DICHOTOMY FOR THE HAUSDORFF DIMENSION OF THE SET OF NONERGODIC DIRECTIONS

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ABSTRACT. Given an irrational $0 < \lambda < 1$, we consider billiards in the table P_λ formed by a $\frac{1}{2} \times 1$ rectangle with a horizontal barrier of length $\frac{1-\lambda}{2}$ with one end touching at the midpoint of a vertical side. Let $\mathrm{NE}(P_\lambda)$ be the set of θ such that the flow on P_λ in direction θ is not ergodic. We show that the Hausdorff dimension of $\mathrm{NE}(P_\lambda)$ can only take on the values 0 and $\frac{1}{2}$, depending on the summability of the series $\sum_k \frac{\log \log q_{k+1}}{q_k}$ where $\{q_k\}$ is the sequence of denominators of the continued fraction expansion of λ. More specifically, we prove that the Hausdorff dimension is $\frac{1}{2}$ if this series converges, and 0 otherwise. This extends earlier results of Boshernitzan and Cheung.

1. INTRODUCTION

In 1969, ([Ve1]) Veech found examples of skew products over a rotation of the circle that are minimal but not uniquely ergodic. These were turned into interval exchange transformations in [KN]. Masur and Smillie gave a geometric interpretation of these examples (see for instance [MT]) which may be described as follows. Let P_λ denote the billiard in a $\frac{1}{2} \times 1$ rectangle with a horizontal barrier of length $\alpha = \frac{1-\lambda}{2}$ based at the midpoint of a vertical side. There is a standard unfolding procedure which turns billiards in this polygon into flows along parallel lines on a translation surface. See Figure 1.

The associated translation surface in this case is a double cover of a standard flat torus of area one branched over two points z_0 and z_1 a horizontal distance λ apart on the flat torus. See Figure 1. We denote it by (X, ω).

The linear flows on this translation surface preserve Lebesgue measure. What Veech showed in these examples is that given θ with unbounded partial quotients in its continued fraction expansion, there is

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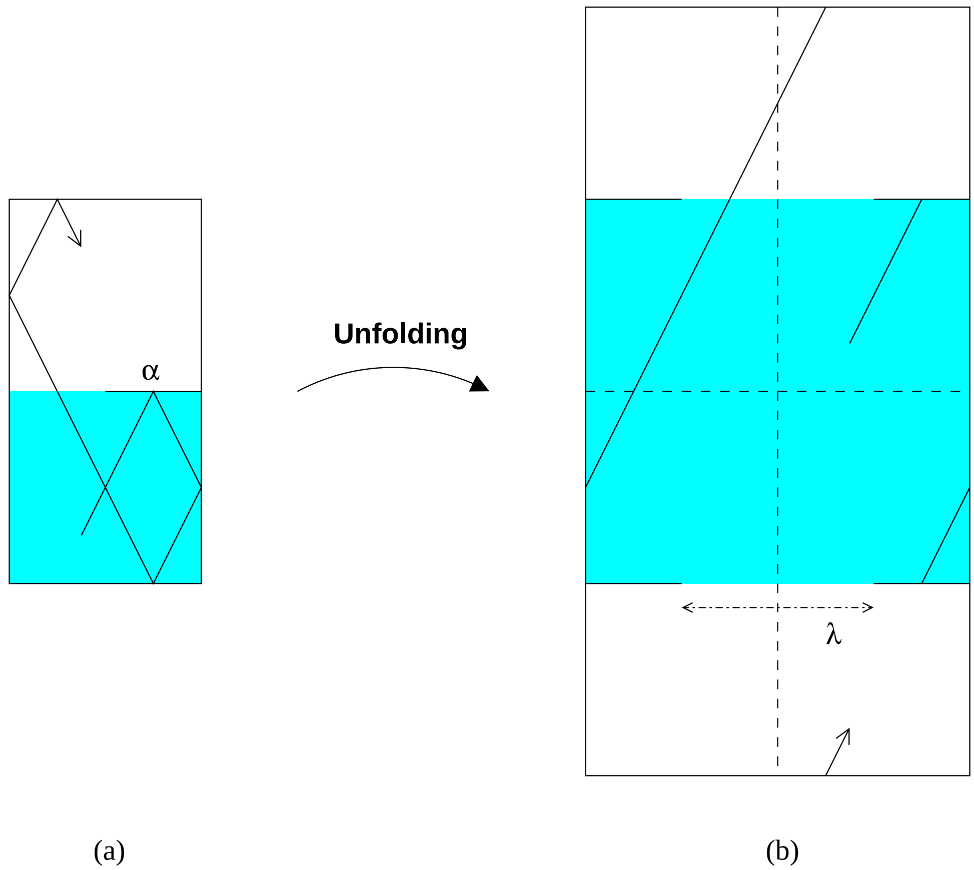

FIGURE 1. Unfolding the table P_λ.

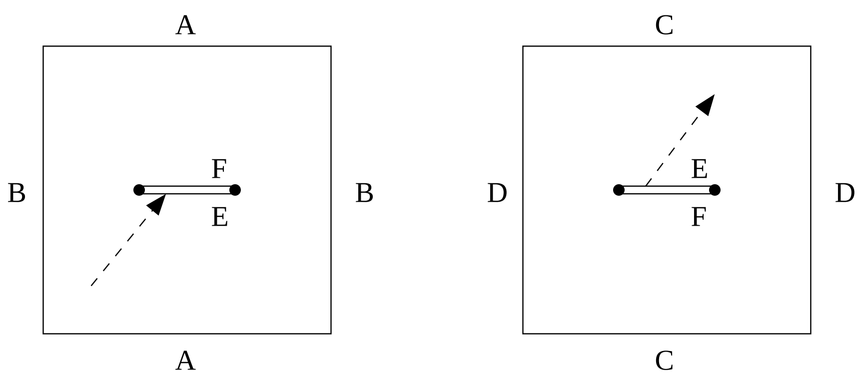

FIGURE 2. The branched double cover (X, ω).

a λ such that the flow on P_λ in direction with slope θ is minimal but not uniquely ergodic.

Let $\mathrm{NE}(P_\lambda)$ denote the set of nonergodic directions, i.e. those directions for which Lebesgue measure is not ergodic. It was shown in [MT] that $\mathrm{NE}(P_\lambda)$ is uncountable if λ is irrational. When λ is rational, a result of Veech ([Ve2]) implies that minimal directions are uniquely ergodic; thus $\mathrm{NE}(P_\lambda)$ is the set of rational directions and is countable. By a general result of Masur (see [Ma2]), the Hausdorff dimension of $\mathrm{NE}(P_\lambda)$ satisfies $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) \leq \frac{1}{2}$.

In [Ch1] Cheung proved that this estimate is sharp. He showed that if λ is *Diophantine*, then $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) \geq \frac{1}{2}$. Recall that λ is Diophantine if there is lower bound of the form

$$\left|\lambda - \frac{p}{q}\right| > \frac{c}{q^s}, \quad c > 0, s > 0$$

controlling how well λ can be approximated by rationals. This raises the question of the situation when λ is irrational but not Diophantine; namely, when λ is a Liouville number. Boshernitzan showed that $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) = 0$ for a residual (in particular, uncountable) set of λ (see the Appendix in [Ch1]) although it is not obvious how to exhibit a specific Liouville number in this set.

In this paper, we establish the following dichotomy:

Theorem 1.1. *Let $\{q_k\}$ be the sequence of denominators in the continued fraction expansion of λ. Then* $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) = 0$ *or* $\frac{1}{2}$*, the latter case occurring if and only if λ is irrational and*

$$\sum_k \frac{\log\log q_{k+1}}{q_k} < \infty. \tag{1}$$ [1]

We briefly outline the proof of Theorem 1.1, which naturally divides into two parts: an upper bound argument giving the dimension 0 result and a lower bound argument giving the dimension $\frac{1}{2}$ result. In §2 we discuss the geometry of the surface (X, ω) associated to P_λ; in particular, the ways it can be decomposed into tori glued together along *slits*. We call this a partition of the surface. The main object of study in both parts of the theorem concerns the summability of the areas of the changes of the partitions, expressed in terms (3) of the summability of the cross-product of the vectors of the slits.

[1] This condition on the denominators of the continued fraction appears in complex dynamics in the work of Pérez Marco ([PM]) in the context of the linearization problem and is commonly referred to as the Pérez Marco condition. An expository account of the history leading up to this work is given in [Mi].

1.1. **Sketch of dimension 0 case.** The starting point for the proof of Hausdorff dimension 0 in the case that

$$\sum_k \frac{\log\log q_{k+1}}{q_k} = \infty \tag{2}$$

is Theorem 4.1 from [CE]. That theorem asserts that to each nonergodic direction $\theta \in \mathrm{NE}(P_\lambda)$ there is an associated sequence of slits $\{w_j\}$ and loops $\{v_j\}$ whose directions converge to θ and satisfy the summability condition (3). The natural language to describe the manner by which a sequence of vectors is associated to a nonergodic direction is within the framework of Z-expansions.[2] (See §3.) Here, Z denotes a closed discrete subset of $\mathbb{R}^2$ satisfying some mild restrictions and in the case when Z is the set of primitive vectors in $\mathbb{Z}^2$ this notion reduces to continued fraction expansions. We also have the notion of *Liouville direction* (relative to Z) which intuitively refers to a direction that is extremely well approximated by the directions of vectors in Z. Under fairly general assumptions, which hold for example if Z is a set of holonomies of saddle connections on a translation surface, the set of Liouville directions has Hausdorff dimension zero. (Corollary 3.9) The proof of Hausdorff dimension 0 then reduces to showing that if λ satisfies (2), then every minimal nonergodic direction is Liouville with respect to the Z expansion. This is stated as Lemma 4.7.

For the proof of Lemma 4.7 the key ingedient is Lemma 4.6, which gives a *lower bound* on cross-products. It is based on the fact that $\frac{p_k+mq_k}{nq_k}$ will be an extremely good approximation to $\frac{\lambda+m}{n}$ provided the interval $[q_k, q_{k+1}]$ is large enough and also contains n not too close to q_{k+1}. (See Lemma 4.4.) This idea is motivated by the elementary fact that for any pair of vectors $w = (\frac{p}{q} + m, n)$ and $v = (m', n')$ where $m, n, m', n', p, q \in \mathbb{Z}$ with $q > 0$ we have

$$|w \times v| = \frac{|(p+m)n' - m'nq|}{q} \geq \frac{1}{q}$$

unless v, w are parallel to each other, in which case the cross-product vanishes.

We apply Lemma 4.6 to the sequence $\{w_j\}$ associated by Theorem 4.1 to a minimal nonergodic direction θ. If one assumes, by contradiction, that θ is *not* Liouville with respect to the Z-expansion, then Lemma 4.6 implies that

$$|w_j \times v_j| \geq \frac{1}{2q_k}$$

[2]This is more of a convenience than an essential tool.

whenever $|w_j|$ falls in a large interval $[q_k, q_{k+1}]$. Moreover, the number of such slits is at least a fixed constant times $\log\log q_{k+1}$. Thus the sum of the cross-products would be at least

$$\sum \frac{\log\log q_{k+1}}{q_k},$$

the sum over those k for which $[q_k, q_{k+1}]$ is large. Since (2) still holds if the sum is restricted to those k, the summable cross-products condition (3) would be contradicted. This will then show that θ is Liouville and we will conclude that $\mathrm{HDim}\,\mathrm{NE}(P_\lambda) = 0$.

1.2. **Sketch of dimension 1/2 case.** The starting point for the dimension $\frac{1}{2}$ argument is Theorem 2.9, which is the specialization of a result from [MS] to the case of (X, ω) that says the summability condition (3) is sufficient to guarantee that the limiting direction of a sequence of slit directions is a nonergodic direction.

One proceeds to construct a Cantor set of nonergodic directions arising as a limit of directions of slits on the torus. Aspects of this construction were already carried out in [Ch1] in the case that λ is Diophantine.

For $r > 1$, let $F(r)$ be the set of limiting directions obtained from sequences $\{w_j\}$ satisfying $|w_{j+1}| \approx |w_j|^r$. It was shown in [Ch1], under the assumption of Diophantine λ, that one can make the series in (3) be dominated by a geometric series of ratio $1/r$, and then $\mathrm{HDim}\,F(r) \geq \frac{1}{1+r}$. The lower bound $\frac{1}{2}$ then follows by taking the limit as r tends to one.

The strategy of bounding cross-products using a geometric series fails if only the weaker Diophantine condition (1) is assumed. In fact, in the large gaps $[q_k . q_{k+1}]$, as we have indicated, the cross-product is bounded below by $\frac{1}{2q_k}$. So if the gaps are large, (where the notion of "large" is to be made precise later) then there are many terms with cross-products bounded below by $\frac{1}{2q_k}$ and these terms would eventually become larger than the terms in the geometric series.

This suggests modifying the strategy in [Ch1] by replacing the geometric series used to dominate the series in (3) with a series whose terms δ_j are $O(1/q_k)$ if $|w_j|$ lies in a large interval $[q_k, q_{k+1}]$ and are otherwise decreasing like a geometric series of ratio $1/r$ for j such that $|w_j|$ lies between successive large intervals. The number of slits in $[q_k, q_{k+1}]$ is $O(\log_r \log q_{k+1})$ so that $\sum \delta_j$ restricted to those j for which $|w_j|$ lies in a large interval $[q_k, q_{k+1}]$ is bounded using the assumption (1). The sum of the remaining terms is bounded by the sum of a geometric series times $\sum_k \frac{1}{q_k}$. This latter sum is finite. The finiteness then of $\sum \delta_j$ and therefore (3) ensures that the resulting set $F(r) \subset \mathrm{NE}(P_\lambda)$.

