\|}. \tag{5.4}$$

Proof. Let u^* denote the centroid of $\tau(u)$. The total length of the spine of $\tau(u)$ is $\|u^*-u_x\|+\|u^*-u_y\|+\|u^*-v\|$ where $\|\centerdot\|$ is the oridinary Euclidean norm. By some simple calculations, we see $u^*-u_x=\frac{1}{6}\log\frac{\|n_x\alpha\|}{\|n\alpha\|}\langle 1,3\rangle$, $u^*-u_y=\frac{1}{6}\log\frac{\|n_y\beta\|}{\|n\beta\|}\langle 1,-3\rangle$ and $u^*-v=\frac{1}{3}\log\frac{q}{n}\langle 1,0\rangle$. Thus, the total length of the spine of $\tau(u)$ equals $\frac{\sqrt{10}}{6}\log\frac{\|n_x\alpha\|}{\|n\alpha\|}+\frac{\sqrt{10}}{6}\log\frac{\|n_y\beta\|}{\|n\beta\|}+\frac{1}{3}\log\frac{q}{n}\geq\frac{1}{3}\log\frac{q\|n_x\alpha\|\|n_y\beta\|}{n\|n\alpha\|\|n\beta\|}$. □

Lemma 5.10. *A plane in* $\mathbb{R}^3$ *that contains the origin and a corner of the standard unit cube, i.e., the unit ball with respect to the sup norm, is disjoint from a pair of open faces of the unit cube.*

Proof. By applying a map of the form $(x,y,z)\to(\pm x,\pm y,\pm z)$, we may reduce to the case when the plane is given by an equation $Ax+By+Cz=0$ where the coefficients A,B and C are nonnegative real numbers. The hypothesis that the plane contains a corner implies that one of the coefficients is the sum of the other two. Without loss of generality, we suppose $A=B+C$. Let (x,y,z) be a point contained in one of the two open x-faces. Then $|x|=1>\max(|y|,|z|)$ and by the triangle inequality

$$|Ax+By+Cz|\geq A|x|-B|y|-C|z|=B(1-|y|)+C(1-|z|)>0.$$

Hence, (x,y,z) does not lie on the plane. □

Lemma 5.11. *Given* $\{\hat{u}, \hat{u_x}, \hat{u_y}, \hat{v}\}$ *as defined above, the set* $\{\hat{u}, \hat{u_x}, \hat{u_y}, \hat{v}\}$ *contains* 3 *linearly independent vectors.*

Proof. Observe that if $\mathcal{P}$ is a plane in $\mathbf{R}^3$ containing the origin, a corner of $R(\hat{u})$ and the x-neighbor of $\hat{u}$, then $\mathcal{P} \cap \partial_x R(\hat{u})$ is nonempty. This observation is also true with the $y-$ and $z-$neighbor of $\hat{u}$, respectively. Let $\mathcal{P} = \mathrm{span}(\hat{u}, \hat{u_x}, \hat{u_y})$. If $\mathcal{P} = (R)^3$, we are done. So assume $\mathcal{P}$ has dimension 2. Since $\hat{u_x} \in \mathcal{P}$ and $\hat{u_y} \in \mathcal{P}$, $\mathcal{P} \cap \partial_x R(\hat{u})$ and $\mathcal{P} \cap \partial_y R(\hat{u})$ are nonempty. Then Lemma 5.10 implies that $\mathcal{P} \cap \partial_z R(\hat{u})$ is empty. Thus, $\hat{v} \notin \mathcal{P}$. Therefore, $\{\hat{u}, \hat{u_x}, \hat{u_y}, \hat{v}\}$ contains 3 linearly independent vectors. Obviously, the result holds also if $\mathcal{P} = span(\{\hat{u}, \hat{u_x}, \hat{v}\})$ or $\mathcal{P} = span(\{\hat{u}, \hat{u_y}, \hat{v}\})$.

Now, suppose $\mathcal{P} = span(\{\hat{u_x}, \hat{u_y}, \hat{v}\})$. Then $\dim \mathcal{P} > 1$. If $\dim \mathcal{P} = 3$, we are done. So assume $\dim \mathcal{P} = 2$. Then Lemma 5.10 implies that $\hat{u} \notin \mathcal{P}$. Hence, $\{\hat{u}, \hat{u_x}, \hat{u_y}, \hat{v}\}$ contains 3 linearly independent vectors. □

Lemma 5.12. *Let* u, v *and* w *be linearly independent vectors and* $S = \{\pm u, \pm v, \pm w\}$. *Define* $T(S) = \mathrm{conv}(S)$. *Then* $\mathrm{vol}(T(S)) \geq \frac{8}{6}$.

Proof. Let S_0 be the set consists of the elementary vectors in $\mathbf{Z}^3$ and their additive inverses. Then the volume of $T(S_0)$, $\mathrm{vol}(T(S_0))$, equals $\frac{8}{6}$. Now consider a linearly independent set $S = \{u, v, w\} \subset \mathbf{Z}^3$ and define the 3×3 integer matrix $A = [u \ v \ w]$. We see that $|\det(A)| \geq 1$. Therefore, we $\mathrm{vol}(T(S)) = |\det(A)| \, \mathrm{vol}(T(S_0)) \geq \frac{8}{6}$. □

Lemma 5.13. *Given* $\{\hat{u}, \hat{u_x}, \hat{u}_y, \hat{v}\}$ *as defined above, then* $q\|n_x\alpha\|\|n_y\beta\| \geq 1/6$.

