Multidimensional Continued Fractions, Tilings, and Roots of Unity

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CERTIFICATION OF APPROVAL

I certify that I have read *Multidimensional Continued Fractions, Tilings, and Roots of Unity* by Therese-Marie Basa Landry and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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First formulated around 1930, the Littlewood Conjecture claims that for any pair of real numbers, (α, β), and any positive ϵ, there are integers (p_1, p_2, q) such that $q|q\alpha - p_1||q\beta - p_2|$ is less than ϵ. It is known that known that the set of exceptions has Hausdorff dimension zero [3]. The Littlewood Conjecture can be geometrically reformulated as a problem in a tiling of the plane [4]. When (α, β) is a pair of irrational numbers, Lui shows that there is an associated tiling of the plane. We investigate the properties of this tiling in the case when α and β are both rationals. The edges of our tilings take on at most three slopes and third order cyclic symmetry can occur in the pattern of slopes along the boundary tiles. We obtain a structure theorem for the special class of rational pairs for which α and β have the same height q and show that these are essentially in one-to-one correspondence with the set of solutions to the equation $abc \equiv 1 (\bmod\ q)$. Via our structure theorem, we show how tilings with cyclic symmetry arise whenever there exists a cube root of unity modulo q. As a multidimensional continued fraction, every one of our tilings is associated to a lattice and we also completely classify an infinite family of lattices that gives rise to tilings with such symmetry.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee   Date

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Chapter 1

Introduction

First formulated by John Edensor Littlewood around 1930, the Littlewood Conjecture claims that for any $\alpha, \beta \in \mathbb{R}$,

$$\inf n\|n\alpha\|\|n\beta\| = 0, \tag{1.1}$$

where $\|\cdot\|$ is the distance to the nearest integer. The Littlewood Conjecture holds in the case of rational pairs as well as pairs of reals wherein at least one member possesses a continued fraction expansion of unbounded partial quotients. Moreover, counterexamples can also be significantly characterized- advancements within the last decade have classified the set of exceptions to the Littlewood Conjecture as one of Hausdorff dimension zero.

The Littlewood Conjecture can be geometrically reformulated as a tiling problem. Our tilings are coverings of $\mathrm{SL}(n, \mathbb{R})$ and our goal is to completely characterize a class of tilings which have cyclic symmetry. Three master's theses form the foundation of this project.

In Samantha Lui's thesis, our tilings are first defined [4]. Lui shows that her construction yields tiles whose diameters are comparable to

$$-\log n\|n\alpha\|\|n\beta\|. \tag{1.2}$$

Since the Littlewood Conjecture holds for rational pairs, identifying the set of exceptions becomes a question of characterizing the tiling associated to irrational pairs. The tiling established in her project can be thought of as a simultaneous generalization of the continued fractions of α and β. Counterexamples to the Littlewood Conjecture would therefore possess a uniform bound on the diameters of nondegenerate tiles.

When $(\alpha, \beta) \in \mathbb{Q}^2$, three degenerate tiles bound the set of nondegenerate tiles and it is in this setting where cyclic symmetry can occur along the slopes of the boundary tiles. In this case, the tiles can overlap. Odom's thesis specifies the additional conditions required to preserve this tiling property [5]. Some of these tilings exhibit exceptional levels of self-similarity along the edges of their boundary tiles.

Through numerical explorations, Damon discovered an infinite family of tilings with cyclic symmetry. Three slopes are possible for the edges of a tile and the pattern along the boundary tiles is repeated for each degenerate tile. Every one of our tilings is associated with a lattice and Damon's thesis determines a framework for understanding the lattices from which our tilings can arise [2]. We improve upon her lexicon, achieve a unified understanding of patterns along the boundary tiles, and for lattices in $\mathbb{R}^3$, obtain a result that generalizes the following:

Theorem 1.1. *Let σ be a cube root of unity modulo q. Then the tiling associated to $(q\mathbb{Z})^3 + \mathbb{Z}(1, \sigma, \sigma^2)$ has cyclic symmetry.*

As the multidimensional continued fraction of a lattice, our tiling encodes its set of shortest vectors with respect to a box norm. We detail in Chapter 2 the construction of our tiling. We begin with the fundamental notion of how every lattice, Λ, in $\mathbb{R}^3$ determines a distinctive sequence of Λ-boxes and show how such subsets of $\mathbb{R}^3$ yield a tiling of $\mathrm{SL}(3, \mathbb{R})$ that is isomorphic to the plane.

In Chapter 3, we classify lattices with the aim of identifying key features of lattices that give rise to tilings with cyclic symmetry. In the process, we define *axial* lattices and uncover an important invariant for such lattices. Most importantly, we identify the class of lattices- namely, simple axial lattices (see Section 3.2)- that is the subject of our main theorem.

Chapter 4 clarifies the role of roots of unity in understanding the structure of simple axial lattices. We identify the Λ-boxes that determine the boundary pattern between bounded and unbounded tiles. In particular, we prove our main theorem, which is a structure theorem for simple axial lattices explaining that the associated tilings are in one-to-one correspondence with solutions to the equation,

$$abc \equiv 1 (\text{mod } q),$$

where q is the level of the lattice (see Definition 3.2). Theorem 1.1 is the special case where a, b, and c are identical roots of unity modulo q.

Insights about lattice structures naturally lead to questions about lattice symmetries. We conclude by considering the role of our tilings and our proposed invariant in lattice classification and discuss potential questions for future research.

Chapter 2

Tiling

Each of our tilings is associated to a lattice. Consider a lattice in $\mathbb{R}^n$ of dimension n. Any such lattice can be written as the $\mathbb{Z}$-span of a linearly independent subset of n vectors in the lattice. The multidimensional continued fraction of a lattice encodes its set of shortest vectors. Our tilings are a type of multidimensional continued fraction. We begin by showing how the set of shortest vectors in a lattice is identified for our tilings.

2.1 Boxes

We are interested in subsets of $\mathbb{R}^n$ of the following form.

Definition 2.1. Given $u \in \mathbb{R}^n$, we define the box, $B(u)$, by

$$B(u) = \{x \in \mathbb{R}^n : \mid x_1 \mid \leq \mid u_1 \mid, \mid x_2 \mid \leq \mid u_2 \mid, \cdots, \mid x_n \mid \leq \mid u_n \mid\}.$$

We say $B(u)$ is degenerate if $u_1 u_2 \cdots u_n = 0$. We refer to the smallest subspace $W \subset \mathbb{R}^n$ such that $B(u) \subset W$ as the support of $B(u)$. By the dimension of $B(u)$, we mean the dimension of W.

Boxes are symmetric with respect to the origin. We allow degenerate boxes and consider the boundary or interior of a box with respect to the smallest subspace containing the box. For any lattice Λ, we consider the set of boxes such that the interior of each box intersects Λ only at $\{0\}$ and the boundary of each box contains at least one nonzero lattice point.

Definition 2.2. Let B be a box and Λ a lattice. Relative to W, let $int(B)$ denote the interior of B and ∂B the boundary of B. Then B is a Λ-box if $int(B) \cap \Lambda = \{0\}$ and $\partial B \cap \Lambda \neq \emptyset$.