Following [Ch1], we seek to build a tree of slits so that by associating intervals about the direction of each slit in the tree, we can give $F(r)$ the structure of a Cantor set to which standard techniques can be used to give lower estimates on Hausdorff dimension. These techniques require certain "local estimates" (expressed in terms of lower bounds on the number of subintervals and the size of gaps between them) hold at each stage of the construction. In §5, we express these local estimates in terms of the parameters r and δ_j.

For slits w whose lengths lie in a "small" interval $[q_k, q_{k+1}]$ we repeat the construction given in [Ch1] to construct "children" slits from "parent" slits. This is carried out in §7. In the current situation we have to combine that construction with a new one to deal with slits lengths that lie between consecutive q_k, q_{k+1} with large ratio. We call this the "Liouville" part of λ. The construction of new slits from old ones in that case is carried out in §6.

The construction of the tree of slits and the precise definition of the terms δ_j are given in §8 and §9. These sections are the most technical part of the paper. The main task is to ensure that the recursive procedure for constructing the tree of slits can be continued indefinitely while at the same time ensuring the required local estimates are satisfied in the case of our two constructions.

Finally, in §10, we verify that the series $\sum \delta_j$ is convergent and that the lower bound on $\operatorname{HDim} F(r)$ can be made arbitrarily close to $\frac{1}{2}$ by choosing the parameter r sufficiently close to one.

1.3. **Divergent geodesics.**

Finally we record the following by-product of our investigation. Associated to any translation surface (or more generally a holomorphic quadratic differential) is a Teichmüller geodesic. For each t the Riemann surface X_t along the geodesic is found by expanding along horizontal lines by a factor of e^t and contracting along vertical lines by e^t. It is known (see [Ma2]) that if the vertical foliation of the quadratic differential is nonergodic, then the associated Teichmüller geodesic is *divergent*, i.e. it eventually leaves every compact subset of the stratum.[3] The converse is however false. There are divergent geodesics for which the vertical foliation is uniquely ergodic. In fact, we have

Theorem 1.2. *Let* $\mathrm{DIV}(P_\lambda)$ *denote the set of divergent directions in* P_λ*, i.e. directions for which the associated Teichmüller geodesic leaves*

[3]In [Ma2], a stronger assertion was proved, namely the *projection* of the Teichmüller geodesic to the moduli space of Riemann surfaces is also divergent.

every compact subset of the stratum.[4] *Then* $\mathrm{HDim}\,\mathrm{DIV}(P_\lambda) = 0$ *or* $\frac{1}{2}$*, with the latter case occurring if and only if* λ *is irrational.*

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2. Loops, slits, and summable cross-products

In this section, we establish notation, study partitions of the surface associated to P_λ, and recall the summable cross-products condition (3) for detecting nonergodic directions.

Let $(T; z_0, z_1)$ denote the standard flat torus with two marked points. A *saddle connection* on T is a straight line that starts and ends in $\{z_0, z_1\}$ without meeting either point in its interior. By a *slit* we mean a saddle connection that joins z_0 and z_1, while a *loop* is a saddle connection that joins either one of these points to itself.

Holonomies of saddle connections will always be represented as a pair of real numbers. In particular,

$$\mathrm{hol}(\gamma_0) = (\lambda, 0)$$

where γ_0 is the horizontal slit joining z_0 to z_1. The set of holonomies of loops is given by

$$V_0 = \{(p, q) \in \mathbb{Z}^2 : \gcd(p, q) = 1\}.$$

Since λ is irrational, the set of holonomies of slits is given by

$$V_1 = V_1^+ \cup \left(-V_1^+\right)$$

where

$$V_1^+ = \{(\lambda + m, n) : m, n \in \mathbb{Z}^2, n > 0\} \cup \{(\lambda, 0)\}.$$

Note that V_0 and V_1 are disjoint and that V_1 is in one-to-one correspondence with the set of oriented slits. When we speak of "the slit w ..." we shall always mean the slit whose holonomy is w, while $w \in V_1^+$ specifies that the orientation is meant to be from z_0 to z_1. Also, each $v \in V_0$ corresponds to a pair of loops, one based at each branch point. The pair of cylinders in T bounded by these loops will be denoted by C_v^1, C_v^2. The core curves of these cylinders also have v as their holonomy.

Definition 2.1. Each slit γ has two lifts in (X, ω) whose union is a simple closed curve. We say γ is *separating* if this curve separates X

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[4]Theorem 1.2 remains valid if $\mathrm{DIV}(P_\lambda)$ is interpreted as the set of directions that are divergent in the sense described in the previous footnote.

into a pair of tori interchanged by the involution of the double cover.[5] We denote the slit tori by T_w^1, T_w^2 where $w = \mathrm{hol}(\gamma)$.

Lemma 2.2. ([Ch1]) *A slit w is separating if and only if $w = (\lambda+m, n)$ for some even integers m, n.*

The collection of separating slits have holonomies given by

$$V_2 = V_2^+ \cup \left(-V_2^+\right)$$

where

$$V_2^+ = \{(\lambda + 2m, 2n) : m, n \in \mathbb{Z}^2, n > 0\} \cup \{(\lambda, 0)\}.$$

The cross-product formula from vector calculus expresses the area of the parallelogram spanned by u and v as

$$|u \times v| = \|u\|\|v\| \sin\theta$$

where $\times$ denote the standard skew-symmetric bilinear form on $\mathbb{R}^2$, $\|\cdot\|$ the Euclidean norm, and θ the angle between u and v. It will be convenient to introduce the following.

Notation 2.3. The distance between the directions of $u, v \in \mathbb{R} \times \mathbb{R}_{>0}$, denoted by $\angle uv$, will be measured with respect to inverse slope coordinates. That is, $\angle uv$ is the absolute value of the difference between the reciprocals of their slopes. We have the folllowing analog of the cross-product formula

$$|u \times v| = |u|\ |v| \angle uv$$

where $|\cdot|$ denotes the absolute value of the y-coordinate.

Remark 2.4. For our purposes, the vectors we consider will always have directions close to some fixed direction and nothing essential is lost if one chooses to think of $|v|$ as the *length* of the vector v (or to think of $\angle uv$ as the angle between the vectors) for these notions differ by a ratio that is nearly constant. In fact, the notations $|v|$ and $\angle uv$ are intended to remind the reader of Euclidean lengths and angles, and in the discussions we shall sometimes refer to them as such. These nonstandard notions are particularly convenient in calculations as they allows us to avoid trivial approximations involving square roots and the sine function that would otherwise be unavoidable had we instead insisted on the Euclidean notions. As will become clear later, the benefits of the nonstandard notions will far outweigh the potential risks of confusion.

[5]This involution, which fixes each branch point, should not be confused with the hyperelliptic involution that interchanges the branch points and maps each slit torus to itself.

Lemma 2.5. *Let C_v^1, C_v^2 be the cylinders in T determined by $v \in V_0$. A slit w is contained in one of the cylinders C_v^i if and only if $|w \times v| < 1$.*

Proof. To prove necessity, we note that the area of the cylinder containing the slit is $|w \times v|$, which is < 1 since the complement has positive area. For sufficiency, let us first rotate the surface so that v is horizontal. If the slit were not contained in one of the cylinders, then the vertical component of w is a (strictly) positive linear combination of the heights h_1, h_2 of the rotated cylinders. However, the vertical component is given by

$$\frac{|w \times v|}{\|v\|} < \frac{1}{\|v\|} = h_1 + h_2$$

which is absurd. □

Definition 2.6. Let $w, w' \in V_1$ and $v \in V_0$. We shall say w and w' are "related by a Dehn twist about v" if they are contained in the same cylinder determined by v. If both lie in V_1^+ (or both in $-V_1^+$) then their holonomies are related by $w' = w + bv$ for some $b \in \mathbb{Z}$. In this case, we refer to $|b|$ as the *order* of the Dehn twist.

Lemma 2.7. *Let $w, w' \in V_1$ and $v \in V_0$. If $|w \times v| + |w' \times v| < 1$ then w and w' are related by a Dehn twist about v.*

Proof. Lemma 2.5 implies each of w and w' is contained in one of the cylinders C_v^1 and C_v^2 determined by v. If they belong to different cylinders, then the sum of the areas of the cylinders would be less than one, which is impossible. Hence, w and w' lie in the same cylinder and, therefore, they are related by a Dehn twist about v. □

Suppose w, w' are a pair of separating slits. Then we may measure the change in the partitions they determine by

$$\chi(w, w') := \text{area}(T_w^1 \Delta T_{w'}^1).$$

There is an ambiguity in this definition arising from the fact that we have not tried to distinguish between T_w^1 and T_w^2. Let us agree to always take the smaller of the two possibilities, which is at most one as their sum represents the area of (X, ω).

Lemma 2.8. *If w, w' are separating slits related by a Dehn twist about v then*

$$\chi(w, w') = |w' \times v| = |w \times v|.$$

Proof. Let C be the cylinder that contains both slits and let $b > 0$ be the order of the Dehn twist relating them. Note that b is even. The slits cross each other, each subdividing the other into b segments of equal length. The symmetric difference between the partitions is a finite union of parallelograms bounded by the lifts of w and w'. There are b parallelograms, each having area $\frac{1}{b^2}|w\times w'|$ and since $w' = w \pm bv$, we have $|w \times w'| = b|w \times v| = b|w' \times v|$, giving the lemma. □

Each separating slit determines a partition of (X, ω) into a pair of slit tori of equal area. The next theorem explains how nonergodic directions arise as certain limits of such partitions. It is a special case, adapted to branched double covers of tori, of a more general condition developed in [MS] that applies to arbitrary translation surfaces and quadratic differentials. We will use it in §10 to identify large subsets of $\mathrm{NE}(P_\lambda)$.

Theorem 2.9. *Let $\{w_j\}$ be a sequence of separating slits with increasing lengths $|w_j|$ and suppose that every consecutive pair of slits w_j and w_{j+1} are related by a Dehn twist about some v_j such that*

$$\sum_j |w_j \times v_j| < \infty. \tag{3}$$

Then the inverse slopes of w_j converge to some θ and this limiting direction belongs to $\mathrm{NE}(P_\lambda)$.

Proof. Since $|w_{j+1}| > |w_j|$, we have $|v_j| \geq 1$ so that

$$\angle w_j w_{j+1} \leq \frac{|w_j \times v_j|}{|w_j||v_j|} + \frac{|v_j \times w_{j+1}|}{|v_j||w_{j+1}|} \leq \frac{2|w_j \times v_j|}{|w_j|}$$

from which the existence of the limit θ follows. Let μ be the normalised area measure on (X, ω) and let h_j be the component of w_j orthogonal to $w_\infty = (\theta, 1)$. Theorem 2.1 in [MS] asserts that θ is a nonergodic direction if the following conditions hold:

(i) $\lim h_j = 0$,
(ii) $0 < c < \mu(T^1_{w_j}) < c' < 1$ for some constants c, c', and
(iii) $\sum \chi(w_j, w_{j+1}) < \infty$.

Since $\mu(T^1_{w_j}) = \frac{1}{2}$, (ii) is clear, while (iii) is a consequence of (3), by Lemma 2.8. It remains to verify (i), but this follows easily from

$$\angle w_j w_\infty \leq \sum_{i\geq j} \angle w_i w_{i+1} = \sum_{i\geq j} \frac{2|w_i \times v_i|}{|w_i||w_{i+1}|} \leq \sum_{i\geq j} \frac{2|w_i \times v_i|}{|w_j|}$$

since then $h_j \leq |w_j| \angle w_j w_\infty \leq \sum_{i\geq j} 2|w_i \times v_i|$ so that $h_j \to 0$, by (3). □

The converse to Theorem 2.9 also holds. That is, to each nonergodic direction θ one can associate a sequence of slits (w_j) whose directions converge to θ and such that all the hypotheses of Theorem 2.9 hold. The definition of this sequence will be explained next.

3. Z-expansions, Liouville directions

In this section we introduce Z-expansions and use them to define the notion of a Liouville direction relative to a closed discrete subset $Z \subset \mathbb{R}^2$. Under fairly general assumptions on Z, the set of Liouville directions is shown to have Hausdorff dimension zero.

Notation 3.1. Given an inverse slope θ and $v = (p, q) \in \mathbb{R}^2$ we define

$$\mathrm{hor}_\theta(v) = |q\theta - p|$$

which we shall refer to as the "horizontal component" of v in the direction θ. It represents the absolute value of the x-coordinate of the vector $h_\theta v$ where $h_\theta = \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix}$ is the horizontal shear that sends the direction of θ to the vertical.

Definition 3.2. Let Z be a closed discrete subset of $\mathbb{R}^2$ and θ an inverse slope. A Z-*convergent* of θ is any vector $v \in Z$ that minimizes the expression $\mathrm{hor}_\theta(u)$ among all vectors $u \in Z$ with $|u| \leq |v|$. Recall that $|v|$ is the absolute value of the y-coordinate. We call it the *height* of v.[6] Thus, Z-convergents are those vectors in Z that minimize horizontal components among all vectors in Z of equal of lesser height. The Z-*expansion* of θ is defined to be the sequence of Z-convergents ordered by increasing height. If two or more Z-convergents have the same height we choose one and ignore the others.

Note that by definition the sequence of heights of Z-expansion is *strictly* increasing and, as a consequence, the sequence of horizontal components is strictly decreasing–if $|v| < |v'|$ then $\mathrm{hor}_\theta(v)$ must be greater than $\mathrm{hor}_\theta(v)$, for otherwise v' would not qualify as a Z-convergent.