Proof. Lemma 5.11 implies that $\{\hat{u}, \hat{u_x}, \hat{u_y}, \hat{v}\}$ contains 3 linearly independent vectors. If $\{\hat{u_x}, \hat{u_y}, \hat{v}\}$ is linearly independent, the result follows directly from Lemma 5.12. If $\{\hat{u}, \hat{u_x}, \hat{u_y}\}$ is linearly independent, then $n\|n_x\alpha\|\|n_y\beta\| \geq 1/6$ and since $n < q$, the result follows. If $\{\hat{u}, \hat{u_x}, \hat{v}\}$ is linearly independent then $q\|n_x\alpha\|\|n\beta\| \geq 1/6$ and the result follows since $\|n_y\beta\| > \|n\beta\|$. In the last case where $\{\hat{u}, \hat{u_y}, \hat{v}\}$ is linearly independent, $q\|n\alpha\|\|n_y\beta\| \geq 1/6$ and the result follows since $\|n_x\alpha\| > \|n\alpha\|$. □

Theorem 5.14. *For any convergent* u *of* (α, β), diam $\tau(u) \geq \frac{1}{9}(\textit{diam } \Delta(u) - \log 6)$.

Proof. By the construction of the spine of $\tau(u)$, we see that the diameter of $\tau(u)$ is at least the average length of the branches of the spine. Thus, together with the implication by Lemma 5.13, *diam* $\tau(u) \geq \frac{1}{3}(\frac{1}{3}\log\frac{q\|n_x\alpha\|\|n_y\beta\|}{n\|n\alpha\|\|n\beta\|}) = \frac{1}{9}(-\log n\|n\alpha\|\|n\beta\| + \log q\|n_x\alpha\|\|n_y\beta\|) \geq \frac{1}{9}(\textit{diam } \Delta(u) - \log 6)$ □

Substituting $\text{diam}(\Delta(u)) = -\log n\|n\alpha\|\|n\beta\|$ in (2.13) into the inequality given in Theorem 5.14,

Corollary. *Let u be a convergent of (α, β), if diam $\tau(u) \le M$ for some $M \in \mathbf{R}$, then*

$$n\|n\alpha\|\|n\beta\| > \frac{e^{-9M}}{6}. \tag{5.5}$$

Theorem 5.15. *Let $\alpha, \beta \in \mathbf{R}$ be two irrational numbers. There exists an $M \in \mathbf{R}$ such that $D_{\alpha,\beta} := \sup\{\text{diam}(\tau(u)) : h_{\alpha,\beta}u$ is a pivot of $\Lambda_{\alpha,\beta}\} \le M$ if and only if (α, β) is a counterexample to Littlewood's Conjecture.*

Proof. Let $\alpha, \beta \in \mathbf{R}$ be two irrational numbers. Suppose (α, β) is a counterexample to Littlewood's Conjecture. Then $\inf_{n \in \mathbf{Z}_+} n\|n\alpha\|\|n\beta\| \ge \delta_0$ for some $\delta_0 > 0$. Hence, $\text{diam}(\Delta(u) = -\log n\|n\alpha\|\|n\beta\| \le -\log \delta_0$ for all pivots $h_{\alpha,\beta}u$ of $\Lambda_{\alpha,\beta}$. Thus $D_{\alpha,\beta} \le -\log \delta_0$.

Conversely, suppose $D_{\alpha,\beta} \le M$. Then by the corollary to Theorem 5.14, $n\|n\alpha\|\|n\beta\| > \frac{1}{6}e^{-9M}$ for any pivot denominator. Suppose q is not a pivot denominator. Let $v = (p_1, p_2, q)$ where p_1 is the nearest integer to $n\alpha$ and p_2 is the nearest integer to $n\beta$. Then the box $R(h_{\alpha,\beta}v)$ contains a pivot u, with $|u| = n < q$, $\|n\alpha\| < |p_1 - q\alpha|$ and $\|n\beta\| < |p_2 - q\beta|$, so that $q|p_1 - q\alpha||p_2 - q\beta| > n\|n\alpha\|\|n\beta\| > \frac{1}{6}e^{-9M}$. Hence, this is a counterexample to the Littlewoord Conjecture. $\square$

6 Conclusion

Our project has come to an end. Having formulated a tiling problem on $\mathbb{R}^2$ that is equivalent to the Littlewood's conjecture and shown that the irrational pair (α, β) is a counterexample if and only if the diameters of the tiles are uniformly bounded, we have achieved our goal. We close our project here by posing a few problems for further research:

- Given an irrational pair (α, β), define the density of the set of pivot denominators by

$$\rho(\pi(\alpha, \beta)) := \lim_{n\to\infty} \frac{\#(\pi(\alpha, \beta) \cap [1, n])}{(\log n)^2}$$

  where # is denoted by the number of elements in the set. What is the density of the set of pivot denominators for (α, β)?

- Given an irrational pair (α, β) and $\epsilon > 0$, what is the probability that the centroids of two tiles associated to two consecutive indices are within distance ϵ?

- Is it possible to extend the idea to a higher dimension? And if it is, is there a similar interpretation of a counterexample of the generalized Littlewood's conjecture? Taking the 3-dimensional case for instance, does $\liminf n\|n\alpha\|\|n\beta\|\|n\gamma\| = 0$ have a corresponding tiling with a uniform bound on the nondegenerate tiles?

- If it is possible to extend the idea to a higher dimension, what is the maximum number of tiles that can be in contact with a given tile?

- And lastly, is there a way to generalize the density of the set of pivot denominators in the k-dimensional case?

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