THE MINKOWSKI CONTINUED FRACTION ALGORITHM FROM A DYNAMICS PERSPECTIVE

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by

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CERTIFICATION OF APPROVAL

I certify that I have read ***The Minkowski Continued Fraction Algorithm From a Dynamics Perspective*** **by Jon Graham and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.**

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The Littlewood Conjecture is an open simultaneous Diophantine approximation problem. It and the stronger Margolis Conjecture, a proof of which would imply Littlewood, are currently viewed using dynamic systems. In this paper we will look at the space of "triple points", also known as the domain for the Minkowski continued fraction algorithm, using a dynamics perspective. We will look at "maximal boxes" and their relationship to lattices that are allowed to vary. We will explore the types of configurations possible for lattices with maximal boxes from different perspectives. We will explore the possible indexes for lattices with "uniquely" maximal boxes. This will be done to attempt to shed some light on the configuration space of triple points.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee **Date**

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Acknowledgements to follow.

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Chapter 1

Introduction

1.1 Introduction to the Littlewood Conjecture and Three Dimensional Staircases

The Littlewood Conjecture is that for any two real numbers α and β : $\liminf_{n\to\infty} n \, \|n\alpha\| \, \|n\beta\| = 0$, where $\|\cdot\|$ is the distance to the nearest integer. It has remained a conjecture for more than 85 years. In the intervening years, it has been shown that any counterexamples must have Hausdorff dimension zero. The conjecture is a famous open simultaneous Diophantine approximations problem. It is currently believed that to solve it, some appropriate generalization of the theory of continued fractions may be necessary. For about the last 35 years, the problem has been viewed using a dynamic systems approach.

One dynamic systems approach towards a resolution of the Littlewood Conjecture is documented in a progress paper written by Kyla Quillan, Progress in a Geometric Interpretation of the Littlewood Conjecture. [5] In her paper, she touches on several areas of investigation in a research program led by Dr. Yitwah Cheung including, Farey tiling, domains of approximation and three dimensional lattices that are associated with "staircases" in $\mathbb{R}^3$. In particular, she draws on two recent Masters Theses, under the supervision of Dr. Cheung. The first thesis was written by Samantha Lui, and was titled : Tiling Problem for the Littlewood Conjecture. [3] Ms. Lui showed that if the Farey tiles mentioned in the title are bounded, this is equivalent to a counterexample to the Littlewood Conjecture. Ms. Lui also introduced the lattice $\Lambda_{\alpha,\beta}$ associated with the two real numbers α and β from the Littlewood Conjecture or "L.C." She defined the lattice the following way: $\Lambda_{\alpha,\beta} := h_{\alpha,\beta}\mathbb{Z}^3$, where $h_{\alpha,\beta} =$

$$\begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathbb{Z}^3$ is the three dimensional integer lattice. This lattice $\Lambda_{\alpha,\beta}$ is involved in the L.C. investigation.

The second thesis was written by Lucy Odom, also under the supervision of Dr. Cheung. It was titled : An Overlap Criterion for the Tiling Problem of the Littlewood Conjecture. [4] In her thesis, Ms. Odum looked into the connection between tiling and lattices where α and β could be rational numbers, whereas in Ms. Lui's paper, α and β were non rational real numbers. Ms. Odum defined what are known as minimal Λ-boxes and maximal Λ-boxes in terms of a given lattice in $\mathbb{R}^3$. A minimal box is associated with what are known as "pivot points" in the tiling and maximal boxes are associated with "triple points" in the tiling. For Λ-boxes, set containment sets up a partial order by inclusion, leading to the definitions of *minimal* and *maximal* Λ-boxes.

Ms. Odom explored the characteristics of the tiling when α and β are rational. She showed that overlapping can occur. This led to our picturing the problem differently to avoid this problem of overlapping. We believe this to be a better intuitive approach that moves away from 2-dimensional "Farey Tiles" of Ms. Lui's Thesis to three dimensional "staircases". These staircases can be pictured as the surfaces in octant 1 of $\mathbb{R}^3$. These 3-dimensional staircases associated with the lattice $\Lambda_{\alpha,\beta}$, are the analogy of continued fractions of an irrational number for a problem that involves two irrational numbers.

We want to better understand these staircases formed by maximal and minimal Λ-boxes. Specifically, we are interested in investigating maximal Λ-boxes associated with lattices.

For any maximal Λ-box $\mathbf{B}$, $\exists!\ a \in \mathbf{A}$: a$\mathbf{B}$ is a cube. The maximal Λ-boxes and their associated triple points are "critical points" of the $\mathbf{A}$-orbit of $\Lambda_{\alpha,\beta}$. Knowing more about these critical points could lead to insight into the Margulis Conjecture, as well as the L.C. The Margulis Conjecture deals with $\mathbf{A}$-ergotic measures where $\mathbf{A}$ is the group of positive diagonal $k\ x\ k$ matrices on SL(k,$\mathbb{R}$)/(SL(k,$\mathbb{Z}$). A proof of the stronger Margulis Conjecture would imply the L.C. [1] The Margulis Conjecture states that, in $\mathbb{R}^3$, if the $\mathbf{A}$-orbit is bounded (in G/Γ = SL(3,$\mathbb{R}$)/SL(3,$\mathbb{Z}$)) then it is compact and takes the form of a 2-dimensional torus.

The investigation of L.C. involves a scaling matrix $g_{t,s}$. It is defined by Ms. Odom by:

$$g_{t,s} = \begin{pmatrix} e^{t+s} & 0 & 0 \\ 0 & e^{t-s} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$$ and a lattice defined by:

$\Lambda := g_{t,s}\Lambda_{\alpha,\beta}$. We can restrict $\mathbf{A}$ to $\mathbf{A}_+$ such that $g_{t,s} \in \mathbf{A}_+$. $\mathbf{A}_+$ is SL(3,$\mathbb{R}$) restricted to positive elements. A counterexample to L.C. may be characterized by $\mathbf{A}_+\Lambda_{\alpha,\beta}$ being bounded.

Any counterexample to the LC would allow us to construct a counterexample to the Margulis Conjecture. Therefore, if the Margulis Conjecture can be proven, no such counterexample to the LC could exist and LC would be true. We are motivated to get a better understanding of the space of triple points by understanding of what forms these maximal boxes with lattices can take in $\mathbb{R}^3$.

We will start in Chapter 1 by defining lattices, maximal-Λ-boxes, *uniquely* maximal-Λ-boxes and how to construct uniquely maximal$-\Lambda$-boxes. We will describe the relationship of uniquely maximal boxes to uniquely maximal cubes and define the moduli space of lattices in $\mathbb{R}^n$ with uniquely maximal cubes.

In Chapter 2 we will define the cube in $\mathbb{R}^3$ and then look at edges and quadrants to see if any patterns emerge in how edges and areas containing lattice points might be configured. We will introduce the "corner condition" and try to establish configurations. We will see that indeed interesting patterns do emerge. I'd like to point out that this approach was inspired by work done many decades earlier by the mathematician Minkowski.
For more on the connection with Minkowski, see Appendix A.

In Chapter 3, the subject will be the "index zero" case for a lattice with a uniquely maximal cube. This manifests itself as the two dimensional lattice embedded in $\mathbb{R}^3$ and a result will be useful for Chapter 4.

In Chapter 4, we will define a set of vectors on a "Standard Cube" with the goal of parameterizing all "degree one" lattices in the space of uniquely maximal cubes in $\mathbb{R}^3$. We want to understand all possible configurations of maximal-Λ-boxes. Finally, in Chapter 5, we want to find the best possible upper bound on the degree of a lattice in that space. Given a degree one lattice in the space, we will consider how it can be properly contained as a sublattice in some of larger degree in the space. Depending on the answer, this may be equivalent to showing that a simple domain exists for a certain function in the L.C. discrete dynamics approach. Working with that function is another area of research in Dr. Cheung's research program.

1.2 Maximal Boxes and Lattices

1. Definitions, Remarks, Lemma

We will start with the tools needed to proceed. First, we will define a lattice of dimension n. Then there will be more definitions and a few lemmas borrowed from Lucy Odom's Thesis that deal with Λ-boxes, maximal-Λ-boxes and constructing cubes from boxes. Finally we will define the uniquely maximal cube and the space of lattices with *uniquely maximal cubes.*

Definition 1.1. *Lattices are regular arrangements of points in Euclidean Space.*

Less formally, we define the lattice, Λ, as follows:

The lattice is a discrete additive subgroup of $\mathbb{R}^n$ i.e. it is a subset of $\mathbb{R}^n$ satisfying the following properties:

Subgroup of $\mathbb{R}^n$ - Λ is closed under addition and subtraction.

Discrete - There is an $\epsilon > 0$ such that any two distinct lattice points $\mathbf{x} \neq \mathbf{y} \in \Lambda$ are at a distance at least $\| \mathbf{x} - \mathbf{y} \| \geqslant \epsilon$.

Remark 1.1. This definition emphasizes the subgroup nature of a lattice in $\mathbb{R}^n$ and the fact that the points cannot be arbitrarily close to each other in the lattice. As an example: $\mathbb{Z}^n$ is an additive subgroup of $\mathbb{R}^n$ and a lattice. $\mathbb{Q}^n$ is an additive subgroup of $\mathbb{R}^n$, but is not a lattice, as it fails the discreteness condition. We will be interested in the index of a subgroup of a particular lattice.

A somewhat more formal definition of lattices:

Definition 1.2.*Let* $\mathbf{B} = [\mathbf{b}_1, ..., \mathbf{b}_k] \in \mathbb{R}^{n\times k}$ *be linearly independent vectors in $\mathbb{R}^n$. The lattice generated by $\mathbf{B}$ is the set* $\mathcal{L}(\mathbf{B}) = \{\mathbf{Bx}\text{:}\ \mathbf{x} \in \mathbb{Z}^k\} = \left\{ \sum_{i=1}^{k} x_i \cdot \mathbf{b_i} \,\text{:}\, x_i \in \mathbb{Z} \right\}$ *of all the integer linear combinations of the columns of $\mathbf{B}$. The matrix $\mathbf{B}$ is called a basis for the lattice $\mathcal{L}(\mathbf{B})$.*

In our case we will mostly deal with lattices where $n = k$, but there will be an example of a lattice where $n > k$ specifically, a 2-dimensional lattice embedded in $\mathbb{R}^3$. Our definition is a shorthand way of stating that our lattice is made up of linear combinations of integer multiples of basis elements. When we construct a lattice, points on the surface of our Λ-box will be used as basis elements. We will be interested in parameterizing these basis elements.

We will define a Λ-box associated with a lattice by paraphasing Lucy Odom's **Odom – Definitions 2.1** and **2.2**, in her thesis as: a box that is a subset of $\mathbb{R}^n$ of the form $B(u) = \{x \in \mathbb{R}^n : |x_i| \leqslant u_i \text{ for } i = 1, ...n\}$ where $u_1, ...u_n$ are non-negative real numbers with the $intB \cap \{0\} = \{0\}$ and $\partial B \cap \Lambda \neq \emptyset$. [4]

Borrowing Ms. Odom's **Odom – Lemma 2.3**: *Let B be a $\Lambda-box$.*

Then B is relatively maximal (uniquely maximal) if and only if it contains a lattice point in each co-dimension open face, i.e.. $\partial B_i \cap \Lambda \neq \emptyset$, for each i. [4]

A Λ-box is constructed in $\mathbb{R}^n$ as follows: Given an arbitrary lattice in $\mathbb{R}^n$, we can move away from the origin until we encounter the closest point from the origin, using distance induced by the "Euclidean" norm. We now switch to the "Sup" norm meaning that we give this point a distance from the origin that is the absolute value of the largest tuple in our *n-tuple*. We now have two faces of our Λ-box because of the symmetrical nature of lattices. Suppose the jth-tuple is the largest tuple for our point. We can construct a face that is orthogonal to the "j-direction" that is the distance of the "jth" coordinate from the origin. We have a point on the "jth" face and the "-(jth)" face due to symmetry. We then repeat this process using the Euclidean norm-induced distance to find the point that is next closest to the origin such that the largest tuple is NOT the "jth" tuple and, the "jth" tuple of our new point is less that the "jth" tuple of our first point. In other words, we have the faces of our box in the "jth" direction, we are no longer interested in points lying beyond those two faces. Suppose the supremum is the "kth" coordinate. We now have two more faces of our Λ-box and they are in the " $\pm$ kth" direction. We keep repeating this until we have all $2n$ faces and the points from the lattice that lie on them. This is easiest to picture in $\mathbb{R}^2$ and $\mathbb{R}^3$, of course, but can be done with any n. Our Λ-box can take on all different sizes and shapes, depending on the configuration of the lattice. In $\mathbb{R}^3$ it might look like a cube, or possibly more like a shoebox. By construction, the only lattice point that can lie inside the

Λ-box is the origin. Informally, the open box is the Λ-box without the faces. More formally, we define the open box such that there exists an open ball around any element in the open box that stays entirely inside the open box, i.e. does not intersect an edge.