In the case when Z is the set of primitive vectors in $\mathbb{Z}^2$, i.e. $Z = V_0$, the notion of a Z-convergent reduces to the notion from continued fraction theory. That is, $v = (p, q)$ is a Z-convergent of θ if and only if p/q is a convergent of θ in the usual sense.[7]A generalisation to higher

[6]The height of a rational is the smallest positive integer that multiplies it into the integers. A rational represented in lowest terms by p/q can be identified with $v = (p, q) \in \mathbb{Z}^2$, so that the height of the vector v coincides with the height of the rational.

[7]There is a trivial exception in the case when θ has fractional part strictly between $\frac{1}{2}$ and 1: the integer part of θ is the *zeroth order* convergent of θ in the usual sense, but nevertheless fails to be a Z-convergent.

dimensions (where Z is the set of primitive vectors in $\mathbb{Z}^n$ for $n > 2$) is given in [Ch3].

Obviously, we should always assume Z does not contain the origin, for otherwise the zero vector is the only convergent, independent of θ. Let us also assume that Z contains some nonzero vector on the x-axis, for this ensures that the heights of Z-expansions are well-ordered. Indeed if $(x, 0)$ is a Z-convergent, then all Z-convergents lie in an infinite parallel strip of width $2x$ about the direction of θ. Since the set of Z-convergents forms a closed discrete subset of this strip, there is no accumulation point. Hence, if there are infinitely many Z-convergents, their heights increase towards infinity.

One last assumption we shall impose is the finiteness of the "Minkowski" constant:

$$\mu(Z) := \frac{1}{4} \sup_K \operatorname{area}(K) < \infty \tag{4}$$

where the supremum is taken over all bounded, 0-symmetric convex regions disjoint from Z. Any direction which is not the direction of a vector in Z will be called *minimal* (relative to Z).

Lemma 3.3. *Assume (4) and that Z contains a non-zero vector on the x-axis. Then the Z-expansion of a direction with inverse slope θ is infinite if and only if θ is minimal.*

Proof. If the Z-expansion is finite, take the last convergent. If it does not lie in the direction of θ, then there is an infinite parallel strip containing the origin with one side the direction of θ containing no points of Z, but this is ruled out by (4). Hence, its direction is θ, so θ is not minimal. Conversely, if θ is not minimal, then there is a vector in Z in the direction of θ and it is necessarily a convergent and no other convergent can beat it, so it is the last one in the Z-expansion. There is also a first convergent; it lies on the x-axis. Let x the horizontal component of the first convergent and y the height of the last convergent. The compact region

$$P_\theta(x, y) = \{v \in \mathbb{R}^2 : \operatorname{hor}_\theta(v) \leq x, |v| \leq y\} \tag{5}$$

contains all the Z-convergents. Since Z is closed, it is compact; by discreteness, it is finite. □

Note that the Z-expansion are defined for all directions except the horizontal. In the sequel, we shall always assume the hypotheses of Lemma 3.3 remain in force.

Notation 3.4. If θ is an inverse slope and u a non-horizontal vector then we shall often write $\angle u\theta$ for the absolute difference between the directions. That is,

$$\angle u\theta = \angle uv = \frac{|u \times v|}{|u||v|}$$

for any vector v whose inverse slope is θ. Similarly, the notation $|u \times \theta|$ will be used to mean

$$|u \times \theta| = \frac{|u \times v|}{|v|} = |u \times v_\theta|$$

where $v_\theta = (\theta, 1)$.

Theorem 3.5. *The sequence of Z-convergents of θ satisfies*[8]

$$\frac{|v_k \times v_{k+1}|}{2|v_k||v_{k+1}|} < \angle v_k\theta \leq \frac{\mu(Z)}{|v_k||v_{k+1}|}. \tag{6}$$

Proof. Consider the parallelogram $P = P(x_k, y_{k+1})$ defined by (5) where $x_k = \mathrm{hor}_\theta(v_k)$ and $y_{k+1} = |v_{k+1}|$. The base is $2|v_k \times \theta|$ and the height is $2|v_{k+1}|$. By definition of v_{k+1}, the interior of P is disjoint from Z so that (4) implies

$$|v_k \times \theta||v_{k+1}| \leq \mu(Z)$$

giving the right hand inequality in (6). Since

$$\angle v_{k+1}\theta = \frac{|v_{k+1} \times \theta|}{|v_{k+1}|} < \frac{|v_k \times \theta|}{|v_k|} = \angle v_k\theta$$

we have $\angle v_k v_{k+1} < 2\angle v_k\theta$, giving the left hand inequality in (6). □

3.1. **Liouville directions.** Recall that an irrational number is Diophantine iff the sequence of denominators of its convergents satisfies $q_{k+1} = O(q_k^N)$ for some N. Otherwise, it is Liouville. This motivates our next definition.

Definition 3.6. We say a minimal direction is *Diophantine* relative to Z if its Z-expansion satisfies

$$|v_{k+1}| = O\left(|v_k|^N\right) \tag{7}$$

for some N. Otherwise, it is *Liouville* relative to Z.

Note that we have a trichotomy: every direction is either Diophantine, Liouville or not minimal, relative to Z.

Definition 3.7. We say Z has *polynomial growth* of rate (at most) d if

$$\#(Z \cap B_R) = O(R^d)$$

where B_R denotes the ball of radius R about the origin.

[8]The notation $\angle v\theta$, as in (6), means $\angle vw$ for any w whose inverse slope is θ.

Lemma 3.8. *Let E_r be the set of (inverse slopes of) directions θ whose Z-expansions satisfy*

$$|v_{k+1}| > |v_k|^r$$

for infinitely many k. If Z has polynomial growth of rate d, then

$$\text{HDim}\, E_r \leq \frac{d}{1+r}.$$

Proof. It is enough to bound the Hausdorff dimension of the set $E_r' = E_r \cap [a, a+1]$ for some arbitrary but fixed $a \in \mathbb{R}$. Let Z_k be the set of $v \in Z$ that arise as Z-convergents of some direction whose inverse slope lies in $[a, a+1]$ and such that

$$2^k \leq |v| < 2^{k+1}.$$

Then Z_k is contained in some ball of radius $2^k R_0$ where R_0 is a constant depending only on a. Let $I(v)$ be the closed interval of length $\frac{2\mu(Z)}{|v|^{1+r}}$ centered about the inverse slope of v. Then Theorem 3.5 implies every $\theta \in E_r'$ is contained in $I(v)$ for infinitely many $v \in \bigcup_k Z_k$. For any k_0 let

$$Z_{k_0}' = \bigcup_{k \geq k_0} Z_k.$$

Then given $\varepsilon > 0$ we can choose k_0 large enough so that $\{I(v) : v \in Z_{k_0}'\}$ is an ε-cover of E_r'. Since the number of elements in Z_k is bounded by

$$\# Z_k \leq C R_0^d 2^{kd}$$

for some $C > 0$ we have

$$\sum_{v \in Z_{k_0}'} |I(v)|^s \leq \sum_{k \geq k_0} \frac{2^s \mu(Z)^s C R_0^d 2^{kd}}{2^{k(1+r)s}}$$

so that the s-dimensional Hausdorff measure is finite for any $s > \frac{d}{1+r}$. This shows $\text{HDim}\, E_r' \leq \frac{d}{1+r}$, from which the lemma follows. □

By [Ma1] (see also [EM], [Vo]) the set of holonomies of saddle connections on any translation surface satisfies a quadratic growth rate.

Corollary 3.9. *The set of Liouville directions relative to the set of holonomies of saddle connections on a translation surface has Hausdorff dimension zero.*

4. Hausdorff dimension 0

In this section we assume the denominators of the convergents of λ satisfy (2) and set

$$Z = V_0 \cup V_2.$$

(Recall the sets V_0 and V_2 were defined in §2.)

We shall need the following characterisation of nonergodic directions in terms of Z-expansions.

Theorem 4.1. *([CE]) Let θ be a minimal*[9] *direction in P_λ. Then θ is nonergodic if and only if its Z-expansion is eventually alternating between loops and separating slits*

$$\ldots, v_{j-1}, w_j, v_j, w_{j+1}, \ldots$$

and satisfies the summable cross-products condition (3).

Our goal is to show that $\operatorname{HDim} \operatorname{NE}(P_\lambda) = 0$ under the assumption (2). By Corollary 3.9, it is enough to show that every minimal nonergodic direction is Liouville relative to Z.

Note that the sufficiency in Theorem 4.1 follows from Theorem 2.9 since the heights of Z-convergents increase and as soon as $|w_{j+1} \times v_j| = |w_j \times v_j| < \frac{1}{2}$ then w_j and w_{j+1} are related by a Dehn twist about v_j, by Lemma 2.7. The main point of Theorem 4.1 is that the converse also holds.

Observe that our main task has been reduced to a question about the set of possible limits for the directions of certain sequences of vectors in Z.

In the sequel we shall need the following two standard facts from the theory of continued fractions.

Theorem 4.2. *([Kh, Thm. 9 and 13]) The sequence of convergents of a real number θ satisfies*

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| \theta - \frac{p_k}{q_k} \right| \leq \frac{1}{q_k q_{k+1}}. \tag{8}$$

Theorem 4.3. *([Kh, Thm. 19]) If a reduced fraction satisfies*

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{2q^2} \tag{9}$$

then it is a convergent of θ.

---

[9]This implies it will also be a minimal direction relative to Z.

4.1. **Liouville convergents.** The next lemma shows that convergents of λ with $q_{k+1} \gg q_k$ give rise to convergents of $\frac{\lambda+m}{n}$.

Lemma 4.4. *Let $w = (\lambda + m, n)$ be a slit and $\frac{p_k}{q_k}$ a convergent of λ such that*

$$|w| = n < \frac{q_{k+1}}{2q_k}. \tag{10}$$

Let $\frac{p}{q}$ denote the fraction $\frac{p_k+mq_k}{nq_k}$ in lowest terms. Then $\frac{p}{q}$ is a convergent of $\frac{\lambda+m}{n}$ and its height satisfies $q_k \leq q \leq |w|q_k$. Furthermore, the height q' of the next convergent of $\frac{\lambda+m}{n}$ is larger than $\frac{q_{k+1}}{2}$.

Proof. Using the right hand side of (8) and (10) we get

$$\left| \frac{\lambda + m}{n} - \frac{p_k + mq_k}{nq_k} \right| < \frac{1}{|w|q_k q_{k+1}} < \frac{1}{2n^2 q_k^2} \tag{11}$$

which implies that $\frac{p}{q}$ is a convergent of $\frac{\lambda+m}{n}$. Clearly, $q \leq |n|q_k = |w|q_k$ and since $\gcd(p_k, q_k) = 1$, n is divisible by $\gcd(p_k + mq_k, nq_k)$ so that $q \geq q_k$. Let q' be the height of the next convergent of $\frac{\lambda+m}{n}$. From the first inequalities in (8) and in (11) we get

$$\frac{1}{2qq'} < \left| \frac{\lambda + m}{n} - \frac{p}{q} \right| < \frac{1}{|w|q_k q_{k+1}}$$

so that

$$q' > \frac{|w|q_k q_{k+1}}{2q} \geq \frac{q_{k+1}}{2}.$$

□

Definition 4.5. When the conclusion of Lemma 4.4 holds, we refer to $\frac{p}{q}$ (or the vector $v = (p, q)$) as the *Liouville convergent* of w indexed by k. (We shall often blur the distinction between the rational $\frac{p}{q}$ and the vector v.)

The terminology of Liouville convergent is justified by the sequel both in the dimension 0 result and in the dimension 1/2 result. In the next lemma we show that if w', w have their lengths in a range defined by the convergents of λ and are related by a twist about a loop v, then if v is not the Liouville convergent of w, the area interchange determined by w, w' will be large. If v is the Liouville convergent, then the next slit after w' will not be in the range. The summability condition on area exchanges will then imply that there cannot be too many slit lengths in the Liouville part of λ (in the range where q_{k+1}/q_k is large). Consequently the lengths of the slits must grow quickly and we can find covers of the nonergodic set that allow us to prove Hausdorff dimension

0 using Lemma 3.8. In §6 we will use Liouville convergents to build new children slits out of parent slits.

Lemma 4.6. *Let w, w' be slits such that w, w' are related by a Dehn twist about $v \in V_0$ and $|w \times v| < \frac{1}{2}$. Suppose further that $|w| < |w'| < \frac{q_{k+1}}{2q_k}$ and let u be the Liouville convergent of w indexed by k. Regarding u as a vector, then either*

(i) *$v \neq u$ and*

$$|w \times v| > \frac{1}{2q_k}, \tag{12}$$

or

(ii) *$v = u$ and for any $v' \in \mathbb{Z}^2 \setminus \mathbb{Z}v$ satisfying $|w' \times v'| < \frac{1}{2}$ we have $|v'| > \frac{q_{k+1}}{4}$.*

Proof. We have $w' = w + bv$ for some nonzero, even integer b, so that

$$|v| = \frac{|w' - w|}{|b|} \leq \frac{|w'| + |w|}{2} < |w'|. \tag{13}$$

Let α' be the inverse slope of w'. Let $v = (p, q)$. Then

$$\left|\alpha' - \frac{p}{q}\right| = \frac{|w' \times v|}{|w'||v|} < \frac{|w \times v|}{|v|^2} < \frac{1}{2q^2}$$

so that $\frac{p}{q}$ is a convergent of α', by (9). Let q' be the height of the next convergent of α'. Then (8) implies

$$\frac{1}{2qq'} < \left|\alpha' - \frac{p}{q}\right| < \frac{1}{qq'}$$

so that

$$\frac{|w'|}{2|w \times v|} = \frac{|w'||v|}{2q|w' \times v|} < q' < \frac{|w'|}{|w \times v|}.$$

The Liouville convergent $u = (m, n)$ cannot have its height $n < q$ because Lemma 4.4 implies the height n' of the next convergent of α' is greater than $\frac{q_{k+1}}{2} > |w'| > |v| = q$, contradicting the fact that q is the height of a convergent of α', namely $\frac{p}{q}$. Thus, $|u| \geq |v|$.