Suppose $B \in \mathbb{R}^n$. Let

$$A = \{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} : a_i > 0 \text{ for every } i, \prod_{i=1}^{n} a_i = 1 \}.$$

Definition 2.3. Given a nondegenerate box, B, let $a \in A$ be the unique diagonal matrix such that $Q = aB$ is an n-dimensional cube. The boundary ∂B is partitioned into the multi-faces,

$$\partial_I B = a^{-1}(\partial Q \cap V_I),$$

where I is a nonempty $\{1, \cdots, n\}$ and

$$V_I = \{x \in \mathbb{R}^n : ||x||_\infty = x_i \text{ if and only if } i \in I\}.$$

We refer to any of the $2^{|I|}$ connected components of $\partial_I B$ as a face of B of "type" I.

Notationally, we shall use $\partial_i B$ in lieu of $\partial_{\{i\}} B$, and allow i to sometimes range over $\{x, y, z\}$ instead of $\{1, 2, 3\}$. When I is the largest possible, we shall write $\partial_\bullet B$.

The set of Λ-boxes can be partially ordered with respect to inclusion.

Definition 2.4. A minimal Λ-box is a box that is minimal with respect to inclusion, i.e. B is a minimal Λ-box if B is a Λ-box and if B' is a Λ-box that is contained in B, then $B' = B$.

Definition 2.5. A maximal Λ-box is a box that is maximal with respect to inclusion, i.e. B is a maximal Λ-box if B is a Λ-box and if B' is a Λ-box that contains B, then $B' = B$.

Minimal and maximal Λ-boxes vary in how they intersect a lattice.

Lemma 2.1. *Let B be a Λ-box. Then B is minimal if and only if $\partial B \cap \Lambda \subset \partial_\bullet B$.*

Proof. Suppose $\partial B \cap \Lambda \subset \partial_\bullet B$. Without loss of generality, assume $\partial_\bullet \neq \{0\}$. Then $B \setminus \partial_\bullet B = \{0\}$. If B' is a box strictly contained in B, then $B' \subset (B \setminus \partial_\bullet B)$. In particular, B' is not a Λ-box unless $B' = B$.

Conversely, consider the condition that B be a minimal Λ-box. If $\partial B \cap \Lambda \not\subset \partial_\bullet B$, then there exists a box, B', such that $\partial_\bullet B' \subset \partial B \cap \Lambda$. In particular, $B' \subset B$, thereby contradicting the minimality of B as a Λ-box. □

Lemma 2.2. *Let B be a Λ-box. Then B is maximal if and only if $\partial_i B \cap \Lambda \neq \emptyset$ for every $i \in I$*

Proof. Suppose B' is a Λ-box that contains B. Then $\partial B \cap int(B')$ is nonempty. If $\partial_i(B) \cap \Lambda \neq \emptyset$ for every $i \in I$, then $int(B')$ is nonempty and B' cannot be a Λ-box. In particular, B is a maximal Λ-box.

Now assume that B be a maximal Λ-box. Suppose that $\partial_i B \cap \Lambda$ is empty. Then there exists a Λ-box, B', such that $\partial B \subset B'$ and $\partial_i B' \cap \Lambda$ is nonempty. In particular, $B \subset B'$, thereby contradicting the maximality of B as a Λ-box. □

Lattice points that lie on the boundary of a Λ-box can also occupy differing components of multifaces of different maximal Λ-boxes whereas such lattice points can only define one minimal Λ-box. This distinction will inform the method by which we identify the shortest vectors in a lattice.

2.2 Pivots and a Covering of A

We choose to distinguish lattice points that define minimal Λ-boxes.

Definition 2.6. Let $\Lambda \subset \mathbb{R}^n$ be an n-dimensional lattice. Then $v \in \Lambda \setminus \{0\}$ is a pivot of Λ if $B(v)$ is a minimal Λ-box.

Definition 2.7. Let $\Pi(\Lambda)$ denote the set of equivalence classes of the pivots of Λ. More precisely,

$$\Pi(\Lambda) := \{v \in \Lambda : v \text{ is a pivot}\}.$$

We say two pivots are equivalent if they define the same minimal $\Lambda - box$. In other words, $u \sim v$ if $u \in B(v) \cap \Lambda$ implies $B(u) = B(v)$.

Remark: If v lies in a corner of $B(u)$, then $B(u) = B(v)$.

The set of pivots of Λ plays the role of the continued fraction of the lattice. Recall that the continued fraction of a real encodes the set of rationals that are closest to that number with respect to several box norms. The shortest vectors in a lattice with respect to the supremum norm define minimal Λ-boxes. Consider the positive diagonal subgroup $A \subset \mathrm{SL}(n, \mathbb{R})$. For each minimal Λ-box B, there is some lattice in the A-orbit of Λ for which aB is an n-dimensional cube and $aB \cap a\Lambda$ are the shortest vectors in $a\Lambda$. We can now define a *tile* that encodes the A-orbit of Λ

via the shortest nonzero vector in $a\Lambda$ with respect to the sup norm $||\cdot||_\infty$ as a varies over A.

Definition 2.8. Let Λ be an n-dimensional real lattice. Given $u \in \Pi(\Lambda)$, we denote the tile associated with a pivot $u \in \Pi(\Lambda)$ by

$$\tau(u) = \{a \in A : ||au||_\infty \leq ||av||_\infty \text{ for any nonzero } v \in \Lambda\}.$$

Note that $\tau(u)$ is well-defined. For every $a \in A$, $||au||_\infty = ||av||_\infty$ if $u \sim v$. In particular, $\tau(u) = \tau(v)$ if $u \sim v$.

This definition of a tile gives rise to a covering of A.

Theorem 2.3. *([5]) For any lattice $\Lambda \subset \mathbb{R}^n$, $A = \cup_{u \in \Pi(\Lambda)} \tau(u)$.*

2.3 Tilings and an Interpretation of the Littlewood Conjecture

Points in $\mathbb{R}^2$ can be identified with matrices in A via the isomorphism,

$$(t, s) \mapsto \begin{bmatrix} e^{t+s} & 0 & 0 \\ 0 & e^{t-s} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}. \tag{2.1}$$

Each of our tiles can therefore be associated with a subset of $\mathbb{R}^2$. In particular, the size of our tiles can be measured using the Euclidean metric on $\mathbb{R}^2$. Lui obtains a geometric interpretation for a counterexample to the Littlewood Conjecture by restricting her attention to lattices of the form,

$$\Lambda_{\alpha,\beta} = \begin{bmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{bmatrix} \mathbb{Z}^3.$$

For a lattice in $\mathbb{R}^3$, the *height* of a pivot refers to the absolute value of its z-coordinate. All pivots in $\Lambda_{\alpha,\beta}$ have integral height. There are exactly two pivots classes of height zero. Every other pivot class is uniquely determined by its height. Therefore,

Definition 2.9. Let $\Pi(\alpha, \beta)$ be the set of positive pivot heights of $\Lambda_{\alpha,\beta}$. For each $n \in \Pi(\alpha, \beta)$, let u_n denote a pivot of height n.