We can go from boxes to cubes with the help of Ms. Odom's **Odom – Lemma 2.1**. It states: *For any non-degenerate box $B \subset \mathbb{R}^n$ there is a unique $a \in A$ such that $Q = aB$ is a cube.* Note that a is a diagonal matrix in SL(n,R) whose elements are greater than zero.[4]

Definition 1.3. *Given any lattice $\Lambda \subset \mathbb{R}^n$*, let $\mathrm{Q}(\Lambda)$ *denote a maximal-Λ box that has been normalized into a cube and centered at the origin whose interior contains no nonzero lattice points. We may refer to it as either a normalized maximal-Λ-box or a maximal cube.*

Lemma 1.3. *For any lattice $\Lambda \subset \mathbb{R}^n$, $\mathrm{Q}(\Lambda)$ is a normalized maximal* Λ-box *iff there is a neighborhood* U *of the identity in A such that* $\mathrm{Q}(a\Lambda) \subset \mathrm{Q}(\Lambda)$ *for all* $a \in \mathrm{U}$.

Proof: If a normalized Λ-box Q is maximal then any sufficiently nearby normalized Λ-box is contained in Q. Conversely, a normalized Λ-box with this property is necessarily maximal. Since normalized Λ-boxes are parameterized by $a \in A \mapsto \mathrm{Q}(a\Lambda)$, the lemma follows. □

Definition 1.4. *We say* $\mathrm{Q}(\Lambda)$ *is uniquely maximal if $\#\Lambda \cap \partial_i Q(\Lambda) = 2\ \forall i$. Let $\mathcal{Q}_n$ denote the moduli space of lattices in $\mathbb{R}^n$ with uniquely maximal-Λ-boxes normalized to cubes, thought of as a subset of* $\mathrm{PSL}(n,\mathbb{R})$ / PSL $(n,\mathbb{Z}.)$

To answer the questions we seek to answer, we are interested in cases where there is a single lattice point on every face of the Λ-box. We are interested in lattices with a *uniquely maximal* Λ-box.

Definition 1.5 *The degree of a lattice Λ in $\mathcal{Q}_n$ will be defined as the index of the subgroup generated by $\Lambda \cap Q(\Lambda)$, provided this index is finite; otherwise the degree is zero.*

The degree is continuous on $\mathcal{Q}_n$, but not on the larger set $\mathcal{Q}'_n$ which consists of all lattices Λ whose maximal cube is Λ-maximal.

Remark 1.5.By the time that this project was completed, Dr. Cheung's Research Project had moved into new areas. As always happens, the lexicon changed as new insights were gained. There is now a more general term for pivot points. What were referred to as "pivot points" in my paper are now called "stable pivot points." There is a new term, "stratification", and a Λ-box has a "principal stratum" iff it is uniquely maximal. I will not go into detail about the new terms in this paper, but I mention them to point out that there are definitions in my paper that may have evolved into new definitions and that others may do so in the future as the research project continues.

Remark 1.6.Because $\mathbb{R}^n$ is a well behaved Hilbert Space with respect to rotations, reflections and translations relative to the origin, any Λ-box can be rotated and reflected with respect to the origin without changing the relationship between the Λ-box and the lattice. It can also be normalized to become a cube without any loss of generality in our investigation of the properties of lattices that have uniquely maximal Λ-boxes.
From this point forward when we assume a uniquely maximal-Λ-box we will be working with a normalized uniquely maximal-Λ-box also known as a uniquely maximal cube.

Chapter 2

The Three Dimensional Case

2.1 Trichotomy of Configurations for Maximal Λ-boxes

The first thing we would like to do is to establish what general forms a degree one lattice with a *uniquely maximal-Λ-box* takes. To do this, we will add a few more defined terms to help with our construction and establish what a select group of configurations looks like and then show that these can be easily generalized to all cases. There are four general kinds of lattices embedded in $\mathbb{R}^3$. There is a dimension zero lattice, which is the origin alone; a one-dimensional lattice, essentially a dotted line passing through the origin; a two-dimensional lattice, a lattice that lies in a plane and includes the origin; and a three-dimensional lattice that includes the origin and fills three space. The first two kinds of lattices are of no interest to us, but because our "triple point" may generate a Λ-box for a two-dimensional lattice, as well as three-dimensional lattices, it is in our interest to understand what forms the two-dimensional lattices take, as well as the three-dimensional lattices, and we will investigate both.

Definition 2.1 *A lattice is generic if it meets every co-ordinate plane only at the origin. This property is preserved by the* **A***-action, i.e.* Λ *is generic iff* $a\Lambda$ *is generic* $\forall a$ *in the diagonal group* **A**

Remark 2.1. The *generic* lattice contains no points where the xy, xz and yz-planes intersect the boundary of the maximal Λ box. The typical lattice satisfies the assumption that it it is *generic* on the boundary of a maximal $\Lambda_{\alpha,\beta}$-box. An example that isn't *generic* is the $[-1,1]^3$ which is a maximal box but not uniquely maximal for α, β non-integers. These examples are of no interest in this investigation. We make the assumption that all of our lattices are *generic* with associated maximal-Λ-boxes.

Definition 2.2 *A uniquely maximal cube in* $\mathbb{R}^3$
is a uniquely maximal cube centered at the origin with an edge length of $2a$*.*

The following argument can easily be generalized to any lattice Λ with a uniquely maximal cube in $\mathbb{R}^n$ but we will make the case in $\mathbb{R}^3$ because this is our Hilbert Space of interest.

The cube centered at the origin with edge lengths of $2a$ has 8 vertices that are the 8 combinations of $(\pm a, \pm a, \pm a)$. We can naturally break up the cube into 8 octants, cubes of side length a. We can associate each octant with the vertex that it contains. We can divide each face of our uniquely maximal cube into four congruent squares that share a common point at the center of the face. See $Figure$ 2.1. We will refer to such a square as a *quadrant*. Each quadrant is a face of one of the cubes of side length a, i.e., one of the octants of our uniquely maximal cube. Our uniquely maximal cube has exactly one lattice point on each face.We will concern ourselves with which quadrant that lattice point occupies. A cube in $\mathbb{R}^3$ has 12 edges. If we have a cube with the origin at the center, we can think of the edges coming in pairs $\mathbf{e}$ and $-\mathbf{e}$.

Definition 2.3 *Each pair of edges,* **e** *and* $-$**e**,

are the edges whose centers are furthest away from each other using the Euclidean norm.

We have six pairs of of this type. Let **S** be the set of 6 pairs of edges as defined above so that **e** and $-$**e** make up a pair.

Because of lattice symmetry, we can do all of our work with quadrants on the three "positive" faces as described in Definition 2.2 above.

Then, assuming Λ has a uniquely maximal cube $Q(\Lambda)$, and given an edge **e** of our cube centered at the origin, we can create the following two definitions, to simplify the picture.

Definition 2.4 *Let* **S** *be the set of* 6 *pairs of edges as defined above so that* **e** and $-$**e** *make up a pair. Let* $\mathbf{E} \subset \mathbf{F}$. *The set* **E**

is 6 edges such that no two edges come from the same pair.

This means if $\mathbf{e} \in \mathbf{E}$, then $-\mathbf{e} \notin \mathbf{E}$ or visa versa.

Each edge is an edge of the three positive faces of the cube ie $x = a$, $y = a$, *and* $z = a$ *and* $a > 0$.

Definition 2.5 *Let* ν *be a function from the edge* **e** *of our maximal cube to the set* $\{0, 1, 2, 3, 4\}$. *The set* $\{0, 1, 2, 3, 4\}$

is the number of possible quadrants adjacent to an edge that are occupied by lattice points. If **S** *is the set of edges,* $\nu : \mathbf{S} \rightarrow \{0,1,2,3,4\}$.

E *is a restriction on* **S** *such that* $\nu\restriction_E : \mathbf{E} \rightarrow \{0,1,2,3,4\}$.

The range is a possible multiset comprised of members of the set $\{0, 1, 2, 3, 4\}$. The next two lemmas, **Lemma 2.5** and **Lemma 2.6** will establish that we can choose either edge of a pair as one of our six edges and that the range is a multiset comprised of six elements from the set $\{0,1,2\}$. The first lemma tells us that the number of lattice points adjacent to both edges in a pair is the same. Note: if one edge in a pair is in **E**, the other is not. The second lemma tells us that the added condition of *uniquely maximal* to a maximal cube allows us to eliminate 3 and 4 from the range of ν .

Lemma 2.5 $\nu(-\mathbf{e}) = \nu(\mathbf{e})$

Proof: This is consequence of the definition of pairs of edges and the symmetric nature of Λ in relation to Q(Λ). □

Lemma 2.6 If Q(Λ) is uniquely maximal $\Rightarrow \nu(\mathbf{e}) \in \{0, 1, 2\}$.

Proof: The definition of uniquely maximal tells us that there is only one lattice point on each face which implies that the maximum number of adjacent lattice points to any edge is 2. □

Definition 2.7. *We define the signature of* Λ, $\nu(\mathbf{E})$,
to be the multiset of six values taken from the set: $\{0, 1, 2\}$.

The signature of Λ will help us establish which configurations are possible for a degree one lattice with a *uniquely maximal systole cube.* The signature will specify which quadrants the lattice points must occupy on the systole cube. Q(Λ) can sit in any orientation in $\mathbb{R}^3$ but for simplicity and without loss of generality, we can orient Q(Λ) with all of its edges parallel and perpendicular to the x, y, z co-ordinates. We are free to choose any 6 edges but the easiest way to proceed is to choose edges for the three faces of the cube where $x = y = z = a > 0$. See *Figure 2.1* below.
We are ready to look at our main result.

Proposition 2.8.Let Λ be uniquely maximal, i.e. Q(Λ) is the uniquely maximal systole cube of Λ, then the signature of Λ is $\{0^3, 2^3\}$ or $\{0, 1^4, 2\}$ or $\{1^6\}$

$$\text{i.e. } \nu(\mathbf{E}) = \begin{cases} \{0^3, 2^3\} \\ \{0, 1^4, 2\} \\ \{1^6\} \end{cases}$$

Note: ν is a piecewise function.

Proof:

By Lemma 2.4 an edge can have at most 2 adjacent lattice points meaning $0, 1$ or 2. This is equivalent to saying that at most it can have 2 adjoining quadrants occupied by lattice points. Likewise, it can have $0, 1$ or 2 adjoining quadrants with lattice points. From this point on, we will refer to "adjoining occupied quadrants" as *"full quadrants"*

Suppose we have an edge from **E** with 2 full quadrants. Without loss of generality, assume the edge is parallel to the $z - axis$. by rotation of the cube we can always make it the edge that runs from (a, a, a) to $(a, a, -a)$. The "mirror image" of this edge, the other edge in the pair, also has 2 full quadrants. That edge runs from $(-a, -a, a)$ to $(-a, -a, -a)$. Consider the quadrants on the positive $x - face$ and the positive $y - face$ in the first octant. The first octant is the one that contains the vertex (a, a, a). Either one quadrant or the other is occupied. They can't both be because of the *"corner condition"*. The *corner condition* is when two or more lattice points are in quadrants that are in the same corner, i.e., share the same vertex. Then the difference of any two will put a lattice point inside the cube, violating *maximality* of the uniquely maximal cube.

Regardless of which quadrant is occupied, we can rotate by $\frac{\pi}{2}$ around the $z - axis$ to find an edge with 0 full quadrants.

Of the 4 edges parallel to the $z - axis$, 1 pair has 2 full quadrants and the other pair has 0 full quadrants.

We now consider the positive $z - face$ of the cube. Because of the corner condition, only the quadrant in the second octant, containing vertex $(-a, a, a)$ or the quadrant in the fourth octant, containing vertex $(a, -a, a)$ can be occupied. Without loss of generality, we will choose the quadrant on the $x - face$ to be the occupied quadrant. This leaves us with a choice on the $z - face$ of either the quadrant in the second or fourth octant. See *Figure 2.1* and note the vertical edge parallel to the z-axis is in red and the quadrants in yellow.

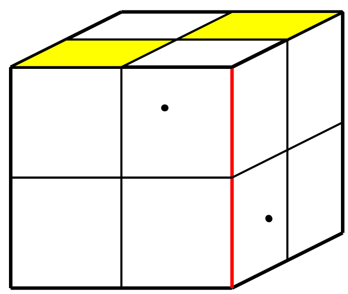

Figure 2.1 Possible quadrants

Suppose we choose the quadrant in the fourth octant. Now all of our points are accounted for on the cube so we can tally up the numbers of edges and adjoining occupied quadrants. I am choosing only one from each pair $\mathbf{e}$ or $-\mathbf{e}$. For the 6 edges we choose, 2 are parallel to the z-axis and 4 are the edges that contain the positive $z - face$. The edge we started with has 2 full quadrants and the one parallel to it has 0 full quadrants. Moving to the edges surrounding the positive $z - face$, 2 edges have 2 full quadrants and 2 edges have 0 full quadrants. This implies: $\nu(\mathbf{E}) = \{\mathbf{0^3}, \mathbf{2^3}\}$

Suppose we choose the quadrant on the positive $z - face$ in the second octant. Only the edges surrounding the positive $z - face$ change. Now each edge has 1 full quadrant. This implies: $\nu(\mathbf{E}) = \{\mathbf{0}, \mathbf{1^4}, \mathbf{2}\}$.