In case (i), $|u| > |v|$ so that $|u| \geq q' > \frac{|w'|}{2|w \times v|}$. Since $|u| \leq |w'|q_k$, the inequality (12) follows.

In case (ii), we have $q' = n' > \frac{q_{k+1}}{2}$, as noted earlier. Given $v' \in \mathbb{Z}^2 \setminus \mathbb{Z}v$, we have

$$1 \le |u \times v'| = |u||v'| \left(\angle w'u + \frac{|w' \times v'|}{|w'||v'|} \right) \le \frac{|v'|}{q'} + \frac{|v|}{2|w'|} < \frac{|v'|}{q'} + \frac{1}{2}$$

from which it follows that $|v'| > \frac{q'}{2} > \frac{q_{k+1}}{4}$. □

The Hausdorff dimension 0 result now follows from

Lemma 4.7. *Assume*

$$\sum_k \frac{\log\log q_{k+1}}{q_k} = \infty$$

holds. Then any minimal $\theta \in \mathrm{NE}(P_\lambda)$ *is Liouville relative to* Z*.*

Proof. Let $n_k > 1$ be defined by $q_{k+1} = q_k^{n_k}$; in other words,

$$n_k = \log_{q_k} q_{k+1} = \frac{\log q_{k+1}}{\log q_k}.$$

Note that since q_k grows exponentially, we have

$$\sum_{n_k \le N} \frac{\log\log q_{k+1}}{q_k} \le \sum_{n_k \le N} \frac{\log N + \log\log q_k}{q_k} < \infty$$

for any $N > 0$. Hence, (2) implies n_k is unbounded; moreover, the series in (2) diverges even if we restrict to terms with $n_k > N$.

Let $\theta \in \mathrm{NE}(P_\lambda)$ be a minimal direction for the flow. Then it is minimal relative to Z and by Theorem 4.1 its Z-expansion eventually alternates $\ldots, w_j, v_j, w_{j+1}, \ldots$ between (separating) slits and loops such that (3) holds. Let J_k be the collection of indices j such that

$$q_k \le |w_j| < |w_{j+1}| < |w_{j+2}| < q_k^{n_k - 2} < \frac{q_{k+1}}{4q_k}.$$

For any $j \in J_k$ we wish to prove that conclusion (i) of Lemma 4.6 holds. Suppose by way of contradiction conclusion (ii) holds so that v_j is the Liouville convergent of w_j indexed by k. Setting $v' = v_{j+1}$ by conclusion (ii) we have $|w_{j+2}| > |v_{j+1}| > \frac{q_{k+1}}{4}$, a contradiction. Thus (i) holds and therefore $|w_j \times v_j| > \frac{1}{2q_k}$.

Suppose θ is Diophantine relative to Z. Then there exists N such that $|w_{j+1}| < |w_j|^N$ for all j. Hence, $|w_j| < |w_0|^{N^j}$ and since

$$\log_N \log_{|w_0|} q^a = \frac{1}{\log N}(\log a + \log\log q - \log|w_0|)$$

we see that the number of j such that $|w_j|$ lies in an interval of the form $[q^a, q^b]$ is at least $\lfloor \log_N(b/a) \rfloor$. It follows that the number of elements in J_k is at least

$$\log_N(n_k - 2) - 3 > \frac{\log n_k}{2\log N} = \frac{\log\log q_{k+1} - \log\log q_k}{2\log N}$$

provided $n_k > N_0$ for some N_0 depending only on N. Since $\sum \frac{\log\log q_k}{q_k} < \infty$ (as heights of convergents grow exponentially) we have

$$\sum_{n_k > N_0} \sum_{j \in J_k} |w_j \times v_j| > \sum_{n_k > N'} \frac{\log\log q_{k+1} - \log\log q_k}{2(\log N) q_k} = \infty$$

which contradicts (3). Hence, θ must be Liouville relative to Z, proving the lemma. □

5. Cantor set construction

We begin the proof of the Hausdorff dimension 1/2 result. To construct nonergodic directions, we use Theorem 2.9. The general idea is as follows. Starting with an initial slit w_0 we will construct a tree of slits. At level j we will have a collection of slits of approximately the same length. For each w in this collection we wish to construct new slits of level $j+1$ each having small cross-product with w. Depending on the relationship of the length of w to the continued fraction expansion of λ, as specified precisely in §8, the construction will be one of two types that will be explained in §6 and §7.

In this section, we associate to this tree of slits a Cantor set. For each j we will define a set F_j which is a disjoint union of intervals. The directions of each slit of level j will lie in some interval in F_j and the intervals at level j will be separated by gaps. The intervals of level $j+1$ will be nested in the intervals of level j. Each nonergodic direction corresponds to a nested intersection of these intervals.

We shall assume the tree of slits satisfy certain assumptions, to be verified later in §9 and §10. These assumptions, expressed in terms of parameters $r > 1$, $\delta_j > 0$ and $\rho_j > 0$, ensure that certain lower bounds on the Hausdorff dimension of the Cantor set will hold.

5.1. **Local Hausdorff dimensions.** To establish lower bounds for Hausdorff dimension we will use an estimate of Falconer [Fa] which we explain next. Let

$$F = \bigcap_{j \geq 0} F_j$$

where each F_j is a finite disjoint union of closed intervals and $F_{j+1} \subset F_j$ for all j. Suppose there are sequences $m_j \geq 2$ and $\varepsilon_j \searrow 0$ such that each interval of F_j contains at least m_j intervals of F_{j+1} and the smallest gap between any two intervals of F_{j+1} is at least ε_j. (Note that $m_j \geq 2$ implies there will always be at least one gap.) Then Falconer's lower bound estimate is

$$\operatorname{HDim} F \geq \liminf_j \frac{\log(m_0 \cdots m_j)}{-\log m_{j+1}\varepsilon_{j+1}}.$$

If $\lim_{j\to\infty} m_j\varepsilon_j = 0$, as is necessarily the case if the length of the longest interval in F_j tends to zero as $j \to \infty$, then

$$\operatorname{HDim} F \geq \liminf_j d_j$$

where

$$d_j := \frac{\log m_j}{-\log \frac{m_{j+1}\varepsilon_{j+1}}{m_j\varepsilon_j}}. \tag{14}$$

Our goal is that for each $\varepsilon > 0$, we make a construction of a Cantor set of nonergodic directions so that each d_j will satisfy

$$d_j > \frac{1}{2} - \varepsilon.$$

5.2. **The parameters r, δ_j, and ρ_j.** Given $r > 1$ and a sequence of positive $\delta_j \to 0$ (which will measure the area interchange defined by consecutive slits), we shall construct a Cantor set F depending on parameters m_j and ε_j that are expressible in terms of r and δ_j. It is based on the assumption, verified later, that we can construct a tree of slits. We start with an initial slit w_0, the unique slit of level 0. Inductively, given a slit w_j of level j we consider slits of the form $w_j + 2v_j$ where $v_j \in \mathbb{Z}^2$ is a primitive vector, i.e. $\gcd(v_j) = 1$, and satisfies

$$|w_j \times v_j| < \delta_j, \quad |w_j|^r \leq |v_j| \leq 2|w_j|^r.$$

We refer to $w_{j+1} = w_j + 2v_j$ of the above form as a *child* of w_j. It satisfies

$$|w_j|^r \leq |w_{j+1}| \leq 5|w_j|^r. \tag{15}$$

The main difficulty in the construction is avoiding slits that have no children at all. To ensure that we can avoid such slits, we shall only use children with "nice Diophantine properties" when we assemble the slits for the next level. However, we shall ensure that at each stage, the number of children (of a parent slit w) used will be at least

$$\rho_j|w|^{r-1}\delta_j \tag{16}$$

where ρ_j is to be determined later.

For w a slit, let $I(w)$ denote the interval of length

$$\operatorname{diam} I(w) = \frac{4}{|w|^{r+1}}$$

centered about the inverse slope of the direction of w. The following lemma allows us to find estimates for the sizes of intervals and the gaps between them.

Lemma 5.1. *Assume $|w_0|^{r(r-1)} \geq 64$ and $\delta_j < \frac{1}{16}$. Let w_{j+1} be a child of a slit w_j of level j. Then*

- *$I(w_{j+1}) \subset I(w_j)$, and*
- *if w'_{j+1} is another child of w_j, then*

$$\operatorname{dist}(I(w_{j+1}), I(w'_{j+1})) \geq \frac{1}{16|w_j|^{2r}}.$$

Proof. Since the distance between the directions of w_j and w_{j+1} is

$$\angle w_j w_{j+1} = \frac{|w_j \times w_{j+1}|}{|w_j||w_{j+1}|} \leq \frac{|w_j \times v_j|}{|w_j||v_j|} < \frac{1}{|w_j|^{r+1}}$$

the first conclusion follows from

$$\frac{1}{|w_j|^{r+1}} + \frac{2}{|w_j|^{r(r+1)}} \leq \frac{2}{|w_j|^{r+1}}$$

which holds easily by the assumption on $|w_0|$.

The distance between the directions of w_{j+1} and v_j is

$$\angle w_{j+1} v_j = \frac{|w_{j+1} \times v_j|}{|w_{j+1}||v_j|} \leq \frac{|w_j \times v_j|}{|w_j|^{2r}} < \frac{\delta_j}{|w_j|^{2r}}.$$

If $w'_{j+1} = w_j + 2v'_j$ is another child of w_j then

$$\angle v_j v'_j = \frac{|v_j \times v'_j|}{|v_j||v'_j|} \geq \frac{1}{4|w_j|^{2r}}$$

so that by the triangle inequality,

$$\angle w_{j+1} w'_{j+1} \geq \frac{1}{4|w_j|^{2r}} - \frac{\delta_j + \delta_{j+1}}{|w_j|^{2r}} \geq \frac{1}{8|w_j|^{2r}}$$

since $\sup \delta_j < \frac{1}{16}$. Therefore,

$$\operatorname{dist}(I(w_{j+1}), I(w'_{j+1})) \geq \frac{1}{8|w_j|^{2r}} - \frac{4}{|w_j|^{r(r+1)}} \geq \frac{1}{16|w_j|^{2r}}$$

since $|w_0|^{r(r-1)} \geq 64$. □

Let

$$F_j = \bigcup_w I(w)$$

where the union is taken over all slits of level j. From (15) we have

$$|w_0|^{r^j} \le |w_j| \le 5^{\frac{r^j-1}{r-1}} |w_0|^{r^j}, \tag{17}$$

so that the number of children given by (16) is at least

$$m_j := \rho_j \delta_j |w_0|^{r^j(r-1)} \tag{18}$$

while the smallest gap between the associated intervals is at least

$$\varepsilon_j := \frac{1}{16 \cdot 5^{2r\frac{r^j-1}{r-1}} |w_0|^{2r^{j+1}}},$$

by Lemma 5.1.

Now we express d_j, given by (14), in terms of r, δ_j and ρ_j. We have

$$m_j \varepsilon_j = \frac{\rho_j \delta_j}{16 \cdot 5^{2r\frac{r^j-1}{r-1}} |w_0|^{r^j(r+1)}}$$

so that

$$\frac{m_{j+1}\varepsilon_{j+1}}{m_j \varepsilon_j} = \frac{\rho_{j+1}\delta_{j+1}/\rho_j\delta_j}{5^{2r^{j+1}} |w_0|^{r^j(r^2-1)}}$$

giving

$$\begin{aligned} d_j &= \frac{r^j(r-1)\log|w_0| + \log(\rho_j\delta_j)}{r^j(r^2-1)\log|w_0| + 2r^{j+1}\log 5 - \log(\rho_{j+1}\delta_{j+1}/\rho_j\delta_j)} \\ &= \frac{1 - \frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|}}{1 + r + \frac{2r\log 5}{(r-1)\log|w_0|} + \frac{\log(\rho_j\delta_j/\rho_{j+1}\delta_{j+1})}{r^j(r-1)\log|w_0|}}. \end{aligned} \tag{19}$$

Now making d_j close to $\frac{1}{2}$ will mean making r close to 1 and making the terms

$$\frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|} \tag{20}$$

and

$$\frac{2r\log 5}{(r-1)\log|w_0|} + \frac{\log(\rho_j\delta_j/\rho_{j+1}\delta_{j+1})}{r^j(r-1)\log|w_0|} \tag{21}$$

small. Notice that if ρ_j and δ_j are constant sequences, then this is easily accomplished by choosing $|w_0|$ large enough. In §9 we shall show that δ_j and ρ_j can be chosen so that (16) is satisfied at each step of the construction. The conditions $\delta_j < \frac{1}{16}$, as required by Lemma 5.1, and $m_j \ge 2$, as required by Falconer's estimate, will be verified in §9 along with the fact that $|w_0|$ can be chosen large enough to ensure that d_j is close to $\frac{1}{2}$.