Recall that the Littlewood Conjecture claims that for any $\alpha, \beta \in \mathbb{R}$,

$$\inf n\|n\alpha\|\|n\beta\| = 0.$$

Lui shows that the diameter of each tile varies with $-\log n\|n\alpha\|\|n\beta\|$. In particular,

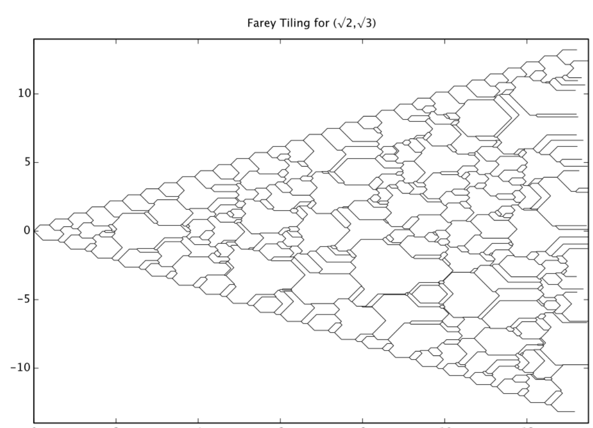
Theorem 2.4. *([4]) For each $n \in \Pi(\alpha, \beta)$, the following inequalities hold:*

$$\frac{1}{9}(-\log n||n\alpha||||n\beta|| - \log 6) \leq diam\tau(u_n) \leq -\log n||n\alpha||||n\beta||.$$

As a consequence,

$$inf n||n\alpha||||n\beta|| > 0 \text{ if and only if } sup_{n\in\Pi(\alpha,\beta)} diam\tau(u_n) < \infty.$$

For $\alpha, \beta \in \mathbb{R}\backslash\mathbb{Q}$, Lui shows that the associated covering of $\mathbb{R}^2$ is a nonoverlapping union, or *tiling* of $\mathbb{R}^2$ [4]. Because the supremum norm is not strictly convex, tiles belonging to inequivalent pivots can sometimes overlap in the absence of this condition. As with all lattices in this class, the tiling associated to $\Lambda_{\sqrt{2},\sqrt{3}}$ contains exactly two tiles of unbounded diameter. These "degenerate" tiles correspond to e_1 and e_2, with $\tau(e_1)$ situated below $\tau(e_2)$. Observe that along the direction of e_1, $\Lambda_{\sqrt{2},\sqrt{3}}$ projects down to $\Lambda_{\sqrt{2}}$, and that the pattern along the boundary of $\tau(e_1)$ reflects the continued fraction expansion of $\sqrt{3}$, which is $[1, 1, 2, 1, 2, 1, 2, 1, \cdots]$.



Similarly, the pattern along $\partial\tau(e_2)$ recalls the continued fraction expansion of $\sqrt{2}$, which is $[1, 2, 2, 2, 2, 2, 2, \cdots]$. We aim to understand the structure of the boundary of tiles like $\tau(e_1)$ and $\tau(e_2)$.

When α and β are rational, $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$, and $\langle 0, 0, q\rangle$ for some integer q are in the lattice and the associated tiling has three or more unbounded tiles. In the next chapter, we introduce a generalization of this class of lattices.

Chapter 3

Axial Lattices

A real number is rational if and only if it has a finite continued fraction representation. Analogously, all lattices with finite tilings have the following property.

Definition 3.1. A lattice, $\Lambda \subset \mathbb{R}^n$, is axial if it has nonzero intersection with each coordinate axis. Equivalently, Λ has a pivot on each coordinate axis. A pivot is said to be trivial if it lies on a coordinate axis.

Every tile is defined with respect to a pivot equivalence class. In particular,

Theorem 3.1. $\Pi(\Lambda)$ *is finite if and only if* Λ *is axial.*

Proof. Let E denote the set of pivots of Λ. A discrete subgroup of $\mathbb{R}^n$ is always closed. For an axial lattice of dimension n, the maximum length of a trivial pivot determines an n-dimensional cube that contains all of its pivots. Let R be the length of such a pivot. Then $[-R, R]^n$ is a compact set and E is a closed subset of a compact set. Every closed, discrete subset of a compact set is finite. Hence E and its set of pivot equivalence classes are finite. Conversely, the absence of trivial pivots on the ith axis guarantees the existence of pivots of arbitrarily large ith coordinate-such a condition would require $\Pi(\Lambda)$ to be infinite. Thus the existence of a pivot on every coordinate axis is equivalent to finiteness for $\Pi(\Lambda)$. □

The index of the subgroup generated by trivial pivots in an axial lattice is also finite.

Definition 3.2. Let $\Lambda_0 \subset \Lambda$ be the subgroup generated by the trivial pivots in an axial lattice. We refer to the index of Λ_0 in Λ as the level of Λ.

3.1 An Invariant of Axial Lattices: The Associated Subgroup F_Λ

Axial lattices admit several normal forms. Suppose Λ is an axial lattice and the level of Λ_0 in Λ is q. Fundamentally, an axial lattice can always be normalized so that the rescaled lattice is contained in $(\frac{1}{q}\mathbb{Z})^n$. Suppose Λ is such a lattice and a_i the length of the shortest nonzero vector on the ith coordinate axis. Then

$$\Lambda_0 = \mathbb{Z}u_1 \oplus \mathbb{Z}u_2 \oplus \cdots \oplus \mathbb{Z}u_n, \text{where } u_i = a_i e_i.$$

Let $a \in A$ be defined by

$$a := \begin{bmatrix} a_1^{-1} & 0 & \cdots & 0 \\ 0 & a_2^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & a_n^{-1} \end{bmatrix}.$$

The *first normal form* of Λ therefore depends on the action of diagonal matrices in $\mathrm{SL}(n, \mathbb{R})$ on $\mathbb{R}^n$ and refers to the lattice, $L_1 = a\Lambda$.

The axial element in Λ, the vector u_i, is mapped to e_i, the standard unit vector, $a\Lambda_0 = \mathbb{Z}^n$. In particular, $\mathbb{Z}^n$ is of index q in L_1 and $L_1/\mathbb{Z}^n$ is a finite group of order q. Lagrange's Theorem requires that the order of the subgroup generated by an element in $L_1/\mathbb{Z}^n$ divide q- if $(x_1, \cdots, x_n) \in L_1$, then $q(x_1, \cdots, x_n) \in \mathbb{Z}^n$. More precisely,

$$\mathbb{Z}^n \subset L_1 \subset (\frac{1}{q}\mathbb{Z})^n.$$

An axial lattice of index q scaled up by a factor of q is an integer lattice. Recall that the height of a rational is the smallest positive integer that multiplies a rational in its reduced form into the integers. If $q(x_1, \cdots, x_n) \in \mathbb{Z}^n$, then each x_i is a rational whose height divides q. Since Λ/Λ_0 is an abelian group of order q, qL_1 is a sublattice of $\mathbb{Z}^n$. Let the *second normal form* of Λ be defined by $L_2 = qL_1$. Then

$$\Lambda_0 = (q\mathbb{Z})^n \subset L_2 \subset \mathbb{Z}^n$$

and the index $(q\mathbb{Z})^n$ in L_2 is q. We therefore study the following subgroup of $(\mathbb{Z}/q\mathbb{Z})^n$.