It is not hard to see that if we start out with an edge with 0 full quadrants that there must be parallel edges with 2 full quadrants. Therefore, we have exhausted the cases involving edges with 0 or 2 full quadrants. This leaves only the case where each edge has 1 full quadrant.

Suppose the edge that runs from (a, a, a) to $(a, a, -a)$ has 1 full quadrant. Without loss of generality, we can choose the quadrant on the $x - face$ of the first octant to be occupied. All of the parallel edges must have 1 full quadrant. Regardless of whether or not the positive $y - face$ in octant 2 is occupied, the positive $z - face$ can only have 2 occupied quadrants that avoid the corner condition. In either case, if 1 quadrant is occupied, we have $\nu(\mathbf{E}) = \{\mathbf{0}, \mathbf{1^4}, \mathbf{2}\}$. If the other quadrant is occupied then all edges of the cube have 1 full quadrant which implies $\nu(\mathbf{E}) = \{\mathbf{1^6}\}$ $\square$

When we start with a uniquely maximal cube, and break the faces down into quadrants, we've shown that there are three basic configurations that can occur with relation to which quadrants can be occupied by lattice points. We will refer to $\nu(\mathbf{E}) = \{\mathbf{0^3}, \mathbf{2^3}\}$ as Signature 0, $\nu(\mathbf{E}) = \{\mathbf{0}, \mathbf{1^4}, \mathbf{2}\}$ as Signature 1 and $\nu(\mathbf{E}) = \{\mathbf{1^6}\}$ as Signature 2. See *Figures 2.2, 2.3* and *2.4*.

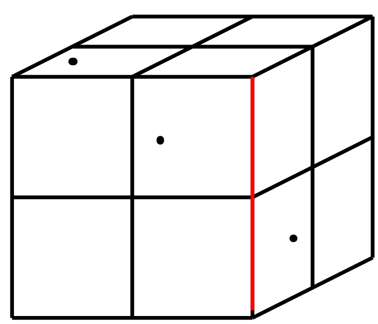

Figure 2.2 Signature 0

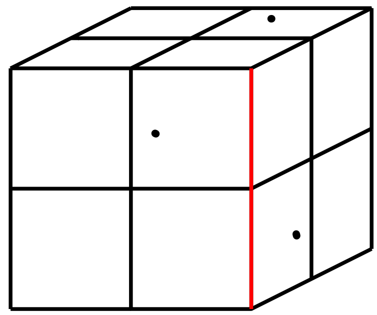

Figure 2.3 Signature 1

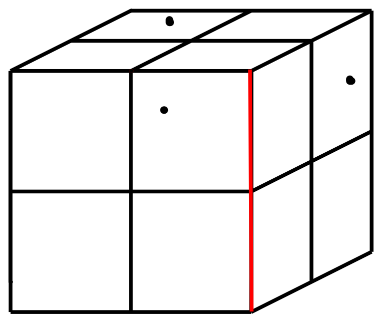

Figure 2.4 Signature 2

Chapter 3 we will talk more about the Signature 0 case, which we will also refer to as Index 0. In Chapter 4, we will explore different types of the Index 1 cases and their configurations parametrically. We will refer to them as *Configurations* to distinguish them from the Signatures of Chapter 2. While there is a relationship between the two, we will not talk about that much in Chapter 4, except to point out how they match up.

Chapter 3

The Index 0 Case

3.1 The Standard Cube and Admissibility

We have described what a uniquely maximal lattice in $\mathbb{R}^3$ looks like in relation to the edges and quadrants of a maximal systole cube, the so-called Signature. Now we can look at the finer structure within those quadrants. We will simplify my uniquely maximal cubes, without loss of generality, by defining a *standard* cube as well as creating other definitions to help us with our constructions.

We will show that if we have a standard cube, and a point on the x-face, the y-face, and the z-face represented by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively, and without loss of generality, if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, the following three statements are equivalent:

i) $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

ii) One is the sum of the other two

iii) The Signature is $\{0^3, 2^3\}$.

This will turn out to be a useful result when parameterizing a degree one lattice in three dimensions in Chapter 4.

Definition 3.1. *We will refer to a standard cube in* $\mathbb{R}^3$ *as one that is centered at the origin, has edges of length 2 and faces at* $x = \pm 1$, $y = \pm 1$ *and* $z = \pm 1$ *that are orthogonal to the* x, y and z *axes respectively.*

Given $\omega = (a_2, a_3, b_1, b_3, c_1, c_2)$ such that $(a_2, a_3, b_1, b_3, c_1, c_2) \in (-1, 1)$, let $\Lambda_\omega = \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$, where $\mathbb{Z}$ are the integers and $\mathbf{u}= (1, a_2, a_3)$, $\mathbf{v}= (b_1, 1, b_3)$ and $\mathbf{w}= (c_1, c_2, 1)$.

Definition 3.2. *I refer to ω as admissible if $\Lambda_\omega \cap (-1, 1)^3 = 0$. Note that Λ_ω defines a degree one lattice in $\mathbf{Q_3}$ iff ω is admissible and dim $\Lambda_\omega = 3$.*

3.2 Linear Dependence, the Special Linear Combination and Signature Zero

Without loss of generality let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be on positive x, y and z faces respectively of the standard cube and above the x, y plane. Let $\mathbf{u}= (1, a_2, a_3)$, $\mathbf{v}= (b_1, 1, b_3)$ and $\mathbf{w}= (c_1, c_2, 1)$. Note that if any one is the sum of the other two we can always rotate the cube on the diagonal axis so that the one on the z-face is the sum of the other two.

Proposition 3.2. $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent $\iff$ $\mathbf{u} + \mathbf{v} = \mathbf{w}$.

Proof:

Let ω be admissible. Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent. Suppose $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent.

This implies that $\mathbf{w}$ in Λ_ω and on the z face of the standard cube, is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Suppose $\mathbf{u} + \mathbf{v}$ is not equal to $\mathbf{w}$. Then $\exists\ a, b$ such that they are positive integers where either a or b or both are greater than 1, without loss of generality, and $a\mathbf{u} + b\mathbf{v} = \mathbf{w}$, by the definition of a lattice. Our three vectors form a triangle in the plane that $\mathbf{u}$ and $\mathbf{v}$ span and that the lattice, Λ_ω, lies in. We can express $\mathbf{u}$+$\mathbf{v}$ in barycentric co-ordinates: $\mathbf{u}+\mathbf{v} = d_1\mathbf{u} + d_2\mathbf{v} + d_3\mathbf{w}$. Where $d_1 + d_2 + d_3 = 1$ and $d_i \geqslant 0$. These conditions place $\mathbf{u}$+$\mathbf{v}$ inside the triangle created by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. To show: it also places $\mathbf{u}$+$\mathbf{v}$ inside the open cube. Substituting $a\mathbf{u} + b\mathbf{v}$ for $\mathbf{w}$, we get $d_1\mathbf{u} + d_2\mathbf{v} + (a\mathbf{u} + b\mathbf{v})d_3$. We can solve to get: $d_1 = \frac{(1-b)}{(1-a-b)}$ $d_2 = \frac{(1-a)}{(1-a-b)}$ $d_3 = \frac{1}{(a+b-1)}$ and rewrite our three solutions as: $d_1 = \frac{(b-1)}{(a+b-1)}$ $d_2 = \frac{(a-1)}{(a+b-1)}$ $d_3 = \frac{1}{(a+b-1)}$. We rewrite $\mathbf{u} + \mathbf{v}$ in barycentric coordinates: $\mathbf{u} + \mathbf{v} =(\frac{(b-1)}{(a+b-1)}, \frac{(a-1)}{(a+b-1)}, \frac{1}{(a+b-1)})$. Using the triangle inequality one can show that unless a

and b are both 1, $\mathbf{u} + \mathbf{v}$ must lie within the "convex hull" hence, inside the open cube $(-1,1)^3$. This contradicts admissibility. Therefore $\mathbf{u} + \mathbf{v} = \mathbf{w}$.

The other direction: Let $\mathbf{u} + \mathbf{v} = \mathbf{w}$ such that $\Lambda_\omega = \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$. Then $\mathbf{u} + \mathbf{v}$ - $\mathbf{w} = 0$. By definition $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ cannot be linearly independent, therefore they are linearly dependent. □

With the conditions we have set for $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ it is visually intuitive that any plane that contains $\mathbf{u}$ and $\mathbf{v}$ that passes through the origin must stay out of the quadrant on the z-face where c_1 and c_2 are both less than zero. However, a short exercise using the cross product for $\mathbf{u} \times \mathbf{v}$ with the dot product for a general (x, y, z) will be useful.
$\mathbf{u} \times \mathbf{v}$= $(a_2b_3 - a_3, a_3b_1 - b_3, 1 - a_2b_1)$ then $(a_2b_3 - a_3)x + (a_3b_1 - b_3)y + (1 - a_2b_1)z = 0$. When we plug in the point (0,0,1) to the dot product we can see that $1 - a_2b_1 \geqslant 0$ and this indicates that the intersection of the plane containing $\mathbf{u}$ and $\mathbf{v}$ cannot intersect the cube in the quadrant on the z-face where c_1 and c_2 are both less than zero.

Proposition 3.3. $\mathbf{u} + \mathbf{v} = \mathbf{w} \iff$ signature $\Lambda_\omega = \{0^3, 2^3\}$. Let ω be admissible. Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent. Suppose $\mathbf{u} + \mathbf{v} = \mathbf{w}$

Proof:

Starting in the forward direction with the equation for the plane passing through the origin containing $\mathbf{u}$ and $\mathbf{v}$, I have $(a_2b_3 - a_3)x + (a_3b_1 - b_3)y + (1 - a_2b_1)z = 0$. Replacing (x, y, z) with $\mathbf{w} = (c_1, c_2, 1)$, I get: $(a_2b_3 - a_3)c_1 + (a_3b_1 - b_3)c_2 + (1 - a_2b_1)1 = 0$..
Note that both $(a_2b_3 - a_3)$ and $(a_3b_1 - b_3)$ are less than zero. Without loss of generality suppose $c_1 > 0$ and $c_2 < 0$. Then $(a_2b_3 - a_3)c_1 + (a_3b_1 - b_3)c_2$ will be greater than -1. This means that if $(a_2b_3 - a_3)c_1 + (a_3b_1 - b_3)c_2 + 1 = a_2b_1$ that $a_2b_1 > 0$.
This implies that either both a_2 and b_1 are negative or both are positive. if they are both positive then the corner condition applies. $\mathbf{u}$ and $\mathbf{v}$ cannot be these quadrants. This implies they are both negative. Now if either c_1 or c_2 is negative and they are not both negative the corner condition applies.
We know that they cannot both be negative because of the remarks above so both c_1 and

c_2 must be positive. This means that the signature of $\Lambda_\omega = \{0^3, 2^3\}$.

In the backward direction I start with the Λ_ω being of Signature Zero as introduced in Proposition 2.5 ie. signature $\Lambda_\omega = \{0^3, 2^3\}$. This implies that both the x and y coordinates of $\mathbf{u} + \mathbf{v}$ are less than 1. Then, in order to stay out of the open cube, the z coordinate of $\mathbf{u}$ + $\mathbf{v}$ must be greater than or equal to 1. Suppose it's greater than 1, ie. $\mathbf{u} + \mathbf{v} \neq \mathbf{w}$ By construction the z coordinate of $\mathbf{u} + \mathbf{v}$ has to be less than 2. This implies that $\mathbf{w}$ - $(\mathbf{u} + \mathbf{v})$ $\in (-1,1)^3$. $\rightarrow|\leftarrow$

$\mathbf{u} + \mathbf{v} = \mathbf{w}$. □

We. have shown the claim: i) → ii) → iii) → i)

For notes on Propositions 3.2 *and* 3.3, *see Appendix B.*

Chapter 4

Determining the Space of Parameters for Index 1

4.1 Introduction

We have described what a uniquely maximal lattice in $\mathbb{R}^3$ looks like in relation to the edges and quadrants of a maximal cube. I have also defined the Standard Cube and admissibility explored the index zero configuration which I will now refer to as Configuration 0. Now we can look at the finer structure within the quadrants of the index one case.

We will start out with two vectors on our Standard Cube that are *admissable* and show the restrictions on the location of the third vector due to *admissability*.

Given $\omega = (a_2, a_3, b_1, b_3, c_1, c_2)$ such that $(a_2, a_3, b_1, b_3, c_1, c_2) \in (-1, 1)$, let $\Lambda_\omega = \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$, where $\mathbb{Z}$ are the integers and $\mathbf{u} = (1, a_2, a_3)$, $\mathbf{v} = (b_1, 1, b_3)$ and $\mathbf{w} = (c_1, c_2, 1)$. We will add one more condition from Chapter 2. Our cube is *generic* in that none of the points lie in the (x, y), (x, z) or (y, z) planes.