6. Liouville construction

The slits of the next level will be constructed from the previous level using one of two constructions. The first construction we call the *Liouville construction* as it uses the Liouville convergents of λ directly to identify new slits. The second construction, introduced in [Ch1], is different. We call it the *Diophantine construction*. It does not use directly the convergents of λ, but rather employs a technique to count lattice points in certain strips.

In this section, we begin with the Liouville construction as it is perhaps the main one of the paper. The Diophantine construction will be explained in §7.

Recall that the Liouville convergent of a slit $w = (\lambda + m, n)$ indexed by k is the vector $u \in \mathbb{Z}^2$ determined by

$$(p_k + mq_k, nq_k) = du, \quad \gcd(u) = 1 \tag{22}$$

where

$$d = d(w, k) = \gcd(p_k + mq_k, nq_k).$$

Note that the height of the Liouville convergent satisfies

$$d|u| = |w|q_k.$$

Choose $\tilde{u} \in \mathbb{Z} \times \mathbb{Z}_{>0}$ so that

$$|u \times \tilde{u}| = 1 \quad \text{and} \quad |\tilde{u}| \leq |u|.$$

Observe that there are exactly 2 possibilites for $\tilde{u}$.

Let

$$\Lambda_1(w, k) = \{w + 2v : v = \tilde{u} + au, a \in \mathbb{Z}_{>0}\}$$

consist of children $w + 2v$ such that v forms a basis for $\mathbb{Z}^2$ together with u, i.e. $\mathbb{Z}^2 = \mathbb{Z}u + \mathbb{Z}v$.

The next lemma gives a bound on the cross-product of a parent with a child, which recall, is a necessary estimate in the construction of nonergodic directions.

Lemma 6.1. *If $w + 2v \in \Lambda_1(w, k)$ for some $|v| < q_{k+1}$ then*

$$|w \times v| < \frac{2|w|}{|u|} = \frac{2d(w, k)}{q_k} \tag{23}$$

where u is the Liouville convergent of w indexed by k.

Proof. From (11) we have

$$\angle uw \le \frac{1}{|w|q_k q_{k+1}}.$$

Since $|v| < q_{k+1}$, $|u \times v| = 1$ and $|u| \le |w|q_k$ we have

$$\angle uv = \frac{|u \times v|}{|u||v|} > \frac{1}{|u|q_{k+1}} \ge \frac{1}{|w|q_k q_{k+1}}$$

so that $\angle vw \le \angle uv + \angle uw < 2\angle uv$. Therefore,

$$|w \times v| = |w||v|\angle vw < 2|w||v|\angle uv = \frac{2|w|}{|u|}.$$

□

The next lemma expresses the key property of slits constructed via the Liouville construction. Note that $d(w,k)$ measures how far $\frac{p+mq}{nq}$ is from being a reduced fraction; namely, it is the amount of cancellation between the numerator and denominator. Since $\gcd(p,q) = 1$ (and $n = |w|$), it is easy to see that $d(w,k) \le |w|$. It is quite surprising that whenever a new slit w' is constructed via the Liouville construction, we have $d(w',k) \le 2$.

Lemma 6.2. *For any $w' \in \Lambda_1(w,k)$, we have $d(w',k) \le 2$. Hence, if $|w'| < \frac{q_{k+1}}{2q_k}$, then the inverse slope of w' has a convergent whose height is either $q_k|w'|$ or $q_k|w'|/2$.*

Proof. Let $w' = (\lambda + m', n')$ where

$$(m',n') - (m,n) = w' - w = 2v.$$

Now $d' = d(w',k)$ is determined by $d'u' = (p_k + m'q_k, n'q_k)$ for some primitive $u' \in \mathbb{Z}^2$. In terms of the basis given by u and $\tilde{u}$ we have

$$d'u' = (p_k + mq_k, nq_k) + 2q_k(\tilde{u} + au) = (2q_k)\tilde{u} + (2aq_k + d)u.$$

Note that $d = \gcd(p_k + mq_k, nq_k)$ is not divisible by any divisor of q_k, since $\gcd(p_k, q_k) = 1$. Therefore,

$$d' = \gcd(2aq_k + d, 2q_k) = \gcd(d, 2q_k) = \gcd(d,2) \le 2.$$

The second statement follows from Lemma 4.4. □

Given $r > 1$ we let

$$\Lambda(w,k) = \{w + 2v \in \Lambda_1(w,k) : |w|^r \le |v| \le 2|w|^r\}.$$

The next lemma gives a lower bound for the number of children constructed in the Liouville construction.

Lemma 6.3. *If* $|w|^{r-1} \geq q_k$ *then*

$$\#\Lambda(w,k) \geq \frac{|w|^{r-1}}{q_k}. \tag{24}$$

Proof. Since there are 2 choices for $\tilde{u}$ the number of slits in $\Lambda(w,k)$ is at least

$$\#\Lambda(w,k) \geq 2\left[\frac{|w|^r}{|u|}\right] \geq \frac{|w|^r}{|u|} \geq \frac{|w|^{r-1}}{q_k}$$

where $|u| \leq |w|q_k \leq |w|^r$ was used in the last two inequalities. □

7. Diophantine construction

Now we explain our next general construction, which is accomplished by Proposition 7.11. Many of the ideas in this section already appeared in [Ch1].

Again given a parent slit w we will construct new slits of the form $w+2v$, where v is a loop satisfying certain conditions on its length and cross-product with w. Not all of these solutions $w+2v$ will be used at the next level for it may happen that some of these will not themselves determine enough further slits. In other words, we will only use some of the slits $w+2v$ of the parent w and the ones used will be called the *children* of w. It will be encumbent to show that there are enough children at each stage in order to obtain lower bounds on the Hausdorff dimension of the Cantor set of §5.

7.1. **Good slits.**

Assume parameters $1 < \alpha < \beta$ be given. In later sections they will each have a dependence on the slit so they are not to be thought of as absolute constants.

Definition 7.1. We say a slit w is (α,β)-*good* if its inverse slope has a convergent of height q satisfying $\alpha|w| \leq q \leq \beta|w|$.

Let $\Delta(w,\alpha,\beta)$ be the collection of slits of the form $w+2v$ where $v \in \mathbb{Z}\times\mathbb{Z}_{>0}$ satisfies $\gcd(v) = 1$ and

$$\beta|w| \leq |v| \leq 2\beta|w| \quad \text{and} \quad \frac{1}{\beta} < |w \times v| < \frac{1}{\alpha}. \tag{25}$$

Notice the right hand inequality gives an upper bound for the cross product of w with $w+2v$. The next lemma gives a lower bound for the number of such $w+2v$ constructed from good slits w.

Lemma 7.2. *There is a universal constant* $0 < c_0 < 1$ *such that*

$$\#\Delta(w,\alpha,\beta) \geq \frac{c_0\beta}{\alpha}. \tag{26}$$

for any (α,β)*-good slit* w *and* $\alpha < c_0\beta$.

Proof. By [Ch1,Thm.3], the number of primitive vectors satisfying

$$\beta|w| \le |v| \le 2\beta|w| \quad \text{and} \quad |w \times v| < \frac{1}{\alpha}. \tag{27}$$

is at least $c_0'\beta/\alpha$ where $c_0' > 0$ is some universal constant.[10] The angle, by which we mean the distance between inverse slopes, between any two solutions $v, \hat{v}$ to (27) is at least

$$\left|\frac{p}{q} - \frac{\hat{p}}{\hat{q}}\right| \ge \frac{1}{q\hat{q}} \ge \frac{1}{4\beta^2|w|^2}.$$

Take an interval J of length $\frac{2}{\beta^2|w|^2}$ centered at the inverse slope of w and divide it into 8 equal subintervals. The inequality above says that there is at most one solution v whose inverse slope lies in each subinterval. Thus, by discarding at most 8 of these solutions, namely those with inverse slopes in J, we can ensure that the remaining solutions satisfy

$$\frac{|w \times v|}{|w||v|} > \frac{1}{\beta^2|w|^2}.$$

These solutions satisfy (25) since

$$|w \times v| > \frac{|v|}{\beta^2|w|} \ge \frac{1}{\beta}.$$

Let $c_0 = c_0'/9$. We may clearly assume $c_0' < 9$ so that $c_0 < 1$. Since $\alpha < c_0\beta$, there are at least $c_0'\beta/\alpha > 9$ solutions to (27). Of these, at least one satisfies (25). Therefore, the number of primitive vectors satisfying (25) is at least

$$\frac{c_0'\beta}{\alpha} - 8 \ge \left(9c_0 - 8\frac{\alpha}{\beta}\right)\frac{\beta}{\alpha} \ge \frac{c_0\beta}{\alpha}.$$

□

Lemma 7.3. *Let w be an (α,β)-good slit. Then every $w' \in \Delta(w,\alpha,\beta)$ is $(\alpha - \frac{1}{2}, \beta)$-good, but not $(1, \alpha - \frac{1}{2})$-good.*

Proof. Let $w' = w + 2v \in \Delta(w,\alpha,\beta)$. Note that v is a convergent of (the inverse slope of) w' since, writing $w' = (\lambda + m', n')$ and $v = (p,q)$, we have

$$\left|\frac{\lambda + m'}{n'} - \frac{p}{q}\right| = \frac{|w' \times v|}{|w'||v|} < \frac{|w \times v|}{2|v|^2}\frac{1}{2\alpha q^2} < \frac{1}{2q^2}$$

and we can use (9).

[10] To apply [Ch1,Thm.3] one needs to assume $\beta \gg \alpha$, but this hypothesis was shown to be redundant in [Ch2]. Indeed, by [Ch2,Thm.4] we can take $c_0' = \frac{4}{27\pi}$.

Let q' be the height of the next convergent of w'. Then by (8)

$$\frac{1}{q(q'+q)} < \left|\frac{\lambda + m'}{n'} - \frac{p}{q}\right| < \frac{1}{qq'}$$

so that

$$\frac{1}{q'+q} < \frac{|w \times v|}{|w'|} < \frac{1}{q'}. \tag{28}$$

From the left hand side above, the fact that $|w'| > 2|v| = 2q$ and $|w \times v| < \frac{1}{\alpha}$, we have

$$q' > \frac{|w'|}{|w \times v|} - q > (\alpha - \frac{1}{2})|w'|.$$

Now, from the right hand side of (28), we have

$$q' < \frac{|w'|}{|w \times v|} < \beta|w'|.$$

This shows that w' is $(\alpha - \frac{1}{2}, \beta)$-good.

Since q and q' are the heights of consecutive convergents of w' (and since $|v| < |w'|$) it follows that w' is not $(1, \alpha - \frac{1}{2})$-good. □

7.2. **Normal slits.** In this subsection, we assume $N > 0$ is fixed and set

$$\ell_N = \{k : q_{k+1} > q_k^N\}. \tag{29}$$

The choice of the parameter N will depend on considerations in §8 and will be specified there, by (40).

Given $N > 0$, we set

$$N' = \frac{(N+1)r}{r-1}. \tag{30}$$

It will also be convienent to set

$$\rho = r + 1/2.$$

Definition 7.4. A slit w is α-normal if it is $(\alpha\rho^t, |w|^{(r-1)t})$-good for all $t \in [1, T]$ where $T > 1$ is determined by $\alpha\rho^T = |w|^{r-1}$. Equivalently, w is α-normal if and only if for all $t \in [1, T]$ we have

$$\Psi(w) \cap [\alpha\rho^t|w|, |w|^{1+(r-1)t}] \neq \emptyset. \tag{31}$$

where $\Psi(w)$ denotes the collection of heights of the convergents of the inverse slope of w.

The following gives a sufficient condition for a slit to be normal.

Lemma 7.5. *Let w be a slit such that*

$$q_{k+1}^{1/N} \leq |w| < q_{k'}^{1/r}$$

where k, k' are consecutive elements of ℓ_N for some $N > 0$. If w is $(\alpha\rho^{N'}, |w|^{r-1})$-good then it is α-normal.

Proof. Suppose on the contrary that w is $(\alpha\rho^{N'}, |w|^{r-1})$-good but not α-normal.[11] Let $\frac{p}{q}$ be the convergent of the inverse slope of w with maximal height $q \leq |w|^r$. Since w is $(\alpha\rho^{N'}, |w|^{r-1})$-good, we have

$$\alpha\rho^{N'}|w| \leq q \leq |w|^r.$$

Let q' the height of the next convergent. If $q' \leq |w|^{1+(r-1)N'}$ then (31) is satisfied by q for all $t \in [1, N']$, and by q' for all $t \in [N', T]$. Since w is not α-normal we must have

$$q' > |w|^{1+(r-1)N'}.$$

Note that

$$\frac{q'}{|w|} > |w|^{(r-1)N'} \geq q^{(1-r^{-1})N'} = q^{N+1}.$$

Writing $w = (\lambda + m, n)$ we have

$$\left|\frac{\lambda + m}{n} - \frac{p}{q}\right| < \frac{1}{qq'}$$

so that

$$\left|\lambda + m - \frac{np}{q}\right| < \frac{|w|}{qq'} < \frac{1}{q^{N+2}} < \frac{1}{2q^2}$$

from which it follows, by (9), that $m - \frac{np}{q}$ is a convergent of λ, say

$$\frac{p_h}{q_h} = m - \frac{np}{q}.$$

Since, by (8),

$$\frac{1}{2q_h q_{h+1}} < \left|\lambda - \frac{p_h}{q_h}\right| < \frac{1}{q^{N+2}}$$

we have

$$q_{h+1} > \frac{q^{N+2}}{2q_h} > q^N \geq q_h^N,$$

from which it follows that $h \in \ell_N$. Since $q_h \leq q \leq |w|^r < q_{k'}$, we must have $q_h \leq q_k$. Hence, $q_{h+1} \leq q_{k+1}$ so that

$$\frac{1}{2q_k q_{k+1}} \leq \frac{1}{2q_h q_{h+1}} < \frac{1}{q^{N+2}}.$$

[11] We remark that $N' > 1$ implies $T > 1$ in the definition of normality.