Definition 3.3. The associated subgroup, $F_\Lambda \subset (\mathbb{Z}/q\mathbb{Z})^n$, of an axial lattice, Λ, of level q, is the image of L_2 under the canonical projection homomorphism.

The associated subgroup of an axial lattice is an invariant of its A-orbit. Axial lattices of level q are in one to one correspondence with the following subgroups of $(\mathbb{Z}/q\mathbb{Z})^n$.

Lemma 3.2. *A subgroup $F \subset (\mathbb{Z}/q\mathbb{Z})^n$ arises from an axial lattice of level q if and only if F arises from an axial lattice of level q and $F \cap \ker \widehat{\pi}_i = \{0\}$ for each $i = 1, \cdots, n$, where $\widehat{\pi}_i$ is the ith "co-rank one projection" of $(\mathbb{Z}/q\mathbb{Z})^n$ given by*

$$\widehat{\pi}_i(x_1, \cdots, x_i, \cdots, x_n) = (x_1, \cdots, \widehat{x}_i, \cdots, x_n).$$

Proof. Suppose Λ is an axial lattice of level q. Then Λ/Λ_0 is an abelian group of order q and $F = F_\Lambda$ is of order q. Because Λ_0 is composed of the trivial pivots of Λ, $F \cap \ker \widehat{\pi}_i = \{0\}$ for each $i = 1, \cdots, n$. Conversely, consider any subgroup, $F \subset (\mathbb{Z}/q\mathbb{Z})^n$, of order q for which $F \cap \ker \widehat{\pi}_i = \{0\}$ for each $i = 1, \cdots, n$. Then

$\Lambda = F + (q\mathbb{Z})^n$ is an axial lattice in second normal form where $\Lambda_0 = (q\mathbb{Z})^n$. In particular, $F = F_\Lambda$. $\square$

Equivalently, an n-dimensional lattice is axial if its associated subgroup under n distinct coordinate hyperplane projections always has a trivial kernel.

3.2 A Special Class of Axial Lattices: Simple Lattices

Axial lattices with the following category of associated subgroups can be completely classified.

Definition 3.4. A subgroup, $F \subset (\mathbb{Z}/q\mathbb{Z})^n$, is simple if each rank one projection $\pi_i : (\mathbb{Z}/q\mathbb{Z})^n \to \mathbb{Z}/q\mathbb{Z}$ restricted to F is surjective; that is, $\pi_i(F_\Lambda) = \mathbb{Z}/q\mathbb{Z}$ for each $i = 1, \cdots, n$. An axial lattice, Λ, of level q is said to be simple if F_Λ is simple.

Associated subgroups of simple level q lattices are always cyclic groups of order q.

Lemma 3.3. *A subgroup $F \subset (\mathbb{Z}/q\mathbb{Z})^n$ arises from a simple lattice of level q if and only if it is a cyclic subgroup generated by an n-tuple of units modulo q.*

Proof. The subgroup, $F \subset (\mathbb{Z}/q\mathbb{Z})^n$, is simple if and only if it is isomorphic to $\mathbb{Z}/q\mathbb{Z}$ under $\pi_i|_F$ for every $i = 1, \cdots, n$. Given this condition, there exists, for every index i, $p_i \in \mathbb{Z}/q\mathbb{Z}$ such that $\langle p_i \rangle = \pi_i(F)$. Every element in $(\mathbb{Z}/q\mathbb{Z})^n$ is of order at most

q. Thus the order of each p_i must divide q. Each p_i generates a cyclic group of order q if and only if $\gcd(p_i, q) = 1$. Hence a subgroup of $(\mathbb{Z}/q\mathbb{Z})^n$ is simple if and only if it is generated by an n-tuple of units modulo q. □

As a consequence, the A-orbits of simple lattices can be enumerated. Since the associated subgroup of an axial lattice is an invariant of its A orbit, simple lattices can be completely classified.

Lemma 3.4. *There are $\phi(q)^{n-1}$ possible simple lattices of level q and dimension n.*

Proof. For a simple level q lattice of dimension n, $F_\Lambda \cap \ker \widehat{\pi}_i = \{0\}$ for all i up to n. Thus $\widehat{\pi}_i$ is injective and every choice of unit for each position in an n-tuple determines a different generator for F_Λ. In particular, there are $\phi(q)^n$ distinct possibilities for generators of simple subgroups, two of which are equivalent if they generate the same subgroup. Since each equivalence class has $\phi(q)$ elements, there are $\phi(q)^{n-1}$ equivalence classes. □

An axial lattice of level q and dimension n is simple if the index of $(q\mathbb{Z})^n$ in Λ is q and the index of Λ in $\mathbb{Z}^n$ is q^{n-1}. The study of simple axial lattices is the study of subgroups of $(\mathbb{Z}/q\mathbb{Z})^n$ of order q for which $\ker \widehat{\pi}_i = \{0\}$. The roots of unity modulo q therefore play a fundamental role in understanding these lattices.

When an axial lattice of level q and dimension n is not simple, $\phi(q)^{n-1}$ is a lower bound on the number of distinct A-orbits, with equality if $n = 2$.

Lemma 3.5. *Every (level q) axial lattice of dimension two is simple.*

Proof. Suppose Λ is an axial lattice of dimension two. When $n = 2$, $\pi_i = \widehat{\pi_i}$ for $i = 1, 2$. The axial condition requires that $F_\Lambda \cap \ker \widehat{\pi_i} = \{0\}$. In particular, $\pi_i(F_\Lambda) \cong \mathbb{Z}/q\mathbb{Z}$ for each index i. □

In the case of three or more dimensions, cyclic subgroups of $(\mathbb{Z}/q\mathbb{Z})^n$ and subgroups that arise from axial lattices no longer coincide.

Example 3.1. Suppose $F_\Lambda = \{(0,0,0), (2,2,0), (2,0,2), (0,2,2)\} \subset (\mathbb{Z}/4\mathbb{Z})^3$. Then Λ is axial but not cyclic.

Cyclic subgroups of $(\mathbb{Z}/q\mathbb{Z})^n$ of order q do not always give rise to axial lattices. The following condition can be used to identify potential such subgroups.

Lemma 3.6. *Given nonnegative q and n and $F = \langle (p_1, \cdots, p_n) \rangle$, $\mid F \mid = \frac{q}{\gcd(p_1, \cdots, p_n, q)}$. In particular, $|F| = q$ if and only if $\gcd(p_1, \cdots, p_n, q) = 1$.*

Proof. Every element in $(\mathbb{Z}/q\mathbb{Z})^n$ is of order at most q and so the order of $(p_1, \cdots, p_n)$ in $(\mathbb{Z}/q\mathbb{Z})^n$ must divide q. In particular, the number of elements in F_Λ is $\frac{q}{\gcd(p_1, \cdots, p_n, q)}$ and as a consequence, $|F_\Lambda| = q$ if and only if $\gcd(p_1, \cdots, p_n, q) = 1$. □

As a consequence, a subgroup of $(\mathbb{Z}/q\mathbb{Z})^n$ is of order q precisely when it is generated by an n-tuple that contains at least one unit modulo q.