4.2 Parameterizing Configuration One

Definition 4.2. *We define $S_{x,y}$ as the open square of side 2 centered at (x, y).*

The region on the the x, y plane of possible x and y values for the $\mathbf{w}$ vector such that ω is admissible, will be called an $\mathbf{L}$ region for its rough shape.

Definition 4.3. *We will define the* $\mathbf{L}$ *region which we will call* $\mathbf{L}_1(a, b)$ *as*

$\mathbf{L}_1(a, b){:=}S_{0,0} \setminus (S_{1,a} \cup S_{b,1} \cup S_{1+b,1+a})$.

Remark 4.3. The description of the previous definition is: This is the open square of side 2 centered at $(0, 0)$ on the (x, y) plane, minus: the open square of side 2 centered at the projection of $\mathbf{u}$ on the (x, y) plane i.e. $(1, a_2)$, and the open square of side 2 centered at the $\mathbf{v}$ projection i.e. $(b_1, 1)$, also on the (x, y) plane and finally, the open square of side 2 centered at the projection of $(\mathbf{u}+\mathbf{v})$ i.e. $(1 + b_1, a_2 + 1)$ on the (x, y) plane.

We are now ready to look at the Configuration 1 form of a degree one lattice with a *uniquely maximal systole cube*. We start with a lattice as described in Remark 3.2. We assume that ω *is admissible*. We then claim the following:

Proposition 4.4. $\Lambda_\omega \in \mathcal{Q}_3 \iff (c_1, c_2) \in \mathbf{L}_1(a_2, b_1)$.

We will start out by showing the necessity of the right side for the left side to be true. It is clear that $a_3 + b_3 > 1$ by the signature of Λ_ω, otherwise admissibility is violated. If $a_3 + b_3 > 1$, this means that in order for $\mathbf{u}$ and $\mathbf{v}$ to be on the faces of the closed cube $[-1, 1]^3$, neither a_3 or b_3 can be equal to zero. We claim it is also necessary for $(c_1, c_2) \in \mathbf{L_1}(a_2, b_1)$. Now it is straightforward to show which areas the first two co-ordinates of $\mathbf{w}$ i.e. (c_1, c_2) must avoid to stay out of the open cube when summed with $\mathbf{u}$ and/or $\mathbf{v}$.

Suppose: $(c_1, c_2) \in S_{1,a_2}$, then $\mathbf{w} - \mathbf{u}$ will end up in $(-1, 1)^3$, $\mathbf{w} - \mathbf{v}$ and $\mathbf{w} - (\mathbf{u} + \mathbf{v})$ do not. Suppose: $(c_1, c_2) \in S_{b_1,1}$, then $\mathbf{w} - \mathbf{v}$ will end up in $(-1, 1)^3$, $\mathbf{w} - \mathbf{u}$ and $\mathbf{w} - (\mathbf{u} + \mathbf{v})$ do not. Suppose: $(c_1, c_2) \in S_{1+b_1,1+a_1}$, then $\mathbf{w} - (\mathbf{u} + \mathbf{v})$ will end up in $(-1, 1)^3$, $\mathbf{w} - \mathbf{u}$ and $\mathbf{w} - \mathbf{v}$ do not.

To recap, c_1 and c_2 are the first two coordinates of $\mathbf{w}$. If c_1 and c_2 are not in the $\mathbf{L_1}$ region of $S_{0,0}$, then one the following: $\mathbf{w} - \mathbf{u}$, $\mathbf{w} - \mathbf{v}$ or $\mathbf{w} - (\mathbf{u} + \mathbf{v})$ will not stay out of $(-1, 1)^3$. Any point not in $\mathbf{L_1}$, must be in the open cube. See *Figure* 4.1

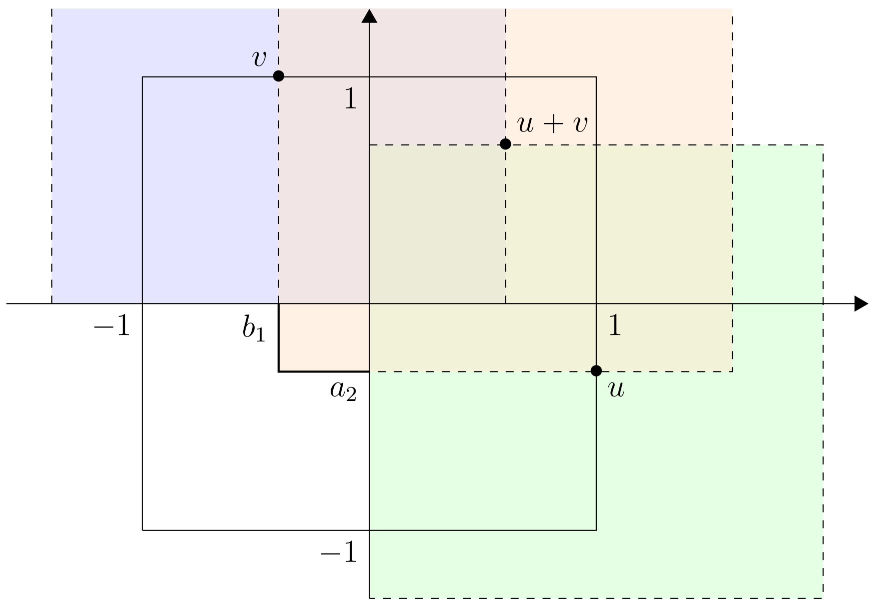

Figure 4.1 The region $L_1(a_2, b_1)$ in Proposition 4.4.

For the other direction, we will show that the right side is sufficient for the left side.

We will do this in two steps. In the first step we will slice our open standard cube with a plane formed by $s\mathbf{u}$ and $t\mathbf{v}$ and pull back to the s, t plane and check for any lattice points in the intersection of the plane and the open standard cube. In our second step, we will examine slices of the open cube by planes parallel to and above the first plane and, using symmetry, establish whether any lattice points are in the remaining standard open cube.

We start by assuming that (c_1, c_2), the first two co-ordinates of $\mathbf{w}$, lie in the area defined as $\mathbf{L_1}$ (a_2, b_1) on the (x, y) plane of $\mathbb{R}^3$. See *Remark* 3.4 and accompanying picture.

StepOne The standard cube being a uniquely maximal cube for Λ_ω can be thought of as being "sliced" by the plane that is spanned by $\mathbf{u}$ and $\mathbf{v}$. Any point on that plane is given by $s\mathbf{u} + t\mathbf{v} = \langle s + tb_1, sa_2 + t, sa_3 + tb_3\rangle$, where s, t are real numbers. (We are using (s, t) to avoid any confusion with (x, y) where our standard cube lies in $\mathbb{R}^3$). We can then look at the inverse image of the open cube under the map: $(s, t) \mapsto s\mathbf{u} + t\mathbf{v}$. Because the origin lies in this plane, there is a symmetry such that the image will be that of an open hexagon with parallel sides, that we will call H.

The parameters for H will be:

$$-1 < (s+tb_1) < 1, \qquad -1 < (sa_2+t) < 1, \qquad -1 < (sa_3+tb_3) < 1$$

, that is for all the points (x, y, z) that lie in H. Let us investigate the relationship between H, and the set of integer points that are our lattice, on the (s,t) plane.

The strategy will be to investigate the boundary lines of H in relation to our lattice points. Starting with our initial assumption that $b_1 < 0$ and using the x parameter:
$-1 < (s + tb_1) < 1$, when we evaluate the left side of the parametric equation and replace the left inequality with equality, we have the line: $t = \frac{-1}{b_1}\text{s} + \frac{-1}{b_1}$, whose slope and t-intercept is greater than 1. When we repeat this evaluating the right side, we have a second line parallel to the first one with a t-intercept of $\frac{1}{b_1}$. H lies in-between. Using our initial assumption that $a_2 < 0$ and using the y parameter: $-1 < (sa_2 + t) < 1$, and evaluating both sides, by replacing inequality with equality, we find that $a_2s - 1 = t$ and then $a_2s + 1 = t$, respectively. The slope is now less than 1 and the lines pass through $t = -1$ and $t = 1$. H lies in-between. We know that the lattice points come in integer pairs in (s,t). We see that the two pairs of parallel lines bound 4 sides of the hexagon such that any lattice points found in H will lie on the line where $s = t$ in the pull back of the image of the map $(s, t) \mapsto s\mathbf{u} + t\mathbf{v}$. The picture, *Figure* 4.2, reinforces this notion. Now we bring in our assumption that $a_3 + b_3 > 1$. There are no integer pairs in (s, t) other than $(0, 0)$ that make the z-parametric equation $-1 < (sa_3 + tb_3) < 1$ true. No lattice points end up in the hexagon H lying on the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ except the origin.

Now that I have shown that there are no non-zero lattice points in the open hexagon, which is the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ intersecting the open cube $(-1, 1)^3$, it remains to be shown that the rest of the open cube is also free from lattice points. See *Figure* 4.2

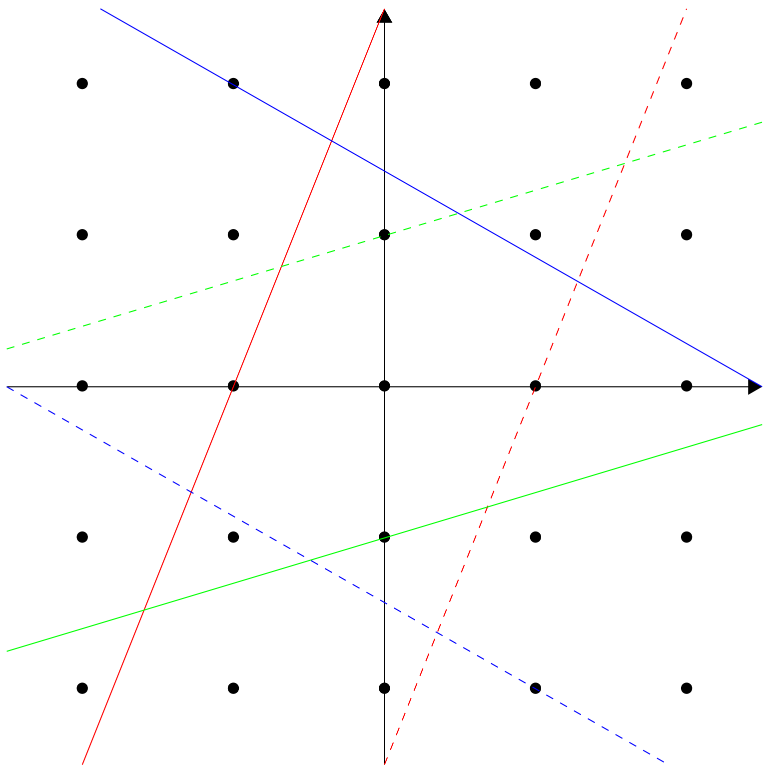

Figure 4.2 The hexagon H with $(a_2, b_1, a_3, b_3) = (-.3, -.4, .4, .7)$.

StepTwo Our strategy will be to move off the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ in integer increments of $\mathbf{w}$ to see whether these lattice points are inside or outside the open cube. Because our ($\mathbf{u}$,$\mathbf{v}$) plane passes through the origin, it is symmetric with regard to our open cube. It is enough for us to look at positive values of n to show that the cube is free from non-zero lattice points. The relationship between the lattice and the plane will be the same on both sides of the plane. We can fix an arbitrary n value, greater than or equal to zero, and call it n_0. We can take a "slice" of the lattice with a plane that runs parallel to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ and intersects the points described by $s\mathbf{u} + t\mathbf{v} + n_0\mathbf{w}$, $s, t, n \in \mathbb{Z}$ and n_0 fixed. The intersection of the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ and the open cube was a hexagon but we don't know a priori the shape of the intersection of this parallel plane and the open cube. We do know, however, that it will be captured by the intersection of three half planes whose edges are formed by three carefully chosen lines. We choose three lines from H to form a triangle we refer to as T_n. Looking again at the inverse image on the (s, t) plane with a fixed n_0, the triangle T_n is formed by the intersection of the three half planes formed by: 1) $s + b_1 t > -1 - c_1 n_0$ 2) $a_2 s + t > -1 - c_2 n_0$ 3)$a_3 s + b_3 t < 1 - n_0$.

See *Figure* 4.3 below.

If our arbitrary n, (n_0), is greater than zero this implies $a_3s + b_3t < 0$ by inequality 3. Now a_3,$b_3 \geqslant 0$, so it isn't possible for both s and t to be greater than zero. T_n is disjoint from the first quadrant. Note: both a_2 and b_1 are elements in the interval $(-1, 0.]$ Suppose that $s > 0$, $t < 0$, $a_2s + t > -1 - c_2n_0$ fails for all respective integers s and t. Suppose we reverse the signs for s and t making $s < 0$, $t > 0$. In this case, $s + b_1t > -1 - c_1n_0$ fails for the respective integers s and t. This leaves only one possibility: $s < 0$, $t < 0$. I claim, not only are s and t both negative integers, but $s = t$.