Since $\alpha > 1$, we have $q > |w| \geq q_{k+1}^{1/N} > q_k$ so that

$$q_{k+1} > \frac{q^{N+2}}{2q_k} > q^N > |w|^N,$$

which contradicts the hypothesis on $|w|$. □

Given a slit w let

$$\beta = |w|^{r-1}.$$

Our goal, Proposition 7.11, is to develop hypotheses on an α-normal slit w that ensures that among the slits $w' = w + 2v \in \Delta(w, \alpha, \beta)$ lots of them are αr-normal. More specifically we wish to show that under suitable hypotheses, an α-normal slit w determines lots of αr-normal $w' = w + 2v$ where $v \in V_0$ and

$$|w|^r \leq |v| \leq 2|w|^r \quad \text{and} \quad |w \times v| < \frac{1}{\alpha} \tag{32}$$

If w' is αr-normal and satisfies (32) then it will be called a *child* of w. The main task will be to bound the number of w' that satisfy (32) but are not αr-normal. We begin with a pair of lemmas that are essentially a consequence of normality.

Lemma 7.6. *Suppose w is α-normal. Let u be the convergent of the inverse slope with maximum height $|u| < |w|^r$ and q the height of the next convergent. Define t_1 by*

$$|u| = \alpha\rho^{t_1}|w|$$

and t_2 by

$$q = |w|^{1+(r-1)t_2}.$$

Then $1 \leq t_1 \leq T$ and $1 \leq t_2 \leq t_1$.

Proof. The first inequality is a consequence of the case $t = 1$ in the definition of normality The left hand part of the second inequality follows from the defintion of q, while the right hand follows from α-normality because there would otherwise be a $t \in (t_1, t_2)$ for which (31) fails. □

Lemma 7.7. *Suppose $w' \in \Delta(w, \alpha, \beta)$ is not αr-normal and again letting u' be the convergent of the inverse slope with maximum height $|u'| < |w'|^r$ and q' the next convergent define t_1' by $|u'| = \alpha r \rho^{t_1'}|w'|$ and t_2' by $q' = |w'|^{1+(r-1)t_2'}$. Suppose $q_{k+1}^{1/N} < |w'| < q_{k'}^{1/r}$. Then $t_1' \leq$* $\min(N', t_2')$.

Proof. We first note again that $t_2' \geq 1$ by definition. Now $t_2' \geq t_1'$ since if $t_1' > 1$ then (31) is satisfied by $|u'|$ for all $t \in [1, t_1']$, and by q' for all $t \in [t_1', T]$, contrary to the assumption that w' is not αr normal. If $t_1' \geq N'$ then Lemma 7.5 applied to the slit w', with (αr) in place of α, implies that w' is αr-normal, contrary to assumption. □

Lemma 7.8. *Suppose $w' \in \Delta(w, \alpha, \beta)$ satisfies the conditions of Lemma 7.7. Let $\bar{t}_1' := \max(t_1', 1)$. Let u' be the convergent of w' as above Then u' determines a (nonzero) integer $a \in \mathbb{Z}$ such that*

$$|(w \times u') + 2a| < \frac{1}{|w|^{r(r-1)\bar{t}_1'}}.$$

Moreover, $|a| < 2\rho^{N'+1}$.

Proof. Write $w' = w + 2v$ and recall that since $|w' \times v| = |w \times v| < 1$ (as in the proof of Lemma 7.2) v is a convergent of w'. Let v' be the next convergent of w' after v. Since $|u'| > |v|$ we either have $u' = v'$ or u' comes after v' in the continued fraction expansion of w'. In any case, we have $u' = av' + bv$ for some *nonnegative* integers $a \geq b \geq 0$ with $\gcd(a, b) = 1$. Since $v \times v' = \pm 1$ we have

$$|w' \times u'| = |(w \times u') + 2(v \times u')| = |(w \times u') \pm 2a|.$$

On the other hand,

$$|w' \times u'| < \frac{|w'|}{q'} = \frac{1}{|w'|^{(r-1)t_2'}} < \frac{1}{|w|^{r(r-1)\bar{t}_1'}}.$$

This proves the first part.

By the first inequality in (28), $|v'| > \frac{|w'|}{2|w \times v|}$ so that

$$a < \frac{|u'|}{|v'|} < 2\alpha r \rho^{t_1'} |w \times v| < 2\rho^{N'+1},$$

by Lemma 7.7 and since $r < \rho$. This proves the second part. □

Suppose $w'' \in \Delta(w, \alpha, \beta)$ also satisfies (32) and is also not (αr)-normal and satisfies $|w''| < q_{k'}^{1/r}$. Let u'' be the convergent of w'' with maximal height $|u''| \leq |w''|^r$. Suppose further that it determines the same integer a determined by u' as in Lemma 7.8. Then we say u' and u'' belong to the same *strip*. The number of strips is bounded by the number of possible values for a. Thus, by Lemma 7.8, the number of strips is bounded by

$$4\rho^{N'+1}. \tag{33}$$

Now suppose u', u'' belong to the same strip. We say u' and u'' lie in the same *cluster* if they differ by a multiple of u.

Lemma 7.9. *If $|u''-u'|<|w|^r$ then they belong to the same cluster.*

Proof. Since u', u'' determine the same a, Lemma 7.8 and the fact that $\bar{t}_1' \geq 1$ implies

$$|w \times (u''-u')| < \frac{2}{|w|^{r(r-1)}} \tag{34}$$

so that writing $u''-u' = d\bar{u}$ where $d = \gcd(u''-u')$ we have

$$\angle w\bar{u} = \frac{|w \times (u''-u')|}{|u''-u'||w|} < \frac{2}{|u''-u'||w|^{r+(r-1)^2}} \leq \frac{1}{2|u''-u'|^2},$$

which implies $\bar{u}$ is a convergent of w. Since $|\bar{u}| \leq |u''-u'| \leq |w|^r$, we have $|\bar{u}| \leq |u|$, by definition of u. Now suppose $|\bar{u}| < |u|$. We will arrive at a contradiction. Since u is a convergent of w coming after $\bar{u}$,

$$|w \times \bar{u}| > \frac{|w|}{2|u|}$$

which together with (34) implies

$$\frac{d|w|}{2|u|} < \frac{2}{|w|^{r(r-1)}}$$

so that

$$|u| > \frac{d|w|^{r+(r-1)^2}}{4} \geq |w|^r,$$

contradicting the definition of u. We conclude that $\bar{u} = u$, so that u', u'' differ by a multiple of u. That is, they belong to the same cluster. □

Pick a representative from each cluster. To bound the number of clusters we bound the number of representatives. Since $|u'| = \alpha r \rho^{t_1'}|w'| < 5\alpha\rho^{N'+1}|w|^r$ and the difference in height of any two representatives is greater than $|w|^r$, the number of clusters is bounded by (since $\alpha > 1$)

$$5\alpha\rho^{N'+1} + 1 \leq 6\alpha\rho^{N'+1}. \tag{35}$$

To bound for the number of u' in each cluster we need an additional assumption.

Lemma 7.10. *Suppose $t_1' \geq t_1 - 1$ (independent of u' within the cluster). Then the number of elements in the cluster is bounded by*

$$5|w|^{(r-1)-(r-1)^2}.$$

Proof. Lemma 7.8 implies for any u', u'' in the cluster

$$|w \times (u''-u')| \leq \frac{2}{|w|^{r(r-1)\bar{t}_1'}}$$

where $\bar{t}'_1$ is the smallest possible within the cluster. On the other hand,

$$|w \times u| > \frac{|w|}{2q} = \frac{1}{2|w|^{(r-1)t_2}} \geq \frac{1}{2|w|^{(r-1)t_1}}$$

so that

$$\frac{|w \times (u'' - u')|}{|w \times u|} < 4|w|^{(r-1)(t_1 - r\bar{t}'_1)}.$$

By definition $u'' - u'$ is a multiple of u. To get the desired bound, using the assumptions $1 < r < 2$ and $|w|^{r-1} \geq 1$, it remains to show that

$$t_1 - r\bar{t}'_1 \leq 2 - r.$$

To see this note that if $t'_1 > 1$ then since $t'_1 \geq t_1 - 1$

$$t_1 - rt'_1 = (t_1 - t'_1) + (1 - r)t'_1 < 2 - r,$$

whereas if $t'_1 \leq 1$ then

$$t_1 - r \leq t'_1 + 1 - r \leq 2 - r.$$

□

We shall now apply our Lemmas to show that, under suitable hypotheses on an α-normal slit w there are lots of children, i.e. αr-normal slits w' satisfying (32).

Proposition 7.11. *Suppose w is an α-normal slit satisfying*

$$q_{k+1}^{1/N} \leq |w| < 5|w|^r < q_{k'}^{1/r}$$

where k, k' are consecutive elements of ℓ_N. Suppose further that

$$240\alpha^2\rho^{3N'+3} \leq c_0|w|^{(r-1)^2}. \tag{36}$$

Then the number of w' satisfying (32) that are αr-normal is at least

$$\frac{c_0|w|^{r-1}}{2\alpha\rho^{N'+1}}$$

Proof. Let t_1 be the parameter associated to the convergent u of w as in (7.6). There are two cases. If $t_1 \geq N' + 1$ then w is $(\alpha\rho^{N'+1}, |w|^{r-1})$-good, so that Lemma 7.2 implies w has at least

$$\frac{c_0|w|^{r-1}}{\alpha\rho^{N'+1}} \tag{37}$$

$w' = w + 2v$ satisfying (32). Moreover, by Lemma 7.3 each w' constructed is $(\alpha\rho^{N'+1} - \frac{1}{2}, |w'|^{r-1})$-good. Since

$$\alpha\rho^{N'+1} - \frac{1}{2} > \alpha r\rho^{N'},$$

by the choice of ρ, every such w' is $(\alpha r\rho^{N'}, |w'|^{r-1})$-good.

Moreover, since each w' has length at most $5|w|^r$, Lemma 7.5 implies each w' constructed is αr-normal. Note that the number in (37) is twice as many as we need.

Now consider the case $t_1 < N' + 1$. In this case w is $(\alpha\rho^{t_1}, |w|^{r-1})$-good, so that Lemma 7.2 implies w has at least

$$\frac{c_0|w|^{r-1}}{\alpha\rho^{t_1}} > \frac{c_0|w|^{r-1}}{\alpha\rho^{N'+1}}$$

w' satisfying (32). Moreover, Lemma 7.3 implies each child w' constructed is $(\alpha\rho^{t_1} - \frac{1}{2}, |w'|^{r-1})$-good, and since

$$\alpha\rho^{t_1} - \frac{1}{2} > \alpha r \rho^{t_1 - 1},$$

again, by the choice of ρ, this means w' is $(\alpha r \rho^{t_1-1}, |w'|^{r-1})$-good.

Moreover, the parameter t_1' associated to the convergent u' of each such w' satisfies $t_1' \geq t_1 - 1$. Applying Lemmas 7.8, 7.9 and 7.10 we conclude the number of w' constructed that are not αr-normal is at most the product of the bounds given in (33), (35), and Lemma (7.10), i.e.

$$120\alpha\rho^{2N'+2}|w|^{(r-1)-(r-1)^2},$$

which is at most half the amount in (37) since (36) holds. □

8. Choice of initial parameters

In this section we specify some parameters that need to be fixed before the construction of the tree of slits can begin. In particular, we shall specify the initial slit. We shall also specify the type of construction that will be used at each level to find the slits of the next level.

8.1. **Choice of initial slit.** Given $\varepsilon > 0$ we first choose $1 < r < 2$ so that

$$\frac{1}{1+r} > \frac{1}{2} - \varepsilon$$

then choose $\delta > 0$ so that

$$\frac{1-\delta}{1+r+2\delta} > \frac{1}{2} - \varepsilon. \tag{38}$$

It will be convenient to set

$$M := \frac{1}{r-1} > 1$$

and let

$$M' = \max(3M^2, Mr/\delta). \tag{39}$$

We set

$$N = M'r^5 \tag{40}$$

and let N' be given by (30).

We assume that ℓ_N, which was defined in (29), has infinitely many elements, for if ℓ_N were finite, then λ is Diophantine and this case has already been dealt with in [Ch1]. Our argument would simplify considerably if we assume ℓ_N is finite and it would essentially reduce to the one given in [Ch1].

Now choose $k_0 \in \ell_N$ large enough so that

$$q_{k_0} > \max\left(5^M, 60c_0^{-1}\rho^{N'+3}, 2\rho^{N'}(\log_r(M') + 4), 2^7\rho N'\right). \tag{41}$$

Lemma 8.1. *There is a slit $w_0 \in V_2^+$ such that $d(w_0, k_0) \leq 2$ and*

$$q_{k_0}^{M'} \leq |w_0| < q_{k_0}^{M'r}. \tag{42}$$

Proof. Let $w \in V_2^+$ be any slit such that $|w| < q_{k_0}/2$. Choose $w_0 \in \Lambda_1(w, k_0)$ with minimal height satisfying the first inequality in (42). Lemma 6.2 implies $d(w_0, k_0) \leq 2$. Let u be the Liouville convergent of w indexed by k_0. Its height $|u| \leq q_{k_0}|w| \leq q_{k_0}^2/2$. Since consecutive elements in $\Lambda_1(w, k_0)$ differ by $2u$, we have

$$|w_0| < q_{k_0}^{M'} + 2|u| \leq q_{k_0}^{M'} + q_{k_0}^2 < q_{k_0}^{M'r}$$

since $M' > 2M$. □

Choose w_0 satisfying the conditions of Lemma 8.1 and let it be fixed for the rest of this paper. It is the unique slit of level 0.