Example 3.2. If $F_\Lambda = \langle(1,2,3)\rangle \subset (\mathbb{Z}/6\mathbb{Z})^3$, then $|F_\Lambda| = 6$, F_Λ is generated by a 3-tuple containing a unit modulo 6, and Λ is axial.

Recall that the axial property is equivalent to triviality in the kernel of every co-rank one projection of the associated subgroup.

Theorem 3.7. *Suppose $F = \langle(p_1, \cdots, p_n)\rangle \subset (\mathbb{Z}/q\mathbb{Z})^n$ has order q. Then F gives rise to an axial lattice if and only if $\gcd(p_1, \cdots, \widehat{p_i}, \cdots, p_n, q) = 1$ for every i.*

Proof. By Lemma 3.2, F is the associated subgroup of an axial lattice if and only if $F \cap \ker \widehat{\pi}_i = \{0\}$ for every i. Suppose $k \in \mathbb{Z}/q\mathbb{Z}$. Equivalently, kp_j, $j \neq i$, is a multiple of q for every i and every k only when $q|kp_i$- in other words, $q|k\gcd(p_1, \cdots, \widehat{p_i}, \cdots, p_n)$ for every i and every k implies that $q|kp_i$. Let $d_i = \gcd(p_1, \cdots, \widehat{p_i}, \cdots, p_n, q)$. Alternately, $\frac{q}{d_i}|k$ for every i and every k indicates that $\frac{q}{\gcd(p_i,q)}|k$. The previous condition occurs precisely when $\frac{q}{d_i}$ is a multiple of $\frac{q}{\gcd(p_i,q)}$ for every i- this is the case if and only if d_i divides $\gcd(p_i, q)$ for every i. If $d_i = 1$, the last equivalence is satisfied and F gives rise to an axial lattice. Since F is of order q, Lemma 3.6 requires that $\gcd(p_1, \cdots, p_n, q) = 1$. In particular, every d_i divides $\gcd(p_1, \cdots, p_n, q)$, or $d_i = 1$. □

Subgroups of $(\mathbb{Z}/q\mathbb{Z})^n$ that give rise to axial lattices can therefore be generated by an n-tuple that contains no units modulo q. More precisely,

Example 3.3. Although $F_\Lambda = \langle(2,3,5)\rangle \subset (\mathbb{Z}/30\mathbb{Z})^3$ is a cyclic subgroup of order 30 that is generated by a 3-tuple containing no units modulo 30, Λ is an axial lattice.

Axial lattices with cyclic associated subgroups can be classified by the number of units modulo q in the generators of those subgroups.

Definition 3.5. Let Λ be an axial lattice of level q. If F_Λ is cyclic, then the degree of Λ is the number of units modulo q in the generator of F_Λ.

Thus axial lattices of dimension n and level q that are simple are precisely those lattices of degree n.

Axial lattices generalize lattices of the form $\Lambda_{\alpha,\beta}$ with $\alpha, \beta \in \mathbb{Q}$. Given $\theta \in \mathbb{R}^d$, let

$$\Lambda_\theta := \begin{bmatrix} 1 & \cdots & & -\theta_1 \\ & \ddots & & \vdots \\ & & 1 & -\theta_d \\ & & & 1 \end{bmatrix} \mathbb{Z}^{d+1}.$$

Lattices like Λ_θ are axial if and only if $\theta \in \mathbb{Q}^d$. Given this condition, there exists an integer q such that $e_1, \cdots, e_d$, and $(0, \cdots, q)$ are in Λ_θ. Suppose $\theta = (\frac{p_1}{q}, \cdots, \frac{p_d}{q}) \in \mathbb{Q}^d$. Then the first normal form of Λ_θ is

$$L_1 = \begin{bmatrix} 1 & \cdots & & -\theta_1 \\ & \ddots & & \vdots \\ & & 1 & -\theta_d \\ & & & \frac{1}{q} \end{bmatrix} \mathbb{Z}^{d+1}.$$

Recall that $\mathbb{Z}^{d+1} \subset L_1 \subset (\frac{1}{q}\mathbb{Z})^{d+1}$.

Corollary 3.8. *If* $\theta = (\frac{p_1}{q}, \cdots, \frac{p_d}{q}) \in \mathbb{Q}^d$ *with* $\gcd(p_1, \cdots, p_d, q) = 1$*, then the level of* Λ_θ *is* q*.*

Proof. A lattice in $\mathbb{R}^{d+1}$ is of level q if the index $(q\mathbb{Z})^{d+1}$ in the second normal form of the lattice is q. For Λ_θ, its second normal form is

$$L_2 = \begin{bmatrix} q & \cdots & & -p_1 \\ & \ddots & & \vdots \\ & & q & -p_d \\ & & & 1 \end{bmatrix} \mathbb{Z}^{d+1}.$$

The image of L_2 in $(\mathbb{Z}/q\mathbb{Z})^{d+1}$ is therefore $F = \langle(-p_1, \cdots, -p_d, 1)\rangle$. Since $\gcd(p_1, \cdots, p_d, q) = 1$, $|F| = q$ by Lemma 3.6. $\square$

In particular, $F_{\Lambda_\theta} = \langle (p_1, \cdots, p_d, -1) \rangle$ and Λ_θ is simple if and only if $\gcd(p_i, q) = 1$ for each index i. Suppose q is the height of θ. The level of

$$\widehat{\pi}_{d+1}\Lambda_\theta = \mathbb{Z}\theta + \mathbb{Z}^d.$$

divides q, with equality if and only if Λ_θ is simple. Lattices like $\mathbb{Z}\theta + \mathbb{Z}^d$ are called *Farey lattices* and they are precisely those axial lattices of level q whose associated subgroups are cyclic. All simple axial lattices are Farey lattices of degree n.

3.3 Rational Tilings and Simple Axial Lattices

In our tilings, two or more tiles of unbounded diameter can occur. Pivots of the following kind define such tiles [4].

Definition 3.6. The pivot u is degenerate if the box $B(u)$ is degenerate; or equivalently, if any coordinate of u is zero.

Shears of three-dimensional integer lattices, like $\Lambda_{\alpha,\beta}$, always contain $\langle 1, 0, 0 \rangle$ and $\langle 0, 1, 0 \rangle$. Lattices of this form corresponding to irrational α and β always have exactly two unbounded tiles. When α and β are rational, $\langle 0, 0, q \rangle$ is in the lattice for some integer q and the associated tiling has three or more unbounded tiles. Tilings that arise from such lattices will be called *rational tilings*.

An axial lattice of dimension n has n trivial pivot classes. Simple axial lattices can be also be characterized as those axial lattices for which all degenerate pivots are trivial. Among axial lattices, such lattices contain the least number of degenerate pivot classes and their associated tilings therefore contain the least number of unbounded tiles.