Starting with equation 1 from above, $s + b_1t > -1 - c_1n_0$, and the conditions $0 > b_1 > -1$, and $c_1 \leqslant 0$, the inequality: $s + b_1t > -1$ implies that $s > -1 - b_1t$ and $t \leqslant -1$. This implies that $s > t - b_1t$ implying that $s - t > -1$, and s and t being integers, s - $t \geqslant 0$ implying, $s \geqslant t$.

Starting with equation 2 from above, $a_2s + t > -1 - c_2n_0$ and the conditions $0 > a_2 > -1$ and $c_2 \leqslant 0$, the inequality: $t + a_2s > -1$ implies that $t > -1 - a_2s$ and $s \leqslant -1$. This implies that $t > s - a_2s$, and $t - s > -1$. Again, s and t being integers, implies that t - $s \geqslant 0$. and $t \geqslant s$. Using both relationships between s and t, $s = t$.

If there are any points in the lattice inside the open cube $(-1, 1)^3$ for $s\mathbf{u} + t\mathbf{v} + n_0\mathbf{w}$, s and t must be negative and they must be equal. In other words, any points in the lattice in the (s, t) plane that lie in the open cube will be of the form: $(-m, -m)$ where $m \in \mathbb{Z}_+$. Because T_n is star convex with respect to the origin, when $n = 0$, if $(-m, -m) \in T_n$, then $(-1, -1) \in T_n$.

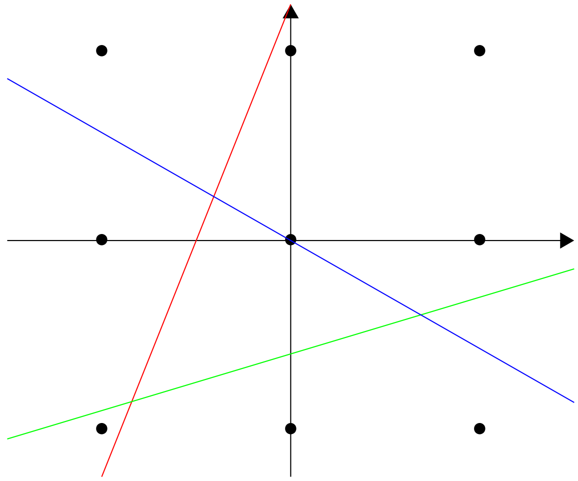

Figure 4.3 The triangle T_1 with $(a_2, b_1, a_3, b_3, c_1, c_2) = (-.3, -.4, .4, .7, -.5, -.4)$.

Using the contrapositive, it will be enough to show that $(-1, -1) \notin T_n$. If I show that, if: $s \leqslant -1$ then $t < -1$ and if: $t \leqslant -1$ $s < -1$, that will do it.

A look at the $\mathbf{L}_1$ box in the appropriate diagram, reveals the following: if $(c_1, c_2) \in \mathbf{L}_1(a_2, b_1)$ either $c_1 < b_1$ or $c_2 < b_2$, when s and t are less than zero. Suppose $s \leqslant -1$. then $t < n_0(\frac{-c_1}{b_1}) < -n_0 \leqslant -1$. Suppose $t \leqslant -1$. Then $s < n_0(\frac{-c_2}{a_2}) < -n_0 \leqslant -1$. In either case, $(s, t) \neq (-1, -1)$ and we are done. There are no cases where s and t are negative and equal therefore there are no lattice points in the open cube $(-1, 1)^3$.

4.3 Parameterizing Configuration Two

I claim that due to the symmetry of the lattice on the uniquely maximal cube, we only have to look at the two general cases, Configuration 1 and Configuration 2. Note that what we find, as far as where the $\mathbf{L}$ region lies on the $z = 1$ face of the standard cube, can easily be generalized to any face on the cube by rotation of the cube around the origin.

Configuration 2 is similar to Configuration 1 except that $a_2 > 0$. Again, *ω is admissible*. We split this into two different cases but we will only need to differentiate between the two cases near the end of the argument. In Case 1, $0 \leqslant b_3 < a_3$ and in Case 2, $0 \leqslant a_3 < b_3$.

Proposition 4.5.

Case One: $\Lambda_\omega \in \mathcal{Q}_3 \iff (c_1, c_2) \in \mathbf{L}_2(a_2, b_1)$.

Case Two: $\Lambda_\omega \in \mathcal{Q}_3 \iff (c_1, c_2) \in \mathbf{L}_2'(a_2, b_1)$.

For <u>Case One</u>: A visual inspection of all the different linear combinations of **u**, **v** and **w** in relation to the standard cube reveals that we need only concern ourselves with three linear combinations: **w** - **u**, **w** - **v** and **w** - (**u** - **v**), with the exception of b_1 and/or b_3 being zero in either Case 1 or Case 2, which are excluded by our cube being generic.
If we stay out of the squares of side 2 that are centered at: **u**, **v** and **u** - **v**, because of the corner condition, the so called "**L**" region, (or "**L**" box), is the entire quadrant. We will define this region as $\mathbf{L}_2$ for consistency. At this point we will only point out that there was a little bit of confusion on this point originally. It was thought originally that $c_1 > 0$ was included in Configuration 2 which made the region have an "L" shape, but it was found that with a simple set of rotations that this was really Configuration 1 again.
It is necessary for **w** to be in this $\mathbf{L}_2$ box (the entire quadrant) to be assured that the three linear combinations of concern stay outside the open cube $(-1, 1)^3$. To say it another way, it isn't possible for a point in the lattice to stay outside the open cube $(-1, 1)^3$ if it is outside the $\mathbf{L}_2$ box.

For <u>Case Two</u>: A similar visual inspection of the vectors and standard cube reveals that the only linear combinations of concern are: **w** - **u**, **w** - **v** and **w** - (**v** - **u**). If we stay out of the squares of side 2 that are centered at: **u**, **v** and **v** - **u** we form another **L** region or **L** box, which we have defined as $\mathbf{L}_2'$. It is necessary for **w** to be in this $\mathbf{L}_2'$ box to be assured that the three linear combinations of concern stay outside the open cube $(-1, 1)^3$. It is not possible for a point in the lattice to stay outside the open cube $(-1, 1)^3$ if it is outside the $\mathbf{L}_2'$ box. One can see this by looking at *Figure* 4.4.

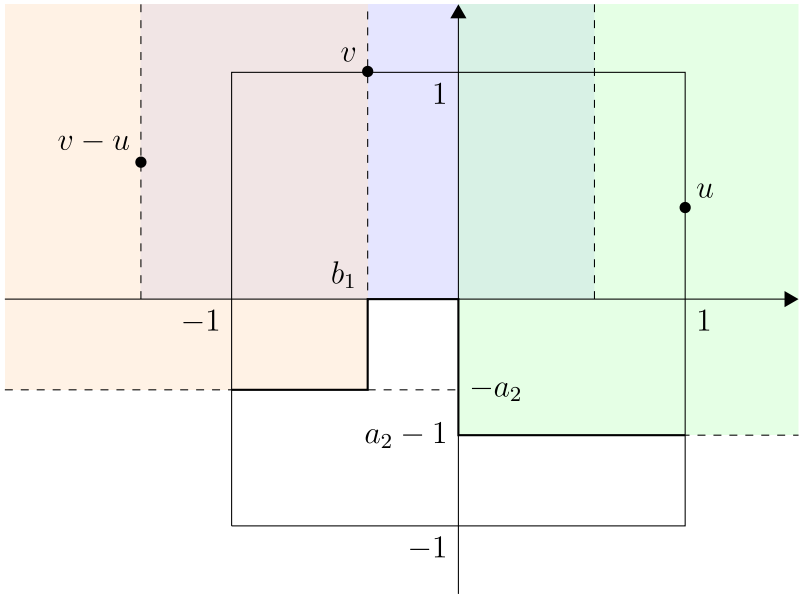

Figure 4.4 The region $L_2'(a_2, b_1)$ in Proposition 4.5 ($c_1 < 0$)

Now we will show the backward direction of the argument by showing the sufficiency of $(c_1, c_2) \in \mathbf{L}_2(a_2, b_1)$ in Case One, and the sufficiency of $(c_1, c_2) \in \mathbf{L}_2'(a_2, b_1)$ in Case Two.

Our argument will resemble our argument for Proposition 4.4 only with altered conditions. To reiterate our conditions: $\mathbf{u}= (1, a_2, a_3)$, $\mathbf{v}= (b_1, 1, b_3)$, $\mathbf{w}= (c_1, c_2, 1)$ where $a_2 > 0$ and $b_1 \leqslant 0$, Case 1: $0 \leqslant b_3 < a_3$, Case 2: $0 \leqslant a_3 < b_3$, we no longer need to meet the condition that $a_3 + b_3 > 1$ as was required in Configuration 1. We start by checking for lattice points in the intersection of the open cube $(-1, 1)^3$ and the plane formed by linear combinations of $\mathbf{u}$ and $\mathbf{v}$: given by $s\mathbf{u} + t\mathbf{v} = \langle s + tb_1, sa_2 + t, sa_3 + tb_3\rangle$, where s, t are real numbers. H is again formed, as in Proposition 4.4 by the parameters:
$-1 < (s + b_1t) < 1, -1 < (a_2s + t) < 1, -1 < (a_3s + b_3t) < 1$. As in Proposition 4.4, six lines can be formed by the inequalities creating H. H may or may not be a hexagon in Configuration 2 when we pull back to the (s, t) plane. The six lines are: $t = -\frac{1}{b_1}s \pm \frac{1}{b_1}$, $t = -a_2s \pm 1$, and $t = -\frac{a_3}{b_3}s \pm \frac{1}{b_3}$ Looking at the two diagrams of the two configurations, Configuration 1 versus Configuration 2 for Cases 1 and 2, we see that the parallel lines from the first set of equations look similar for Configuration 1 and in both cases of Configuration 2. The second set of parallel lines in both cases of Configuration 2 look different from Configuration 1 of Proposition 4.4 because now a_2 is positive. The third set

of lines is also similar in both configurations. The diagrams give us a sense of how hexagon H is situated in the (s, t) plane in relation to the lattice points. In Signature 2, it appears that in quadrants 1 and 3, H contains no lattice points. We verify this by noting that $a_2 s + t < 1$ eliminates all of the points in quadrant 1 where $t \geqslant 1$, which effectively eliminates quadrant 1. We can use the other side of this equation to eliminate quadrant 3, "i.e," $-1 < a_2 s + t$ implies that when $s < 0$, $t > -1$. When $b_1 < 0$ and t is positive, $-1 < (s + b_1 t)$ tells us that $s > -1$. This effectively eliminates any lattice point in quadrant 2. Finally, for quadrant 4, we use the other side of the equation that we used to eliminate quadrant 2, "i.e," $(s + b_1 t) < 1$, where both b_1 and t are negative. This implies $s < 1$, effectively eliminating quadrant 4. Finally, we note that: $-1 < (s + tb_1) < 1$, also tells us that when $t = 0$, $-1 < s < 1$, eliminating any non-zero point on the s-axis. So any lattice points with non-zero pairs of integers is ruled out of our new shape H. It turns out our new shape H, may not a hexagon at all, but, as the diagram suggests, it can be a parallelogram, symmetric around the origin. Whether H is a parallelogram or a hexagon is depends on which equation, equation 1 or equation 3 intersects equation 2 closest to the origin. Let me point out that the shape of H depends only on the equations, $t = -\frac{1}{b_1}s \pm \frac{1}{b_1}$, $t = -a_2 s \pm 1$, and is completely independent of the values of a_3 and b_3. This means up to this point in this direction of the proof, there was no reason to split up Configuration 2 into cases. In fact, it will turn out that both cases can be handled together until the final steps in the other direction of the proof as stated earlier.

As we did with Proposition 4.4, we will build the lattice up off of the plane that $\mathbf{u}$ and $\mathbf{v}$ span and take some arbitrary slice parallel to the plane. We will fix an arbitrary n value, we will call n_0, greater than or equal to zero. The points on the slicing plane are described by $s\mathbf{u} + t\mathbf{v} + n_0\mathbf{w}$, $s, t, n \in \mathbb{Z}$. As in Proposition 4.4 we don't know what the shape is, but we can contain it in the intersection of three half planes. The three lines are chosen from the parameters that formed H, as in Proposition 4.4. They form a triangle I will again refer to as T_n. Our goal is to show that for any $n > 0$ no lattice points can be found.

Our region is bound by the following three inequalities: *(1*)* $-1 - c_1 n < s + b_1 t$ *(2*)* $-1 - c_2 n < a_2 s + t$ *(3*)* $a_3 s + b_3 t < 1 - n$. If we look at the $n = 0$ case where T_n sits on the plane spanned by **u** and **v**, we can see that, unlike the Proposition 4.4 case, T_n can be quite large. We note that the three lines do not remain stationary as n increases, but their projection onto the (s, t) plane sweeps across that plane, eventually shrinking to zero as we move away from the plane spanned by **u** and **v** and pass out of the open cube $(-1, 1)^3$. When $n = 1$, we can see by the diagram of T_n that the boundary of *(3*)* now passes through the origin. We can now look at all $n \geqslant 1$. We add two more inequalities that come from the parameters that formed H. They are: *(1**)* $s + b_1 t < 1 - c_1 n$ and *(2**)* $a_2 s + t < 1 - c_2 n$. Then we will assume that $n \geqslant 1$. We know by *(3*)* that a_3 and b_3 cannot both be zero.