Note that the choice of k_0 in (41) gives various lower bounds on the length of w_0, by virtue of the first inequality in (42). For example, since $M' > M$, the first relation in (41) implies

$$|w_0|^{(r-1)^2} \geq q_{k_0}^{M'/M^2} > q_{k_0}^{1/M} > 5. \tag{43}$$

8.2. **Choice of indices.** Next, we shall specify for each level $j \geq 0$ the type of construction that will be applied to the slits of level j to construct slits of the next level. (The same type of construction will be applied to all slits within the same level.) We shall define indices j_k^A for each $k \in \ell_N$ with $k \geq k_0$ and for $A \in \{B, C, D\}$ such that whenever $k < k'$ are consecutive elements of ℓ_N we have (see Lemma 8.5(i) below)

$$j_k^B < j_k^C < j_k^D < j_{k'}^B.$$

For $j_k^C \leq j < j_k^D$ we use the construction described in §6, while for all other j we use the techniques described in §7. The precise manner in which these types of constructions will be applied is described in the next subsection.

The primary role of these indices is to ensure that various conditions on the lengths of all slits in some particular level are satisfied. (See Lemma 8.6.) Specifically, the conditions in Lemmas 6.2 and 6.3 are needed for the levels $j_k^C \leq j \leq j_k^D$ and those in Proposition 7.11 are needed for the levels $j_k^D \leq j \leq j_{k'}^B$. It will also be important that the number of levels between j_k^B and j_k^C be bounded (Lemma 8.5.ii) whereas the number between j_k^C and j_k^D (or between j_k^D and $j_{k'}^B$) will generally not be bounded.

Let $H_0 = \{|w_0|\}$ and for $j > 0$ set

$$H_j = \left[|w_0|^{r^j}, 5^{\frac{r^j - 1}{r-1}} |w_0|^{r^j} \right]$$

so that the lengths of all slits of level j lie in H_j, by (15).

Lemma 8.2. *For all $j \geq 0$*

$$\sup H_j < \inf H_{j+1} = (\inf H_j)^r. \tag{44}$$

Proof. The condition $\sup H_j < \inf H_{j+1}$ is equivalent to

$$5^{\frac{r^j-1}{r-1}} < |w_0|^{r^j(r-1)},$$

which is implied by

$$5^{r^j} < |w_0|^{(r-1)^2 r^j},$$

which in turn is implied by (43). □

The choice of the indices j_k^A will depend on the position of H_j relative to that of the following intervals:

$$I_k^C = \left[q_k^{M'}, q_{k+1}^{1/r} \right), \qquad \text{and} \qquad I_k^D = \left[q_{k+1}^{1/r^5}, q_{k'}^{1/r} \right).$$

Here, again, k' is the element in ℓ_N immediately after k. These intervals overlap nontrivially and the overlap cannot be too small in the sense that there are at least three consecutive H_j's contained in it.

Lemma 8.3. *For any $k \in \ell_N$ with $k \geq k_0$*

$$\#\{j : H_j \subset I_k^C \cap I_k^D\} \geq 3. \tag{45}$$

Proof. Note that $f(x) = \log_r \log_{|w_0|}(x)$ sends $x = \inf H_j$ to a nonnegative integer and

$$f(q^a) = \frac{\log a + \log\log q - \log\log|w_0|}{\log r}.$$

For any q the image of $[q^a, q^b)$ under f contains exactly $\lfloor \log_r(b/a) \rfloor$ integers, all of them nonnegative if $f(q^a) > -1$; or equivalently, if

$|w_0| < q^{ar}$. Under this condition, the fact in Lemma 8.2 that $\inf H_{j+1} = (\inf H_j)^r$ implies

$$\#\{j \geq 0 : H_j \subset \left[q^a, q^b\right)\} \geq \lfloor \log_r(b/a) \rfloor - 1.$$

Since $N \geq M'r^5$ and $q_{k'} \geq q_{k+1} > q_k^N$, we have

$$I_k^C \cap I_k^D = \left[q_{k+1}^{1/r^5}, q_{k+1}^{1/r}\right)$$

and since $q_{k+1}^{1/r^4} > q_k^{N/r^4} \geq q_k^{M'r} > |w_0|$, (45) follows. □

By virtue of the fact that the quantity in (45) is at least one, we can now give two equivalent definitions of the index j_k^A.

Definition 8.4. For $k < k'$ consecutive elements of ℓ_N with $k \geq k_0$, let

$$\begin{aligned} j_k^C &= \min\{j : H_j \subset I_k^C\} = \min\{j : \inf H_j \geq q_k^{M'}\} \\ j_k^D &= \max\{j : H_{j+1} \subset I_k^C\} = \max\{j : \sup H_j < q_{k+1}^{1/r}\} \\ j_{k'}^B &= \max\{j : H_j \subset I_k^D\} = \max\{j : \sup H_j < q_{k'}^{1/r}\} \end{aligned}$$

Note that $j_{k_0}^C = 0$ and that $j_{k_0}^B$ is not defined.

The main facts about these indices are expressed in the next two lemmas.

Lemma 8.5. *For any $k \in \ell_N$, $k \geq k_0$*

(i) $j_k^B < j_k^C < j_k^D \leq j_{k'}^B$
(ii) $j_k^C \leq j_k^B + \log_r(M') + 4$.

Proof. For (i) we note that

$$\inf H_{j_k^B} \leq \sup H_{j_k^B} \leq q_k^{1/r} < q_k^{M'}$$

so the first inequality follows by the (second) definition of j_k^C. From the first definitions of j_k^C and j_k^D, we see that the second inequality is a consequence of Lemma 8.3. The third inequality follows by comparing the second definitions of j_k^D and $j_{k'}^B$ and noting that $q_{k'} \geq q_{k+1}$.

For (ii) first note that

$$\inf H_{j_k^B} = \left(\inf H_{j_k^B+1}\right)^{1/r} \geq \left(\sup H_{j_k^B+1}\right)^{1/r^2} \geq q_k^{1/r^3}$$

by Lemma 8.2 and the second definition of j_k^B. Thus, we have

$$\inf H_{j_k^B+n} = \left(\inf H_{j_k^B}\right)^{r^n} \geq q_k^{r^{n-3}} \geq q_k^{M'}$$

where $n = \lceil \log_r(M') + 4 \rceil$. The second definition of j_k^C now implies $j_k^C \leq j_k^B + n \leq \log_r(M') + 4$. □

Lemma 8.6. *For any slit w of level j we have*

$$\text{(i)} \quad j_k^C \le j \le j_k^D \implies |w| \in I_k^C \implies q_k^M \le |w| < \frac{q_{k+1}}{2q_k}$$

$$\text{(ii)} \quad j_k^D \le j \le j_{k'}^B \implies |w| \in I_k^D \implies q_{k+1}^{1/N'} \le |w| < q_{k'}^{1/r}.$$

Proof. By definition, $\inf H_{j_k^C} \ge q_k^{M'}$ and $\sup H_{j_k^D} < q_{k+1}^{1/r}$, giving the first implication in (i). Since $N \ge 2Mr$ we have

$$q_{k+1}^{1-1/r} > q_k^{N-N/r} \ge q_k^2 > 2q_k$$

so that $q_{k+1}^{1/r} < \frac{q_{k+1}}{2q_k}$. This, together with $M' \ge M$, implies the second implication in (i).

For (ii) note that (45) implies $H_{j_k^D} \subset I_k^C \cap I_k^D$, giving the first implication, while the second implication follows from $N' > r^5$. □

9. Tree of slits

In this section we specify exactly how the slits of level $j+1$ are constructed from the slits of level j. As before, we refer to any slit constructed from a previously constructed slit w as a *child* of w. The parameters δ_j and ρ_j are also specified in this section. At each step, we shall verify that the choice of δ_j and ρ_j is such that all cross-products of slits of level j with their children are $< \delta_j$ while the number of children is at least $\rho_j |w|^{r-1}\delta_j$, as required by (16) in §5.

Depending on the type of construction to be applied, there will be various kinds of hypotheses on all slits within a given level that we need to verify. These hypotheses can be one of two kinds. The first kind involve inequalities on lengths of slits and these will always be satisfied using Lemma 8.6. We will not check these hypotheses explicitly. The second kind is more subtle and involve conditions related to the continued fraction expansions of the inverse slopes of slit directions. The fact that we need such hypotheses on slits is evident from Lemma 7.2, which is one of the main tools we have for determining whether a slit will have lots of children.

One of the main tasks of this section will be to check the required hypotheses of the second kind at each step. For the levels between consecutive indices of the form j_k^A, these hypotheses will hold by virtue of the results in §6 and §7. Special attention is needed to check the relevant hypotheses of the second kind for the levels $j_k^A, k \in B, C, D$ when the type of construction used to find the slits of the next level changes.

In what follows, it will be implicitly understood that $k < k'$ denote consecutive elements of ℓ_N, with $k \geq k_0$. If $k > k_0$, then $\tilde{k}$ will denote the element of ℓ_N immediately before k.

9.1. **Liouville region.** For the levels j satisfying $j_k^C \leq j < j_k^D$, the slits of level $j+1$ will be constructed by applying Lemma 6.3 to all slits of level j. In other words, the slits of level $j+1$ consist of all slits $w' \in \Lambda(w,k)$ where w is a slit of level j and v is a loop such that $w' = w + 2v$.

Recall that an initial slit w_0 has been fixed using Lemma 8.1. Lemma 6.1 implies the cross-products of w_0 with its children are all less than $4/q_{k_0}$, while Lemma 6.3 implies the number children is at least $|w|^{r-1}/q_{k_0}$. Therefore, we set

$$\delta_0 = \frac{4}{q_{k_0}} \quad \text{and} \quad \rho_0 = \frac{1}{4}.$$

For the levels $j_k^C < j < j_k^D$, we set

$$\delta_j = \frac{4}{q_k} \quad \text{and} \quad \rho_j = \frac{1}{4}.$$

Lemma 9.1. *For $j_k^c < j \leq j_k^D$, every slit w of level j satisfies $d(w,k) \leq 2$. Moreover, if $j < j_k^D$ then the cross-products of each slit of level j with its children are less than δ_j and the number of children is at least $\rho_j |w|^{r-1} \delta_j$.*

Proof. Since all slits of level j were obtained via the Liouville construction, the first part follows from the first assertion of Lemma 6.2. Suppose w is a slit of level j with $j_k^C < j < j_k^D$. Lemma 6.1 now implies the cross-products of w with its children are less than $4/q_k$, and the number of children is at least $|w|^{r-1}/q_k$, by Lemma 6.3. □

It will be convenient to set

$$\alpha_k = \frac{q_k}{2\rho^{N'}}.$$

Lemma 9.2. *Every slit of level j_k^D is α_k-normal.*

Proof. Let w be a slit of level j_k^D. Since $H_{j_k^D} \subset I_k^C$, we have

$$2\alpha_k \rho^{N'} = q_k \leq |w|^{r-1}.$$

By Lemma 9.1, we have $d(w,k) \leq 2$ and since w was obtained via the Liouville construction, Lemma 6.2 implies the inverse slope of w has a convergent with height between $q_k|w|/2$ and $q_k|w|$, or, by the above, between $\alpha_k \rho^{N'}|w|$ and $|w|^r$. This means w is $(\alpha_k \rho^{N'}, |w|^{r-1})$-good, and therefore, α_k-normal, by Lemma 7.5. □

9.2. **Diophantine region.** For the levels j satisfying $j_k^D \le j < j_{k'}^B$, the slits of level $j+1$ will be constructed by applying Proposition 7.11 with the parameter $\alpha = \alpha_k r^{j-j_k^D}$ to all slits w of level j. In other words, the slits of level $j+1$ consist of all αr-normal children of all slits of level j, where $\alpha r = \alpha_k r^{j-j_k^D+1}$.

For the levels $j_k^D \le j < j_{k'}^B$, we set

$$\delta_j = \frac{2\rho^{N'}}{q_k r^{j-j_k^D}} \quad \text{and} \quad \rho_j = \frac{c_0}{2\rho^{N'+1}}.$$

Lemma 9.3. *For $j_k^D \le j \le j_{k'}^B$, every slit w of level j is $\alpha_k r^{j-j_k^D}$-normal. Morevover, if $j < j_{k'}^B$ then the cross-products of each slit of level j with its children are less than δ_j and the number of children is at least $\rho_j |w|^{r-1}\delta_j$.*

Proof. The case $j = j_k^D$ of the first assertion follows from Lemma 9.2 while the remaining cases follow from Proposition 7.11.