Lemma 3.9. *Let* Λ *be an axial lattice in* $\mathbb{R}^n$*. Then* Λ *is simple if and only if there are exactly* n *degenerate pivots in* $\Pi(\Lambda)$*.*

Proof. If Λ is simple, $F_\Lambda \cap \ker \pi_i = \{0\}$ for each $i = 1, \cdots, n$. Suppose u is a degenerate pivot of Λ. Then for some index i, the image of u in $(\mathbb{Z}/q\mathbb{Z})^n$ belongs to $F_\Lambda \cap \ker \pi_i$. In particular, $u \in \Lambda_0$ and so u is a trivial pivot. Now consider the converse. If every degenerate pivot is trivial, then $F_\Lambda \cap \ker \pi_i = \{0\}$ for every i. Equivalently, $\pi_i|_{F_\Lambda}$ is surjective for every i. Hence Λ is simple. □

We study tilings in which exactly three unbounded tiles occur. In the general case, the tiles may overlap. Consider $\Lambda_{\alpha,\beta} \in \mathbb{R}^3$ with $p_1, p_2, q \in \mathbb{Z}$ such that $\alpha = \frac{p_1}{q}$ and $\beta = \frac{p_2}{q}$. Odom shows that tilings with our desired property occur when $\gcd(p_1, q) = \gcd(p_2, q) = 1$ [5]. Her work also demonstrates that this condition guarantees the non-overlapping property.

As in the case of $\tau(e_1)$ and $\tau(e_2)$, $\tau(\langle 0, 0, q\rangle)$ is degenerate. The tile corresponding to $\langle 0, 0, q\rangle$ occupies the western section of the tiling. For tilings associated to simple lattices, the pattern along $\partial\tau(\langle 0, 0, q\rangle)$ entirely depends on those along $\partial\tau(e_1)$ and $\partial\tau(e_2)$. More precisely, the sequence of slopes along the boundary of the degenerate tiles has cyclic symmetry- Damon first discovered this relationship. In the next chapter, we seek to explain the existence of such structure in our tilings.

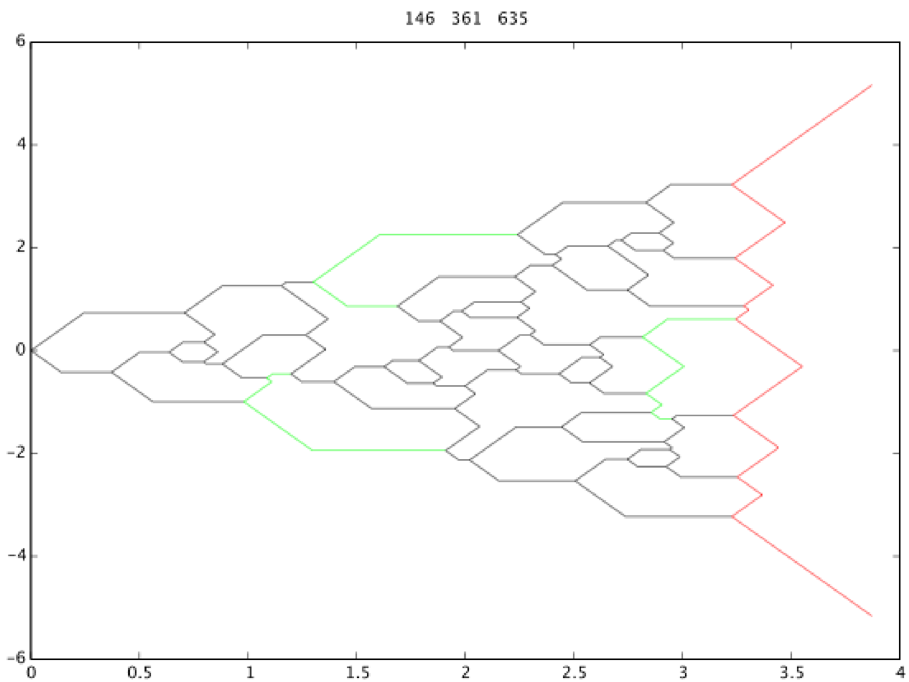


Figure 3.1: 146 and 361 are both cube roots of unity modulo 635.

Chapter 4

ABC Theorem

Patterns in the tiling associated to a lattice point to possible symmetries in the lattice itself. In the case of simple axial lattices, patterns in the boundary of trivial tiles, and therefore of all degenerate tiles, can be completely understood in terms of their associated subgroups.

4.1 Slopes of Two-Dimensional Lattices

The associated subgroups of simple level q lattices are in one to one correspondence with $F \subset (\mathbb{Z}/q\mathbb{Z})^n$ of order q. Recall that if Λ is a two-dimensional axial lattice, then it is also simple. In particular, F_Λ, can be generated by some $(p_1, p_2) \in (\mathbb{Z}/q\mathbb{Z})^2$, where p_1 and p_2 are units modulo q. Therefore,

Definition 4.1. For an axial lattice, Λ, of level q and dimension 2, with $F_\Lambda = \langle (p_1, p_2) \rangle$, we define the slope, σ, of Λ to be the residue class of the unit modulo q

given by $p_1^{-1}p_2$.

The slope of a lattice is independent of the choice of generator and can be recovered from any of its nonzero elements.

Lemma 4.1. *Suppose Λ is a simple reduced axial lattice of level q and dimension 2. Then there exists unique $\sigma \in \mathbb{Z}/q\mathbb{Z}$ such that $(m,n) \in F_\Lambda$ and $n = \sigma m$.*

Proof. Since the surjective image of F_Λ is a cyclic group, F_Λ can be generated by an element of the form $(1,\sigma)$. Now consider uniqueness. Let $(m,n) \in F_\Lambda$ be a generator. Then $(m,\sigma m) \in F_\Lambda$. Let π_1 signify projection onto the x-axis. Because $\pi_1|_{F_\Lambda}$ is surjective, m must be a unit modulo q. Suppose that $(1,\sigma_1),(1,\sigma_2) \in F_\Lambda$. Since $\pi_1|_{F_\Lambda}$ is also injective, $\sigma_1 = \sigma_2$. □

Lemma 4.2. *If a lattice in $\mathbb{Z}^2$ has a slope of σ, then $(1,\sigma)$ is in the lattice.*

Proof. The slope of a lattice is defined only in the case of simple axial lattices of dimension 2. Suppose the lattice is of level q. Recall that π_1 signifies projection onto the x-axis. Since $\sigma \in (\mathbb{Z}/q\mathbb{Z})^*$ and $\pi_1(F_\Lambda) = \mathbb{Z}/q\mathbb{Z}$, $\sigma^{-1} \in \pi_1(F_\Lambda)$. Let $n = 1$ in the previous proof. Then $m = \sigma^{-1}$ and $(1,\sigma)$ is in the lattice. □

The slope of a two-dimensional axial lattice can be related to its level via the language of continued fractions. Recall that the Euler bracket is a recurrence relation wherein

$$[] = 1,$$

$$[a] = a,$$

and

$$[a_1, .., a_k] = [a_1, ..., a_{k-1}]a_k + [a_1, ...a_{k-2}].$$

Notably,

$$[a_1, .., a_k] = [a_k, ..., a_1].$$

When expressed as an Euler bracket, the level of an axial lattice in $\mathbb{R}^2$ coincides with the following relationship between its nontrivial pivot classes.