The general strategy is to start by assuming $s < 0$ and show that this leads to a contradiction. Then we suppose that $s \geqslant 0$ that this also leads to a contradiction. In short, no value of s works so that all of the inequalities are satisfied. This means that there are no lattice points in the intersection of the slices and the open cube $(-1, 1)^3$.

Step 1- Let $n \geqslant 1$ and suppose $s < 0$, then by *(2*)*, $t \geqslant 0$ and by *(1*)* and $b_1 < 0$, if $n \geqslant 1, c_1 > 0$. $\rightarrow\!|\!\leftarrow$ By the definition of $c_1 \in \mathbf{L}_2(a_2, b_1)$ or $c_1 \in \mathrm{L}'_2(a_2, b_1)$.

Step 2- So we can assume $s \geqslant 0$. This means by *(3*)* that $t < 0$.

The s-intercept of *(1**)* is such that: $c_1 > -1$ means $s + b_1 t < 1 - c_1 n < 1 + n$, i.e. what is the maximum that $1 - c_1 n$ can be? So, $s < 1 + n$ which means, s being an integer in the lattice, $s \leqslant n$. The t-intercept of *(3*)* is such that $a_3 s + b_3 t < 1 - n$. Then $t < \frac{-a_3}{b_3} s + \frac{1-n}{b_3}$. Note that $b_3 < 1$ makes $\frac{1-n}{b_3} < 1 - n$. Then $s = 0$ means $t < \frac{1-n}{b_3} < 1 - n$. So $t < -n + 1 \Rightarrow t \leqslant -n$ and $s \leqslant n, t \leqslant -n$, or $s + t \leqslant 0$.

By *(2*)*, $-1 < a_2 s + t, c_2 < 0$ or $-1 - a_2 s < t \Rightarrow 1 + a_2 s > -t$ then, $s > a_2 s > -t - 1 \Rightarrow s > -t - 1 \Rightarrow s \geqslant -t \Rightarrow s + t \geqslant 0$. Putting the two results together, $s + t \geqslant 0, s + t \leqslant 0 \Rightarrow s + t = 0, s = -t$. Because we are assuming that $s \geqslant 0$, we only need to rule out $(n, -n)$.

We will rule out the last remaining possibility:$(n, -n)$. If $a_3 \geqslant b_3$ as in Case 1, then because, $s \geqslant 0, t < 0$ and $-a_3 s < 0, -s < 0$, we have, $a_3 s + b_3 t < 0\ \forall n$ $\Rightarrow b_3 t < -a_3 s \Rightarrow t < \frac{b_3}{a_3} t < -s \Rightarrow t < -s$ and $(n, -n)$ is not possible. So we can assume that $b_3 > a_3$. We have taken care of Case 1 and move to Case 2.

Let's suppose that $c_1 \geqslant b_1$ then $-c_1 \leqslant -b_1$. Then $s + b_1 t < 1 - c_1 n < 1 - b_1 n$. So, $s + b_1 t < 1 - b_1 n$ or $b_1(t + n) < 1 - s$. Let $t = -n, s = n$, then $0 < 1 - n \Rightarrow n < 1 \rightarrow\mid\leftarrow$ We assumed $n \geqslant 1$.

Let's suppose $c_1 < b_1$, by the definition of the $\mathbf{L}_2'(a_2, b_1)$. box, $c_2 < -a_2 \Rightarrow -c_2 n > a_2 n$ and $-1 - c_2 n > -1 + a_2 n \Rightarrow a_2 s + t > -1 - c_2 n > -1 + a_2 n \Rightarrow a_2 s + t > -1 + a_2 n \Rightarrow t + 1 > a_2 n - a_2 s \Rightarrow t + 1 > a_2(n - s)$. Now let $s = n, t = -n \Rightarrow -n + 1 > a_2(n - n) = 0. \Rightarrow 1 > n$ $\rightarrow\mid\leftarrow$ We assumed $n \geqslant 1$. So $(n, -n)$ cannot be a possibility for (s, t). No integer values of s will give us points that lie in the open cube $(-1, 1)^3$. We have shown that both $(c_1, c_2) \in \mathbf{L}_2(a_2, b_1)$. and $(c_1, c_2) \in \mathbf{L}_2'(a_2, b_1)$. are sufficient.

Although "Signatures" and "Configurations" are arrived at differently, they do match up. Configuration 1 matches up with Signature 1, and Configuration 2 matches up with Signature 2. One can almost think of Configurations as being a "fine tuning" of the Signatures. We now have a sense of a one degree lattice with a uniquely maximal cube. This brings us to a most interesting question. What are the possible indexes for a lattice with a uniquely maximal cube? One might guess that they must be of index 1, but there is nothing a priori that makes it so. In fact, it was computer modeling that Dr. Yitwah Cheung performed revealing determinants not equal to one, that motivated further investigation. This is what led directly to my work with Dr. Cheung. It is now our goal to show that all lattices in $\mathcal{Q}_3$ are index 1 or index 0.

Chapter 5

Ruling Out Higher Index for Lattices with Uniquely Maximal Cubes in $\mathbb{R}^3$

5.1 Introduction

To show that all lattices in $\mathcal{Q}_3$ are index 1 or index 0, we will take an arbitrary lattice Λ in $\mathbb{R}^3$ with a uniquely maximal systole cube such that its volume, $\mathrm{vol}(\Lambda) \leqslant 1$ and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are on the respective x, y, z faces of $(\mathcal{Q}(\Lambda))$, and a sub-lattice, $\Lambda_0 = \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w} \subset \Lambda$. Without loss of generality, we can scale up Λ so that the unique systole cube becomes $[-1, 1]^3$, the standard cube. We claim that, because of the symmetry of the standard cube, with some possible rotations, Λ_0 now resembles one of the configurations we investigated earlier in Propositions 4.4 or 4.5. Then, again without loss of generality, we can say $\Lambda_0 = \Lambda_\omega$ where $\omega = (a_2, a_3, b_1, b_3, c_1, c_2)$ making the scaled up basis elements: $\mathbf{u}= (1, a_2, a_3)$, $\mathbf{v}= (b_1, 1, b_3)$,$\mathbf{w}= (c_1, c_2, 1)$. We can now talk about admissibility in terms of ω as we defined it earlier. We can rephrase: We assume that Λ_ω is a subgroup of Λ and show it must either be index 1 or index 0.

We will do this in three steps. Step one, using Minkowski's Convex Body (MCB) Theorem and a short Lemma, Lemma 4.1, we will show that the index of Λ_ω in Λ, $i.e.[\Lambda_\omega : \Lambda] \leqslant 4$. Step two, we will show that $[\Lambda_\omega : \Lambda] \neq 2$, and following from that, $[\Lambda_\omega : \Lambda] \neq 4$,. Step three, we will show that $[\Lambda_\omega : \Lambda] \neq 3$, Then what remains is that $[\Lambda_\omega : \Lambda] = 1$, or it is equal to 0.

5.2 Step One

StepOne : Minkowski's theorem is the statement that any convex set in $\mathbb{R}^n$ which is symmetric with respect to the origin and with volume greater than 2^n d(Λ), contains a non-zero lattice point. Note that d(Λ) is the determinant of Λ. The determinant of Λ is equal to the volume of what is known as the *Fundamental Parallelepiped* of the lattice, see diagram. The theorem was proved by Hermann Minkowski in 1889 and became the foundation of the branch of number theory called the geometry of numbers. The MCB Theorem tells us that if the vol($\mathcal{Q}(\Lambda)$) $\leqslant 8 \Rightarrow$ vol(Λ) $\geqslant 1$ before scaling up. In words, if the convex body has a volume less than or equal to 1, then the determinant of Λ must be greater than or equal to 1, in order for non zero lattice points to stay out of the convex body.

Lemma 4.1.$[\Lambda_\omega : \Lambda] \leqslant 4$

The vol(Λ_ω) is given by the expanded determinant

det(Λ_ω) =$(1 - a_2b_1) - b_3(c_2 - a_2c_1) + a_3(b_1c_2 - c_1)$. We want to show that this is less than 5. If it is greater than 5, than all five terms that follow 1 must be positive because each is less than 1. If just one of the terms is negative, then we start out less than 1, and adding four terms that must add to less than 4 will never reach 5. All five terms being positive means: if $a_2 < 0 \Rightarrow b_1 > 0$, but $\Rightarrow c_1c_2 > 0$ <u>and</u> $c_1c_2 < 0$. If: $a_2 > 0 \Rightarrow b_1 < 0$, but $\Rightarrow c_1c_2 < 0$ <u>and</u> $c_1c_2 > 0$. So the vol(Λ_ω) < 5. That leads directly to this: $\frac{volume(\Lambda_\omega)}{5} < 1$. Now $[\Lambda_\omega : \Lambda] = \frac{volume(\Lambda_\omega)}{volume(\Lambda)}$. Suppose $[\Lambda_\omega : \Lambda] \geqslant 5 \Rightarrow \frac{volume(\Lambda_\omega)}{volume(\Lambda)} \geqslant 5 \Rightarrow \frac{volume(\Lambda_\omega)}{5} \geqslant volume(\Lambda)$. So $1 > \frac{volume(\Lambda_\omega)}{5} \geqslant volume(\Lambda)$. MCB gives us the opposite result. $\rightarrow\mid\leftarrow$ So, $[\Lambda_\omega : \Lambda] < 5$. But it is a non negative integer so we can say $[\Lambda_\omega : \Lambda] \leqslant 4$.

Having dispatched any index greater than 4, we now focus briefly on index 4. Any group of order 4 is isomorphic to either $\mathbb{Z}_4$ or ($\mathbb{Z}_2$ x $\mathbb{Z}_2$). Both groups have a quotient group of order 2. Therefore, if Λ_ω is index 4 in Λ, it must have a quotient group of order 2 in Λ. By the contrapositive, if $[\Lambda_\omega : \Lambda] \neq 2$ then $[\Lambda_\omega : \Lambda] \neq 4$. If no such group of order 2 can exist, no group of order 4 can exist.

5.3 Step Two

StepTwo: $[\Lambda_\omega : \Lambda] \neq 2$

To investigate whether some admissible Λ_ω of index 2 in Λ might exist, it will be helpful to create a map from Λ_ω to a new lattice, $(2\mathbb{Z})^3$. The new basis elements for this lattice are: $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$. So $\rho(\mathbf{u})= (2,0,0), \rho(\mathbf{v})= (0,2,0), \rho(\mathbf{w})= (0,0,2)$. This linear map ρ, can be created knowing that we are going from one basis to another in $\mathbb{R}^3$ by first finding ρ^{-1} and then solving for ρ. We let $\rho^{-1}(2,0,0) = \mathbf{u}$, $\rho^{-1}(0,2,0) = \mathbf{v}$, $\rho^{-1}(0,0,2) = \mathbf{w}$ A quick visual inspection tells us that the map ρ^{-1} can be accomplished by the following matrix: $M_{\rho^{-1}} = \begin{pmatrix} \frac{1}{2} & \frac{b_1}{2} & \frac{c_1}{2} \\ \frac{a_2}{2} & \frac{1}{2} & \frac{c_2}{2} \\ \frac{a_3}{2} & \frac{b_3}{2} & \frac{1}{2} \end{pmatrix}$ From this we can figure out M_ρ .

M_ρ is a rather ugly matrix: $\frac{2}{1+a_2b_3c_1+a_3b_1c_2-a_2b_1-a_3c_1-b_3c_2} \begin{pmatrix} 1-b_3c_2 & b_3c_1-b_1 & b_1c_2-c_1 \\ a_3c_2-a_2 & 1-a_3c_1 & a_2c_1-c_2 \\ a_2b_3-a_3 & a_3b_1-b_3 & 1-a_2b_1 \end{pmatrix}$ but is nonetheless our map from Λ_ω to $(2\mathbb{Z})^3$.

Now that we have our map that identifies Λ_ω with $(2\mathbb{Z})^3$, i.e. $\rho : (\Lambda_\omega) \to (2\mathbb{Z})^3$, we can think of the lattice we began with, Λ, as corresponding to some subgroup of $\mathbb{Z}^3$. We can state it this way: $\rho(\Lambda_\omega) \subseteq \rho(\Lambda) \subseteq \mathbb{Z}^3$ or, equivalently, $(2\mathbb{Z})^3 \subseteq \rho(\Lambda) \subseteq \mathbb{Z}^3$. Let us assume that Λ_ω is index two in Λ. Equivalently: $[(2\mathbb{Z})^3 : \rho(\Lambda)] = 2$. (As an aside, that would imply that $[\rho(\Lambda) : (\mathbb{Z})^3] = 4$.)