For children constructed via Proposition 7.11 applied to an α-normal slit, the cross-products are less than $1/\alpha$, which is δ_j if $\alpha = \alpha_k r^{j-j_k^D}$. The number of children is at least

$$\frac{c_0|w|^{r-1}}{2\alpha_k \rho^{N'+1}} = \frac{c_0 r^{j-j_k^D}}{2\rho^{N'+1}} |w|^{r-1}\delta_j \ge \rho_j |w|^{r-1}\delta_j$$

provided we verify that the inequality (36) holds, i.e. if

$$60 q_k^2 r^{2(j-j_k^D)} \rho^{N'+3} \le c_0 |w|^{(r-1)^2}. \tag{46}$$

To check this inequality, we first note that $|w| \ge q_{k+1}^{1/r} > q_k^{N/r} > q_k^{M'}$ so that

$$|w|^{(r-1)^2} > q_k^{M'/M^2} \ge q_k^3,$$

since $M' \ge 3M^2$, by the first relation in (39). Next, we note that it is enough to check (46) in the case $j = j_k^D$ since the left hand side increases by a factor r^2 as j increments by one, while the right hand side increases by a factor $|w|^{(r-1)^3} > q_{k_0}^{3(r-1)} > 5 > r^2$. Moreover, since $|w|^{(r-1)^2} > q_k^3$, (46) in the case $j = j_k^D$ follows from $60\rho^{N'+3} < c_0 q_{k_0}$, which is guaranteed by the second term in (41). □

Lemma 9.4. *Every slit w of level $j_{k'}^B$ is $(\alpha_k, |w|^{r-1})$-good.*

Proof. Let w be a slit of level $j_{k'}^B$. Lemma 9.3 implies that w is α-normal for some $\alpha > \alpha_k$. By the case $t = 1$ in the definition of normality, this means w is $(\alpha, |w|^{r-1})$-good, i.e. its inverse slope has a convergent whose height is between $\alpha|w|$ and $|w|^r$. Since $\alpha > \alpha_k$ the height of this convergent is between $\alpha_k|w|$ and $|w|^r$. Hence, w is $(\alpha_k, |w|^{r-1})$-good. □

9.3. **Bounded region.** For the levels j satisfying $j_k^B \le j < j_k^C, k > k_0$, the slits of level $j+1$ will be constructed by applying Lemma 7.2 to all slits w of level j with the parameters

$$\alpha = \alpha_{\tilde{k}} - \frac{j - j_k^B}{2} \quad \text{and} \quad \beta = |w|^{r-1}. \tag{47}$$

In other words, the slits of level $j+1$ consist of all slits of the form $w + 2v$ where w is a slit of level j and $v \in \Delta(w, \alpha, \beta)$ where α and β are the parameters given in (47).

For the levels $j_k^B \le j < j_k^C, k > k_0$, we set

$$\delta_j = \frac{4\rho^{N'}}{q_{\tilde{k}}} \quad \text{and} \quad \rho_j = \frac{c_0}{2}.$$

Lemma 9.5. *For $j_k^B \le j \le j_k^C$, every slit w of level j is $(\alpha_{\tilde{k}}/2, |w|^{r-1})$-good. Morevover, if $j < j_k^C$ then the cross-products of each slit of level j with its children are less than δ_j and the number of children is at least $rho_j |w|^{r-1}\delta_j$.*

Proof. First we note that every slit w of level j is (α, β)-good, where α and β are the parameters given in (47). Indeed, for $j = j_k^B$ this follows from Lemma 9.4 while for $j_k^B < j \le j_k^C$ it follows from Lemma 7.3. Lemma 8.5.ii and the third relation in (41) imply

$$j - j_k^B \le j_k^C - j_k^B \le \log_r(M') + 4 \le \frac{\alpha_{\tilde{k}}}{2}$$

from which we see that the first assertion holds.

For children constructed via Lemma 7.2 applied to an (α, β)-good slit, the cross-products are less than $1/\alpha$, which is $< \delta_j$, since $\alpha > \alpha_{\tilde{k}}/2$. And since $\alpha \le \alpha_{\tilde{k}}$, the number of children is at least

$$\frac{c_0 |w|^{r-1}}{\alpha_{\tilde{k}}} = \rho_j |w|^{r-1} \delta_j$$

giving the second assertion. □

Finally, for the levels $j = j_k^C$ with $k > k_0$, we set

$$\delta_j = \frac{8\rho^{N'}}{q_{\tilde{k}}} \quad \text{and} \quad \rho_j = \frac{q_{\tilde{k}}}{8\rho^{N'} q_k}.$$

Lemma 9.6. *For any slit w of level $j = j_k^C$ with $k > k_0$, the cross-products of w with its children are less than δ_j and the number of children is at least $\rho_j |w|^{r-1}\delta_j$.*

Proof. Suppose w is a slit of level j_k^C with $k > k_0$. The case $j = j_k^C$ of Lemma 9.5 implies w is $(\alpha_{\tilde{k}}/2, |w|^{r-1})$-good. Let u be the Liouville convergent of w indexed by k. By Lemma 4.4 the height q' of the next convergent is

$$q' > \frac{q_{k+1}}{2} > q_{k+1}^{1/r} > \left(\sup H_{j_k^C}\right)^r \geq |w|^r.$$

Since w is $(\alpha_{\tilde{k}}/2, |w|^{r-1})$-good, we must have $|u| \geq \alpha_{\tilde{k}}|w|/2$ so that, by Lemma 6.1 the cross-products of w with its children are

$$< \frac{2d(w,k)}{q_k} = \frac{2|w|}{|u|} \leq \frac{4}{\alpha_{\tilde{k}}} = \frac{8\rho^{N'}}{q_{\tilde{k}}}.$$

By Lemma 6.3, the number of children is at least $|w|^{r-1}/q_k = \rho_j|w|^{r-1}\delta_j$. □

The construction of the tree of slits is now complete.

10. Hausdorff dimension 1/2

We gather the definitions of δ_j and ρ_j (for $j > 0$) in the table below.

	$j_k^B \leq j < j_k^C$	j_k^C	$j_k^C < j < j_k^D$	$j_k^D \leq j < j_{k'}^B$
δ_j	$\frac{4\rho^{N'}}{q_{\tilde{k}}}$	$\frac{8\rho^{N'}}{q_{\tilde{k}}}$	$\frac{4}{q_k}$	$\frac{2\rho^{N'}}{q_k r^{j-j_k^D}}$
ρ_j	$\frac{c_0}{2}$	$\frac{q_{\tilde{k}}}{8\rho^{N'}q_k}$	$\frac{1}{4}$	$\frac{c_0}{2\rho^{N'+1}}$
$\rho_j\delta_j$	$\frac{2c_0\rho^{N'}}{q_{\tilde{k}}}$	$\frac{1}{q_k}$	$\frac{1}{q_k}$	$\frac{c_0/\rho}{q_k r^{j-j_k^D}}$

First, we verify the hypotheses needed for Falconer's estimate. Recall the definition $m_j = \rho_j|w_0|^{r^j(r-1)}\delta_j$ in (18).

Lemma 10.1. *$\delta_j < \frac{1}{16}$ and $m_j \geq 2$ for $j \geq 0$.*

Proof. From the fourth relation in (41) we see that $\delta_j \leq \frac{8\rho^{N'}}{q_{k_0}} < \frac{1}{16}$. For $j_k^C \leq j < j_k^D$ we have $\rho_j\delta_j = 1/q_k$ and since $|w_0|^{r^{j_k^C}} \geq q_k^{M'}$, by the definition of j_k^C, we have $|w_0|^{r^j(r-1)} \geq q_k^{M'/M} \geq q_k^{3M}$, from which it easily get

$$m_j \geq \frac{2\rho}{c_0}$$

and, in particular, $m_j \geq 2$. For $j_k^D \leq j < j_{k'}^C$ the expression $|w_0|^{r^j(r-1)}$ increases faster than $\rho_j\delta_j$ decreases, so it is enough to check the case $j = j_k^D$, for which, by the above, we have $m_j \geq \frac{c_0}{\rho} m_{j-1} \geq 2$. □

Next, we obtain the lower bound on the Hausdorff dimension of F. Recall the expression for the local Hausdorff dimensions d_j given in (19). The next lemma shows it is close to $\frac{1}{2}$ by the choices made in §8.

Lemma 10.2. $\liminf_{j\to\infty} d_j > \frac{1}{2} - \varepsilon$.

Proof. By (38) it is enough to show that the term (20) and both of the terms in (21) are bounded by δ. By the choice of M' in (39), it would be enough to show that each term is bounded by $\frac{Mr}{M'}$ for all large enough j. It will be convenient to write

$$A_j \lesssim B_j$$

as an abbreviation for $\liminf A_j \leq \liminf B_j$.

We consider the expression (21) first. Using (43) and the fact that $M' > 2M^2$ we see that the first term in (21) satisfies

$$\frac{2r \log 5}{(r-1)\log|w_0|} \leq \frac{2r}{M} < \frac{Mr}{M'}.$$

From the last row of the table, we see that for $j \neq j_{k-1}^C$ we have

$$\frac{\rho_j\delta_j}{\rho_{j+1}\delta_{j+1}} \in \left\{1, \frac{\rho}{c_0}, r, \frac{1}{2\rho^{N'+1} r^{j_{k'}^B - j_k^D - 1}},\right\}$$

while for $j = j_{k-1}^C$ we have

$$\frac{\rho_j\delta_j}{\rho_{j+1}\delta_{j+1}} = \frac{2c_0\rho^{N'} q_k}{q_{\tilde{k}}}.$$

Then, in the second case, we have

$$\frac{\log(\rho_j\delta_j/\rho_{j+1}\delta_{j+1})}{r^j(r-1)\log|w_0|} \leq M\left(\frac{\log q_k + \log 2c_0\rho^{N'}}{r^{j_k^C - 1}\log|w_0|}\right) \lesssim \frac{Mr}{M'}$$

since $\inf H_{j_k^C} \in I_k^C$; in the first case the left hand side above is $\lesssim 0$.

We now turn to the expression (20). For $j_k^C \leq j < j_k^D$ we have

$$\frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|} \leq M\left(\frac{\log q_k}{r^{j_k^C}\log|w_0|}\right) \leq \frac{M}{M'}.$$

Next consider $j_k^D \le j < j_{k'}^B$. Using $jr^{-j}\log r \le 1$, we have

$$\frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|} \le M\left(\frac{\log q_k + (j-j_i^D)\log r + \log(\rho/c_0)}{r^j\log|w_0|}\right)$$
$$\lesssim \frac{M(\log(q_k)+1)}{r^{j_k^D}\log|w_0|} \lesssim \frac{Mr^5\log q_k}{\log q_{k+1}} < \frac{Mr^5}{N} = \frac{M}{M'}.$$

Finally, we turn to the possibility that $j_i^B \le j < j_i^C$ $(i \ge 1)$. Since $j_k^C - j_k^B \le \log_r(M') + 4$, we have

$$\frac{-\log(\rho_j\delta_j)}{r^j(r-1)\log|w_0|} \le M\left(\frac{\log q_{\tilde{k}} - \log(2c_0\rho^{N'})}{r^{j_i^B}\log|w_0|}\right) \lesssim \frac{MM'r^4\log q_{\tilde{k}}}{r^{j_k^C}\log|w_0|}$$
$$\le \frac{Mr^4\log q_{\tilde{k}}}{\log q_k} < \frac{Mr^4}{N} < \frac{M}{M'}$$

and the lemma follows. □

The proof of Theorem 1.1 will be complete with the proof of the following lemma.

Lemma 10.3. *If λ satisfies (1) then $F \subset \mathrm{NE}(P_\lambda)$.*

Proof. It suffices to check that $\sum \delta_j < \infty$ for in that case, every sequence $\dots, w_j, v_j, w_{j+1}, \dots$ constructed above satisfies (3) and $F \subset \mathrm{NE}(P_\lambda)$, by Theorem 2.9. We break the sum into three intervals: $j_k^B \le j \le j_k^C$, $j_k^C < j < j_k^D$, and $j_k^D \le j < j_{k'}^B$.

Let $n_k = \log_{q_k} q_{k+1}$ so that $q_{k+1} = q_k^{n_k}$. It follows easily from the definitions that

$$j_k^D - j_k^C < \log_r n_k < \frac{\log\log q_{k+1}}{\log r}$$

so that (1) implies

$$\sum_{k\in\ell_N}\sum_{j_k^C<j<j_k^D}|w_j \times v_j| \le \frac{4}{\log r}\sum_{k\in\ell_N}\frac{\log\log q_{k+1}}{q_k} < \infty.$$

Since $j_i^C - j_i^B \le \log_r(M') + 4$ we have

$$\sum_{k\in\ell_N}\sum_{j_k^B\le j\le j_k^C}|w_j \times v_j| \le \sum_{k\in\ell_N}\frac{8\rho^{N'}(\log_r(M')+5)}{q_{\tilde{k}}} < \infty.$$

Finally,

$$\sum_{k\in\ell_N}\sum_{j_k^D\le j<j_{k'}^B}|w_j \times v_j| \le \sum_{k\in\ell_N}\frac{2R\rho^{N'}}{q_k} < \infty$$

where $R = \sum_{j\ge 0} r^{-j}$. □

Proof of Theorem 1.2. The construction of the set F as well as the lower bound 1/2 estimate on its Hausdorff dimension remains valid for any irrational λ. (Note that when $\sum_k \frac{\log\log q_{k+1}}{q_k} = \infty$, F cannot be a subset of $NE(P_\lambda)$ since the latter has Hausdorff dimension 0). On the other hand the fact that $\lim_{j\to\infty} \delta_j = 0$ implies $F \subset \mathrm{DIV}(P_\lambda)$, by [Ch2,Prop. 3.6]. Therefore, $\mathrm{HDim}\,\mathrm{DIV}(P_\lambda) \geq \frac{1}{2}$ for all irrational λ. The opposite inequality follows from a more general result in [Ma2]. Lastly, when $\lambda \in \mathbb{Q}$, the set $\mathrm{DIV}(P_\lambda)$ is countable, so that its Hausdorff dimension vanishes. □

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