Definition 4.2. Let Λ be a two-dimensional lattice, ν the number of nontrivial pivot classes in Λ, and $\{v_k\}_{k=0}^{\nu+1}$ an enumeration of pivots with nonnegative y-coordinate (one for each pivot class) by increasing height. Choose v_0 so that v_0 makes an obtuse angle with v_1. By the strand of Λ, we mean the ν-tuple of positive integers $(a_1, \cdots, a_\nu)$ where a_k is the index of $\mathbb{Z}v_{k-1} + \mathbb{Z}v_{k+1}$ in Λ.

In [2], Damon shows that

Theorem 4.3. *([2]) Given an axial lattice,* $\Lambda \subset \mathbb{R}^2$*, of level* $q > 2$*, there exists a strand,* $(a_1, \cdots, a_n)$*, such that* $\min(a_1, a_n) \geq 2$*,* $q = [a_1, \cdots, a_n]$*, and for each* $k = 1, \cdots, n$*,*

$$v_{k+1} = a_k v_k + v_{k-1}.$$

Two A-orbits correspond to each strand. If σ is the slope of a two-dimensional lattice of level q and dimension 2, then σ is in $\mathbb{Z}/q\mathbb{Z}$. More precisely,

Definition 4.3. Let σ be a residule class modulo q. We say σ is positive if $\sigma \cap \{1, \cdots, \lfloor \frac{q}{2} \rfloor\} \neq \emptyset$, zero if $0 \in \sigma$, and negative if $-\sigma$ is positive. We define its absolute value to be the unique integer in $\{0, \cdots, \lfloor \frac{q}{2} \rfloor\} \cap (\sigma \cup -\sigma)$ and denote it by $|\sigma|$.

The pivot equivalence classes of a lattice is a multidimensional continued fraction of the lattice that distinguishes vectors that map to shortest vectors in its A-orbit. The slope of an axial lattice can be used to generate representatives from each equivalence class while its strand recursively yields such a set of vectors after specification of a particular nontrivial pivot. Via continued fraction theory, the strand of a two-dimensional axial lattice can be recovered from its slope.

Theorem 4.4. [1] *Let* $\Lambda \subset \mathbb{R}^2$ *be an axial lattice of level* q*, slope* σ *and strand*

$q = [a_1, \cdots, a_n]$. *Then*

$$\frac{q}{|\ \sigma\ |} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + a_1}}.$$

In two dimensions, every root of unity modulo q is realized as the slope of an axial lattice of level q. In particular, the slope of a lattice is invariant under its A-orbit and precisely one A-orbit corresponds to each slope.

Lemma 4.5. *For any pair of integers* $1 \leq \sigma \leq q$ *with* $\gcd(\sigma, q) = 1$*, there exists an axial lattice,* $\Lambda \subset \mathbb{R}^2$*, of level* q *and slope* σ*, unique up to its* A*-orbit.*

Proof. Suppose $F \subset (\mathbb{Z}/q\mathbb{Z})^2$. If F gives rise to an axial lattice of level q, then Lemma 3.3 requires that F be a cyclic group generated by a 2-tuple of units modulo q. Without loss of generality, consider $F = \langle 1, \sigma \rangle$. Then $|F| = q$, the lift of F under the canonical projection homomorphism, $\Lambda = F + (q\mathbb{Z})^2$, is an axial lattice of level q, and the slope of Λ is the residue class of $1^{-1}\sigma$, or σ. Since F by construction is the associated subgroup of Λ and an invariant of its A-orbit, an axial lattice of level q and slope σ is unique up to its A-orbit. $\square$

In general, axial lattices are not always simple. We next consider the three-dimensional case, where this equivalence vanishes.

4.2 Canonical Slopes and the ABC Theorem

The tiling associated to a lattice, Λ, captures its pattern of Λ-boxes. For a simple axial lattice, maximal Λ-boxes are in one to one correspondence with maximal boxes in any of its coordinate hyperplane projections. Let the lift of each such projection, $\eta_i : \mathbb{R}^{n-1} \to \mathbb{R}^n$, be defined by

$$\eta_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n).$$

Now

Theorem 4.6. *Let $\Lambda \subset \mathbb{R}^n$ be an axial lattice and fix $i \in \{1, \cdots, n\}$. Let $\Lambda_i = \widehat{\pi}_i \Lambda$ and let v be a trivial pivot of Λ on the i-th axis. Then $\widehat{\pi}_i$ maps any Λ-box containing v to a Λ_i-box. Conversely, any Λ_i-box $B \subset \mathbb{R}^{n-1}$ lifts to a Λ-box $B' \subset \mathbb{R}^n$ that contains v via the formula $B' = \eta_i(B) + [-1, 1]v$.*

Proof. Assume B is a Λ-box. Then $v \in \partial B$. If $\widehat{\pi}_i(B)$ is a Λ_i-box, then every nonzero lattice point in $\widehat{\pi}_i(B)$ is in $\partial\widehat{\pi}_i(B)$. Suppose z is a nonzero lattice point in $\widehat{\pi}_i(B)$ and u is a vector in B such that $\widehat{\pi}_i u = z$. Since z is nonzero, u cannot lie on the i-axis. Thus u is not v. Because a suitable multiple of v added to u always yields a lattice point in B with i-coordinate between $-\frac{q}{2}$ and $\frac{q}{2}$, u can be assumed to have this property. If z were in the interior of $\widehat{\pi}_i(B)$, then u would be in the interior of B, contradicting the assumption that B is a Λ-box. Hence z must be in $\partial\widehat{\pi}_i(B)$.

Conversely, suppose B is a Λ_i-box and consider the box, $B' = \eta_i(B) + [-1,1]v$. Because v is in the lattice, B' has nonzero intersection with Λ. If the interior of B' contains no nonzero lattice points, then B' is a maximal Λ-box. Observe that $\widehat{\pi}_i$ maps the interior of B' to the interior of B. Let u be a lattice point in the interior of B'. Then $\widehat{\pi}_i u$ is in the interior of B. Since B is a Λ_i-box, the image of u must be the origin. In particular, u lies on the i-axis, which contains only integer multiples of v. Hence $u = 0$. □

When Λ is a simple level q lattice in $\mathbb{R}^3$, any generator of F_Λ is a triplet of units modulo q. The projections of simple lattices of level q are also simple lattices of level q. If (x, y, z) is any generator of F_Λ, then $x^{-1}y$, $y^{-1}z$, and $z^{-1}x$ are units modulo q that are invariants of the lattice. We refer to these three quantities as the *canonical slopes* of a three-dimensional lattice. We now prove our main theorem.

Theorem 4.7. *To each simple lattice $\Lambda \subset \mathbb{R}^3$, there are units, a, b, c modulo q such that*

$$abc \equiv 1 (mod\ q),$$

and

$$F = < a, b^{-1}, 1 > + (q\mathbb{Z})^3 = < a, ac, 1 > + (q\mathbb{Z})^3.$$

Proof. By Lemma 3.5, $\widehat{\pi}_i\Lambda$ is a simple axial lattice of dimension two for $i = 1, 2, 3$.