The fact that we have our lattice $(2\mathbb{Z})^3$ and want to picture it as a subgroup of $\rho(\Lambda)$ with one coset, means we need to find another point independent of $(2\mathbb{Z})^3$ that we can form a union with, in order to build the lattice $\rho(\Lambda)$, keeping in mind that $\rho(\Lambda)$ is a subset of $\mathbb{Z}^3$. There are 7 points that it makes sense to look at that are a subset of $\mathbb{Z}^3$. Any one of them when added to $(2\mathbb{Z})^3$ will give us our desired lattice Λ such that $[(2\mathbb{Z})^3 : \rho(\Lambda)] = 2$. The points are: $(1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1)$ and $(1,1,1)$. Any other choice for a point will build the same lattice as one of the seven. Notice that the the ρ^{-1} map tells us that we are looking at the equivalent of: $\mathbf{u}/2$, $\mathbf{v}/2$, $\mathbf{w}/2$, $(\mathbf{u}+\mathbf{v})/2$, $(\mathbf{v}+\mathbf{w})/2$,

$(\mathbf{u}+\mathbf{w})/2$, $(\mathbf{u}+\mathbf{v}+\mathbf{w})/2$. Now we know by our construction of Λ_ω that $\mathbf{u}/2$, $\mathbf{v}/2$ and $\mathbf{w}/2$ are between $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively and the origin, and therefore are in the open cube $(-1,1)^3$. We also know, by a little bit of geometry, that $(\mathbf{u}+\mathbf{v})/2$, $(\mathbf{v}+\mathbf{w})/2$ and $(\mathbf{u}+\mathbf{w})/2$ each lie on the line joining the vectors in each case. This places them inside that open cube also. This leaves us with one point $(\mathbf{u}+\mathbf{v}+\mathbf{w})/2$, or equivalently, $(1,1,1)$. This one is not so obvious to discard, so it will be our candidate. We construct our Λ such that it is: $\Lambda_\omega \cup \{\ \frac{\mathbf{u}+\mathbf{v}+\mathbf{w}}{2} + \Lambda_\omega\ \}$ There is nothing a priori that restricts this from being a lattice that is uniquely systolic on the closed cube $[-1,1]^3$ and having Λ_ω as a subgroup of index 2 within it.

The points $\frac{\mathbf{w}-\mathbf{u}+\mathbf{v}}{2}$, $\frac{\mathbf{w}+\mathbf{u}-\mathbf{v}}{2}$, and $\frac{\mathbf{w}-\mathbf{u}-\mathbf{v}}{2}$ are "useful points" for us in this "constructed" lattice. We want to show that one of these points ends up in the open cube for the conditions in Proposition 4.4 and Proposition 4.5.

We start with the conditions for Proposition 4.4. This means that $\mathbf{u}= (1,a_2,a_3)$, $\mathbf{v}= (b_1,1,b_3)$, $\mathbf{w}= (c_1,c_2,1)$, $a_3+b_3>1 \Rightarrow \min(a_3,b_3)>0$. The $\max(c_1,c_2,a_2,b_1)\leqslant 0$, given. Looking at our "useful points": $\frac{\mathbf{w}-\mathbf{u}+\mathbf{v}}{2}$, $\frac{\mathbf{w}+\mathbf{u}-\mathbf{v}}{2}$, and $\frac{\mathbf{w}-\mathbf{u}-\mathbf{v}}{2}$, we immediately see that for the z-coordinates, the points must always lie between -1 and 1 in the z direction. This means we only have to check the x and y directions. If we find a point where both x and y coordinates also lie between -1 and 1, we have our counterexample. Taking $\frac{\mathbf{w}-\mathbf{u}-\mathbf{v}}{2}$, in the x-direction: $c_1-(1+b_1)>-2$ and because of $\max(c_1,c_2,a_2,b_1)<0$, must be less than 2. Dividing this by 2 means that x must lie between -1 and 1. Taking it in the y-direction: $c_2-(a_2+1)>-2$ and once again for the same reason as in the x direction, must be less than 2. As with the x direction we divide by two and now x,y and z lie between -1 and 1. We have our counterexample. This lattice does not stay out of the open cube.

Conditions are changed in Proposition 4.5 but, once again we only need check x and y coordinates.

As one can see from the diagrams, we need only check $c_1<0$.

For $c_1 < 0$ in the x direction: $-2 < c_1 + 1 - b_1 < 2$ and in the y direction: $-2 < c_2 + a_2 - 1 < 2$ which means $\frac{\mathbf{w}+\mathbf{u}-\mathbf{v}}{2}$ stays inside the open cube. Once again we have our counterexamples.

In conclusion, when we assume that Λ_ω is index two in Λ we end up with a Λ that has non zero points in the systole cube and violates our initial assumption for Λ. Λ_ω cannot be index two in Λ. Because of the argument above regarding groups and quotient groups, Λ_ω cannot be index four in Λ.

5.4 Step Three

StepThree: $[\Lambda_\omega : \Lambda] \neq 3$

This leaves us with index three. For the index 3 case we will use a similar approach. We will use a linear map $\rho : \Lambda_\omega \to (3\mathbb{Z})^3$ such that $[(3\mathbb{Z})^3 : \rho(\Lambda)] = 3$. (This implies that $[\rho(\Lambda)] : \mathbb{Z}^3] = 9$.) At this point, it is helpful to think of each coordinate in terms of mod 3. If we remove the origin, starting with $(0, 0, 1)$, $(0, 0, 2,)$, $(0, 1, 2)$... we have 26 possible candidates to choose from until we get back to something equivalent to (0,0,0) mod 3, in each coordinate. We have 26 choices, from which to pick two representatives for cosets to form a union with the lattice $(3\mathbb{Z})^3$. We are constructing our lattice $\rho(\Lambda)$ that $\rho(\Lambda_\omega)$ is a subset of. We can see that any 3-tuple that contains a 2 can be combined with something from $[(3\mathbb{Z})^3$ so that we always have a 3-tuple in that lattice with only 0's and 1's. This means that no matter what 3-tuple we choose, we could have chosen a 3-tuple made of up 0's and 1's to create our coset. We can combine two members of our initial coset to create our other coset, which is the additive inverse of the initial coset. This must be true for any group order 3. For our purposes we are going to choose one of those tuples that contain no 2's. We will take the union of this with $(3\mathbb{Z})^3$ and then take the additive inverse of this tuple to get the other coset, for instance, if we choose: $(1, 1, 1)$ for our initial coset, we choose: $(-1, -1, -1,)$ as the representative of the other coset.

We refer to our first tuple as: ϵ and the other one as: $-\epsilon$ such that $\rho : \Lambda_\omega \cup \{\Lambda_\omega + \epsilon\} \to (3\mathbb{Z})^3 \cup \{(3\mathbb{Z})^3 + (1,1,1)\}$ and $\rho : \Lambda_\omega \cup \{\Lambda_\omega + -\epsilon\} \to (3\mathbb{Z})^3 \cup \{(3\mathbb{Z})^3 + (-1,-1,-1)\}$.

Our $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ where $|\epsilon_i| \leqslant 1$ which is to say ϵ_i is 0 or ± 1. Mapping back to Λ_ω, and its superset Λ, for $[\Lambda_\omega : \Lambda] = 3$, it means that there will be elements of Λ that are: $\frac{\epsilon_1 \mathbf{u} + \epsilon_2 \mathbf{v} + \epsilon_3 \mathbf{w}}{3}$. This implies that by the construction of $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ that $|\frac{\epsilon_1 \mathbf{u} + \epsilon_2 \mathbf{v} + \epsilon_3 \mathbf{w}}{3}| < 1$, the vector length being less than 1, a counterexample. Which means that if Λ is a lattice such that $[\Lambda_\omega : \Lambda] = 3$, elements of Λ must end up in the open cube $(-1,1)^3$ which is contrary to our initial assumption about Λ so $[\Lambda_\omega : \Lambda] \neq 3$. Therefore $[\Lambda_\omega : \Lambda]$ is equal to 1 or 0.

Chapter 6

Conclusion

The Littlewood Conjecture (L.C.) is one of several problems that lie at the intersection of the theory of dynamical systems and number theory.
The tiling approach has raised interesting questions about the structure of the tiles in relation to pivot points and triple points and how much information is contained with the structure. In our investigation we have attempted to shed some light on the configuration space of the triple points in the context of uniquely maximal Λ-boxes and the associated lattices. With this developed configuration space, it may lead to the possible answer of how triple points can arise.

Starting with a uniquely maximal Λ-box $\mathbf{B}$ associated with lattice $\Lambda_{\alpha,\beta}$, there exists a unique element $a \in \mathbf{A}$: $a\mathbf{B}$ equals some uniquely maximal cube. Suppose the volume of our box $\mathbf{B}$ is $\mu_1\mu_2 n \leqslant 1$ for $n \geqslant 1$ Then there exists $a =$

$$\begin{pmatrix} e^{t+s} & 0 & 0 \\ 0 & e^{t-s} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}.$$

.

such that $a\mathbf{B} = \begin{pmatrix} e^{t+s} & 0 & 0 \\ 0 & e^{t-s} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ n \end{pmatrix}$.

Because we now have a cube, this leads us to : $e^{t+s}\,\mu_1 = e^{t-s}\,\mu_2 = e^{-2t}\,n = r \leqslant 1$

This leads us to $r^2 = e^{-4t}\,n^2$ and then $e^{6t} = \frac{n^2}{\mu_1\mu_2}$ Also $\frac{e^{t+s}\,\mu_1}{e^{t-s}\,\mu_2}$ leads to $e^{2s} = \frac{\mu_2}{\mu_1}$. Finally, we can express t and s in terms of our uniquely maximal Λ-box parameters.

$t = \frac{1}{6}\log(\frac{n^2}{\mu_1\mu_2})$

$s = \frac{1}{2}\log(\frac{\mu_2}{\mu_1})$

Each uniquely maximal box leads to a unique point (t, s) in $\mathbb{R}^2$.

The triple points form a discrete subset of $\mathbf{A} \cong \mathbb{R}^2$. We would like to understand the density of triple points in the tiling. If we denote $\mathbb{N}$(R) as the number of triple points in some disk of radius R, can we come up with a good rough estimate of the number of triple points for a disk of radius R? As R $\to \infty$ does

$$\lim_{R\to\infty} \frac{\mathbb{N}(R)}{\pi R^2}$$

exist? If so, what is it? This is an area of possible investigation in the future.

Appendix A: The Minkowski Connection

The original work that I began doing with Dr. Cheung on lattices and their associated uniquely maximal boxes and/or normalized cubes became the basis for part of Chapter 3 and for all of Chapters 4 and 5.

Right around the time that this work was being completed, Dr. Cheung, came upon an article by Gerhard Ramharter that was a synopsis of a book by Harris Hancock. Dr. Cheung subsequently came upon a copy of the book titled:
Development of the Minkowski Geometry of Numbers.[2] Dr Cheung asked me to take a look at a particular section that he had found. Although the book was somewhat difficult to decipher, it was somewhat like running into an old friend when, in a Chapter entitled, "The Theory of Continued Fractions", I came across a familiar looking illustration. It looked very much like a shoe box with lattice points on the faces i.e., a unique maximal box in $\mathbb{R}^3$. Apparently, much of what I was doing had been touched upon by Minkowski many decades earlier and expanded upon by a Dr. Paul Pepper.
Although the idea of having quadrants on a uniquely maximal box was not new to us, Dr. Pepper's emphasis on the quadrants was. This got us thinking in terms of splitting the faces of the uniquely maximal cubes into quadrants as is explained in Chapter 2 and then searching for patterns in terms of the edges and the quadrants.

Using the *corner condition*, the pattern that emerged was what I have referred to as *Signatures*. This is an easy way to separate out exactly what configurations of quadrants are possible to avoid the corner condition.

Before this pattern was understood, I became curious as to what all the possible configurations would be. To investigate, I used the uniquely maximal cube of length 2, defined in Chapter 3.

Each face of the standard cube has only one lattice point by definition. Because of symmetry I only needed to look at the positive faces of the standard cube. There are 64 (4^3) possibilities of three quadrants at a time, one from each positive face, representing where lattice points can be found. Of the 64 possibilities, 40 were eliminated because of the corner condition.

I decided on a system of numbering the quadrants for each face, which was somewhat arbitrary, but any bookkeeping device might be likewise. I also realize that my bookkeeping system was not a "natural" one, but I chose it at the time in order to be able to group certain configurations into permutations of three, in order to try to make the list all of the possibilities a shorter list. I chose to start in the corner of the cube where the quadrants that meet in the corner at the point (1,1,1). I numbered those three quadrants on the x, y and z faces respectively, number 1. I then moved counterclockwise around each face numbering the other three quadrants, $2, 3$, and 4. See *Figure A1.1*

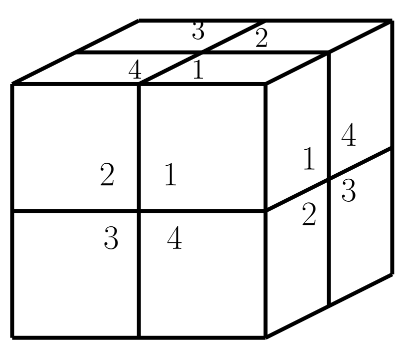

Figure A1.1 Signature 1 initial notation

Of the 64 possibilities, 40 were immediately eliminated because of the corner condition. The permutations were created by rotating the cube around the axis that runs in through (1,1,1) diagonally down through the origin and out through (-1,-1,-1). See diagrams. There were two types of basic permutations. In the first kind, that I refer to as "identical", the quadrants were all the same number, "i.e", (2, 2, 2). For instance : $(2,2,2) \longrightarrow (2,2,2) \longrightarrow (2,2,2)$. After the rotation, nothing has changed quadrant-wise. In the second kind this wasn't the case, "i.e", (1, 3, 3). Each rotation produces a new look quadrant-wise. For instance: $(1,3,3) \longrightarrow (3,1,3) \longrightarrow (3,3,1)$. This second kind I refer to as "differing". I can then use $(1,3,3)$ to represent all three of those configurations. Of the total of 24 possible configurations, there were 3 that were created by the identical kind of permutation and 21 that were created by 7 of the differing kind of permutations. The 3 identicals are: $(2,2,2),(3,3,3)$ and $(4,4,4)$. There are 7 differings: $(1,2,2),(1,3,3),(1,4,4),(2,4,3),(1,4,3),(1,3,2),(1,2,4)$.

Now this can be related to Signatures. We can describe Signature 1 using 4 *differings*. Signature 1 is where the each edge in the set of edges **E** has one adjacent occupied quadrant, "i.e", $\nu(\mathbf{E}) = \{1^6\}$. The four *differings* are: $(1,2,2),(1,3,3),(1,4,4),(2,4,3)$, with 12 configurations in total. We can describe Signature 2 using 2 *differings* and 2 *identicals*. Signature 2 is where 1 edge in the set of edges **E** has 0 adjacent occupied quadrants, 1 edge has 2 adjacent occupied quadrants and 4 edges have 1 adjacent occuied quadrant, "i.e", $\nu(\mathbf{E}) = \{0,1^4,2\}$. The 2 *differings* are: $(1,4,3),(1,3,2)$ and the 2 *identicals* are $(2,2,2),(4,4,4)$, with 8 configurations in total. Finally, we can describe Signature 0 with 1 *differing* and 1 *identical*. Signature 0 is where 3 edges in the set of edges **E** have 0 adjacent occupied quadrants, 3 edges have 2 adjacent occupied quadrants, "i.e", $\nu(\mathbf{E}) = \{0^3,2^3\}$..The *differing* is: $(1,2,4)$ and the *identical* is $(3,3,3)$, with 4 configurations in total. Those are the 24 different configurations that the quadrants can take for a lattice on a maximal systole cube. One can easily see the advantage of Signatures over my initial system of bookkeeping. My initial system was somewhat cumbersome and unnatural. However, the initial path I took did allow me to see something I might have missed had I not gone down it.

As I pointed out earlier, the book by Hancock was not easy to decipher, but much like my earlier experience of running into the familiar shoebox, I came upon the 24 combinations of quadrants and immediately recognized that I was going down a similar path that Dr. Pepper had taken.

This inspired me to take another look at Dr. Pepper's extension of Minkowski's work. What we called *maximal boxes*, Dr. Pepper called *free* and *extreme* parallelepipeds. He went on to construct a *free* and *extreme* parallelepiped with only 1 lattice point on each open face that we would refer to as a *uniquely maximal box*. We assume *generic* maximal boxes and he assumes *generic* parallelepipeds. Whereas we assumed the "corner condition" is true, he proved it for a *free*, *extreme* parallelepiped with only 1 lattice point on each open face.

His notation was different. He used matrices of positive and negative signs to denote the quadrants. For instance, his notation vs. mine for his 1st and 24th configurations:

$$\mathbf{1}: \begin{pmatrix} + & + & + \\ + & + & - \\ + & - & + \end{pmatrix} \equiv (1,2,4)$$

$$\mathbf{24}: \begin{pmatrix} + & - & - \\ - & + & - \\ + & + & + \end{pmatrix} \equiv (2,4,3)$$

He listed all 24 and then and gathered them into 6 groups of 4.

He labeled his six groups as: $\{$**I**, **II**, **III**, **IV**, **V**, **VI** $\}$ and called them *normal systems*. "**I**" in the normal systems was "#16" in the group of 24. So **I** $\equiv$ #16. For instance, then: #16 was reflected through the (x, y)-plane to become #19, reflected through the (x, z)-plane to become #9 and finally through the (y, z)-plane to become #6.

In his notation:

$$\mathbf{16}: \begin{pmatrix} + & + & + \\ - & + & - \\ - & - & + \end{pmatrix} \longrightarrow \mathbf{19}: \begin{pmatrix} + & + & - \\ - & + & + \\ + & + & + \end{pmatrix}$$

$$\mathbf{16}: \begin{pmatrix} + & + & + \\ - & + & - \\ - & - & + \end{pmatrix} \longrightarrow \mathbf{9}: \begin{pmatrix} + & - & + \\ -+ & + & + \\ - & + & + \end{pmatrix}$$

$$\mathbf{16}: \begin{pmatrix} + & + & + \\ - & + & - \\ - & - & + \end{pmatrix} \longrightarrow \mathbf{6}: \begin{pmatrix} + & - & - \\ + & + & - \\ + & - & + \end{pmatrix}$$

He did this with all 6 normal systems to get the 24 configurations. I want to point out that it is clear that Harris Hancock knew about the 3 fundamental forms that a free, extreme parallelepiped with 6 lattice points could take. He has drawings of 3 such cubes on the last page of volume 1 of his two 2 volume book.

Dr. Pepper's notation was more natural than mine and did a better job of shortening the list of 24 configurations. Having said that, my notation help me greatly in deciphering his, and I'm not sure that I would have been able to without it. I think that the Signature approach that we devised is an improvement on Dr. Pepper's system. In a sense, there is a path leading from Minkowski to Dr. Pepper, and to this investigation of maximal boxes, lattices and triple points. I have felt privileged to take part in it. This is rough draft 5.

Appendix B: Notes on Linear Dependence and Signature 0

In this appendix, we will show the simple algebra steps not shown in Proposition 3.2 and another way to show the forward direction of Proposition 3.3.

Starting with our three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ that lie in a 2-dimensional plane, we can describe any point in that plane in terms of barycentric co-ordinates. We can represent the point $\mathbf{u} + \mathbf{v}$ with the vertices of the triangle made up by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. The barycentric co-ordinates of the point $\mathbf{u} + \mathbf{v}$ are (d_1, d_2, d_3) i.e. $\mathbf{u}+\mathbf{v} = d_1\mathbf{u} + d_2\mathbf{v} + d_3\mathbf{w}$. Adding the condition $d_1 + d_2 + d_3 = 1$ and $d_i \geqslant 0$ will make them unique. We suppose $\mathbf{w}$ is a linear combination of the other two. Without loss of generality, we can write $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ for a, b elements of $\mathbf{Z}$. Substituting $a\mathbf{u} + b\mathbf{v}$ for $\mathbf{w}$ we can write: $\mathbf{u}+\mathbf{v} = d_1\mathbf{u} + d_2\mathbf{v} + d_3(a\mathbf{u} + b\mathbf{v})$. Then rearranging the right side: $\mathbf{u}+\mathbf{v} = (d_1 + ad_3)\mathbf{u} + (d_2 + bd_3)\mathbf{v}$ we create two more equations involving the barycentric co-ordinates and a and b.

$d_1 + ad_3 = 1$ and $d_2 + bd_3 = 1$. We can now solve for all three co-ordinates in terms of a and b.

Letting $\mathbf{A}$ be the matrix representing our three equations:

$d_1 + d_2 + d_3 = 1$

$d_1 + ad_3 = 1$

$d_2 + bd_3 = 1.$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}$$

$$\text{Then } \mathbf{A^{-1}} = \begin{pmatrix} \frac{-a}{1-a-b} & \frac{1-b}{1-a-b} & \frac{a}{1-a-b} \\ \frac{-b}{1-a-b} & \frac{b}{1-a-b} & \frac{1-a}{1-a-b} \\ \frac{1}{1-a-b} & \frac{-1}{1-a-b} & \frac{-1}{1-a-b} \end{pmatrix}$$

and $d_1 = \frac{1-b}{1-a-b}$, $d_2 = \frac{1-a}{1-a-b}$, $d_3 = \frac{-1}{1-a-b}$

We can rewrite as $d_1 = \frac{b-1}{a+b-1}$, $d_2 = \frac{a-1}{a+b-1}$, $d_3 = \frac{1}{a+b-1}$

Unless two of the three bayrcentric co-ordinates are zero, in which case, $d_1\mathbf{u} + d_2\mathbf{v} + d_3\mathbf{w}$ is equal to $\mathbf{u}$ or $\mathbf{v}$ or $\mathbf{w}$ and then by construction is not contained in the open cube $(-1, 1)^3$, we write: $d_1(1, a_2, a_3) + d_2(b_1, 1, b_3) + d_3(c_1, c_2, 1) = (x, y, z)$.

Using the triangle inequality we have:

$x = |d_1 + d_2b_1 + d_3c_1| \leqslant d_1 + d_2|b_1| + d_3|c_1| < d_1 + d_2 + d_3 = 1$

$y = |d_1a_2 + d_2 + c_3d_3| \leqslant d_1|a_2| + d_2 + d_3|c_3| < d_1 + d_2 + d_3 = 1$

$z = |d_1a_3 + d_2b_3 + d_3| \leqslant d_1|a_3| + d_2|b_3| + d_3 < d_1 + d_2 + d_3 = 1$

This implies that, except for the above case of two barycentric coordinates being zero, $(x, y, z) \in (-1, 1)^3$. Unless a and b are both 1, $\mathbf{u} + \mathbf{v}$ must lie within the open cube $(-1, 1)^3$. This gives us the result we want in Proposition 3.2.

There is another way to get the forward direction of Proposition 3.3:

$\mathbf{u} + \mathbf{v} = \mathbf{w} \Rightarrow$ signature $\Lambda_\omega = \{0^3, 2^3\}$, in two steps. I will include them here in Appendix B as Lemma 3.4 and Lemma 3.5.

Lemma 3.4. $\mathbf{u} + \mathbf{v} = \mathbf{w} \implies a_2 < 0$ and $b_1 < 0$.

Let $\mathbf{u} + \mathbf{v} = \mathbf{w}$. Let $a_2 > 0$. Suppose $b_1 > 0$. This is the corner condition and violates admissibility.

Suppose $b_1 < 0$. We know that $\mathbf{u_z} + \mathbf{v_z} = \mathbf{1}$. We also know that $\mathbf{u_y} + \mathbf{v_y} > \mathbf{1}$ but, for $\mathbf{w}$ to sit on the open z-face of the open cube, $\mathbf{u_y} + \mathbf{v_y} < \mathbf{1}$. $\rightarrow\mid\leftarrow$ The argument for $a_2 < 0$, $b_1 > 0$ is similar to the $a_2 > 0$, $b_1 < 0$ argument due to symmetry of the cube. $a_2 \nless 0$ or $b_1 \nless 0 \implies \mathbf{u} + \mathbf{v} \neq \mathbf{w}$ when $\mathbf{w}$ sits on the open z-face of the standard cube. □

Lemma 3.5. If $a_2 < 0$ and $b_1 < 0 \implies c_1$ and c_2 are both positive. Note that if $\mathbf{w}$ is on the z-face it is some $(c_1, c_2, 1)$. In order to avoid the corner condition, c_1 and c_2 have to have the same sign. In other words c_1 and c_2 will encounter the corner condition with $\mathbf{u}$ or $\mathbf{v}$ unless they have the same sign. Suppose c_1 and c_2 are both negative. Now, both a_2 and b_1 are greater than -1 and less than 0. This means that if $\mathbf{u} + \mathbf{v} = \mathbf{w}$ then $a_2 + 1 = c_2$. and $1 + b_1 = c_1$. This can't happen. So c_1 and c_2 must both be positive.

$\mathbf{w}$ must sit in the quadrant where c_1, c_2, c_3 are all positive which is Signature 0 ie. $\Lambda_\omega = \{0^3, 2^3\}$. □

Bibliography

[1] M. Einsiedler, A. Katok, and E. Lindenstrauss, *Invariant measures and the set of exceptions to Littlewood's conjecture*, Annals of Mathematics, 2006.

[2] Harris Hancock, *Development of the Minkowski geometry of numbers. Vols. One, Two*, Dover Publications, Inc., New York, 1964. MR 0169821

[3] Samantha Lui, *Tiling problem for Littlewood's conjecture*, MA Thesis, San Francisco State University, 2014.

[4] Lucy Odom, *An overlap criterion for the tiling problem of the Littlewood conjecture*, MA Thesis, San Francisco State University, 2015.

[5] Kyla Quillin, *A geometric approach to Littlewood's conjecture*, Littlewood Conjecture Notes, San Francisco State University, 2014.