Without loss of generality, let $c = x^{-1}y$, $b = y^{-1}z$, and $a = z^{-1}x$. Lemma 4.2 guarantees that $(1, c) \in \widehat{\pi}_3(\Lambda)$, $(1, b) \in \widehat{\pi}_1(\Lambda)$, and $(a, 1) \in \widehat{\pi}_2(\Lambda)$. Now consider a generator of F_Λ. Theorem 4.6 requires that $(1, c, bc) \in F_\Lambda$. Since $(a, ac, abc) \in F_\Lambda$, $abc \equiv 1 \mod q$. Then $(a, ac, 1)$ is equivalent to (a, ac, abc) in F_Λ. Furthermore, b^{-1} is equivalent to ac in F_Λ and therefore, $(a, b^{-1}, 1) \in F_\Lambda$. □

Our tilings cover A and tilings associated to simple axial lattices have the least complicated behavior. All tilings with finite numbers of tiles belong to axial lattices. Degenerate pivots define degenerate tiles, and, as shown in Lemma 3.9, simple lattices possess the least number of degenerate pivots amongst axial lattices. For a simple axial lattice of level q, the pattern along the boundary of its degenerate tiles can be explained by the ABC Theorem.

The A-orbit of a lattice in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$. Let Λ be a lattice in $\mathbb{R}^3$ and v be a representative of one of its pivot equivalence classes. Because every $(t, s) \in \mathbb{R}^2$ can be uniquely associated to some $a \in A$ by 2.1, every point in the plane can be identified with some $a\Lambda$. Given $a \in \tau(v)$, av is a shortest vector in $a\Lambda$ with respect to the supremum norm. In particular, a belongs to multiple tiles when pivots from different equivalence classes are all shortest vectors in $a\Lambda$. When the tiles defined by each pivot equivalence class of Λ do not overlap, such a strictly occupy the boundaries of tiles.

Via the ABC Theorem, the boundaries of any two degenerate tiles in the tiling associated to a simple lattice in $\mathbb{R}^3$ determine the structure of the third. Slopes for level q lattices are units modulo q. The slope of every lattice projection is defined if and only if the lattice is simple. By the ABC Theorem, the canonical slopes are all cube roots of unity modulo q. Combined with the level of the lattice and expressed as a continued fraction, the slope of a two-dimensional lattice projection determines the coefficients of a recursion relation between its pivot equivalence classes (Theorem 4.4). Coordinate hyperplane projections are $\mathbb{R}$-module homomorphisms. Since Theorem 4.6 guarantees that maximal Λ-boxes of axial lattices are in one to one correspondence with maximal Λ_i-boxes, these same coefficients also recursively describe nontrivial pivots of Λ. In particular, these pivots intersect precisely those Λ-boxes whose boundaries also contain trivial pivots on the lattice's i-th axis. If u and v signify such pivots, then there exists $a \in A$ such that u, v, e_i, and pivots equivalent to these vectors are all of the shortest vectors in $a\Lambda$. More precisely, a occupies the boundaries of $\tau(u)$, $\tau(v)$, and $\tau(e_i)$.

Consider the canonical slopes of simple lattices in $\mathbb{R}^3$ and let a, b, and c, be as defined in the proof of the ABC Theorem. When $i = 3$, $\widehat{\pi}_3(\Lambda)$ is a lattice projection of slope c and the continued fraction expansion of $\frac{q}{c}$ recursively supplies the succession of pivots whose tiles border that of e_3. Similarly, the continued fraction expansion of $\frac{q}{b}$ determines the pattern along $\partial\tau(e_1)$ while that of $\frac{q}{a}$ influences the pattern along $\partial\tau(e_2)$. Our tilings are multidimensional continued fractions of lattices, and symmetries in a lattice give rise to symmetries in its associated tiling.

Chapter 5

Conclusion: Lattice Symmetries

Our project originates in the following family of continued fractions.

$$\langle 2, \underbrace{1, \cdots, 1}_{n\ times}, 2, \underbrace{1, \cdots, 1}_{n\ times}, 2\rangle = \frac{q_n}{a_n}.$$

Curiously,

$$a_n^3 \equiv (-1)^{n+1} (\bmod q_n)$$

for all nonnegative n. Damon discovered that the tiling in $\mathbb{R}^2$ associated to the lattice,

$$\Lambda = \mathbb{Z}\langle 1, \sigma, \sigma^2\rangle + (q\mathbb{Z})^3,$$

where σ is a cube root of unity modulo q, has cyclic symmetry along the boundary of its trivial tiles.

Does symmetry in the associated tiling guarantee symmetry in the lattice? Given any $(x, y, z) \in \Lambda$,

$$x \equiv n(\mod q),$$

$$y \equiv n\sigma(\mod q),$$

$$z \equiv n\sigma^2(\mod q).$$

The transformation for which $e_1 \to e_3$, $e_2 \to e_1$, and $e_3 \to e_2$, is described by

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

An application of this transformation to (x, y, z) yields $(n\sigma^2(\mod q), n(\mod q), n\sigma(\mod q))$. To test whether this triplet is in the lattice, let $m = n\sigma^2$. Then

$$m \equiv n\sigma^2(\mod q),$$

$$m\sigma \equiv n\sigma^3 \equiv n(\mod q),$$

$$m\sigma^2 \equiv n\sigma(\mod q).$$

Replacing σ with a cube root of unity of -1 modulo q also yields a lattice with symmetry, this time described by the transformation,

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

In summary,

Theorem 5.1. *The lattice,* $\Lambda = \mathbb{Z}\langle 1, \sigma, \sigma^2 \rangle + (q\mathbb{Z})^3$, *where* $\sigma \equiv \pm 1(\mod q)$ *is a lattice with order 3 cyclic symmetry described by*

$$\begin{bmatrix} 0 & 0 & 1 \\ \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \end{bmatrix}.$$

Viewed from its associated tiling, a lattice in $\mathbb{R}^3$ can be characterized as axial, level q, cyclic, or simple. We propose F_Λ as a lattice invariant with respect to which these properties can be defined. The slopes of a lattice depend on F_Λ. Our tilings have cyclic symmetry if all three slopes are cube roots of unity modulo q. Damon's original examples of such tilings includes those belonging to lattices whose slopes are cube roots of minus one modulo q.

For some q, there are no cube roots of unity, or of minus one. Conditions on $\phi(q)$ can be specified to determine whether such cube roots exist. We hypothesize that if there are no cube roots (of unity or minus one) mod q, then no level q lattice has an associated tiling with cyclic symmetry.

Lattices are fundamental tools and objects of study in Diophantine approximation and dynamics. So far, we only know how to define slopes for simple axial lattices. In the general class of axial lattices, F_Λ does not have to be cyclic and associated tilngs can have overlapping tiles. All of the examples of lattices associated with rationals, $\Lambda_{\frac{p_1}{q},\frac{p_2}{q}}$, are cyclic, because they are "partially" reduced–the projection to the z-axis is surjective. We do not yet see how to proceed to widen our understanding of the general axial lattice. How do conditions on q affect the structure of F_Λ? If q is square-free, do axial Λ give to rise to cyclic F_Λ? When q is a prime power, do cyclic associated subgroups occur only in the case of simple axial lattices?

As generalizations of Farey lattices, simple axial lattices are an important class of lattices. The ABC Theorem is a structure theorem for simple axial lattices. With this tool in hand, we can now begin to formulate questions about multidimensional continued fractions for other categories of lattices and perhaps eventually construct lattices with a desired suite of properties.

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