WINNING GAMES FOR BOUNDED GEODESICS IN MODULI SPACES OF QUADRATIC DIFFERENTIALS

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Abstract. We prove that the set of bounded geodesics in Teichmüller space are a winning set for Schmidt's game. This is a notion of largeness in a metric space that can apply to measure 0 and meager sets. We prove analogous closely related results on any Riemann surface, in any stratum of quadratic differentials, on any Teichmüller disc and for intervals exchanges with any fixed irreducible permutation.

1. Introduction

In the 1966 paper [15] W. Schmidt introduced a game, now called a Schmidt game, to be played by two players in $\mathbb{R}^n$. He showed that winning sets for his game are large in the sense that they have full Hausdorff dimension, and that the set of badly approximable vectors in $\mathbb{R}^n$, which were known to have measure zero, is a winning set for this game. Schmidt's game and a modified version of it were used in [3] and [9] to establish that the set of bounded trajectories of nonquasiunipotent flow on a finite volume homogeneous space has full Hausdorff dimension, a result first established in [7] using different methods. The dynamical significance of badly approximable vectors is well-understood: in terms of the flow on the moduli space of $(n+1)$-dimensional tori $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ induced by the left action of the one-parameter subgroup $g_t = \mathrm{diag}(e^t,\dots,e^t,e^{-nt})$, a vector $\mathbf{x} \in \mathbb{R}^n$ is badly approximable if and only if it determines a bounded trajectory via $g_t U(\mathbf{x})\,\mathrm{SL}(n+1,\mathbb{Z})$ where $U(\mathbf{x})$ is the unipotent matrix whose (i,j)-entry is 1 if $i=j$, $-x_i$ if $i \le n$ and $j = n+1$, and 0 otherwise.

This paper is concerned with higher genus analogues of the same circle of ideas. Let $\mathcal{PMF}$ be Thurston's sphere of projective measured foliations on a closed surface of genus $g>1$. Let $D \subset \mathcal{PMF}$ consist of those foliations F such that for some (hence all) quadratic differentials q whose vertical foliation is F, the Teichmüller geodesic defined by q stays in a compact set in $\mathcal{M}_g$, the moduli space of genus g. Following [11], we use the terminology *Diophantine* to describe the foliations that lie in the set D. It was shown in [11] that a foliation F is Diophantine if and only if

$$\inf_{\beta} |\beta| i(F,\beta) > 0.$$

Here, the infimum is over all homotopy classes of simple closed curves β, $i(F,\beta)$ is the standard intersection number, and for $|\beta|$ one can take it to be the length of the geodesic in the homotopy class with respect to some fixed hyperbolic structure on the surface. The notion of Diophantine does not depend on which hyperbolic metric is

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chosen. Alternatively, we may fix a triangulation of the surface and take the number $|\beta|$ to be the minimum number of edges traversed by any curve in the homotopy class. In the moduli space of genus one, a.k.a. the modular surface, Teichmüller geodesic rays are represented by arcs of circles or vertical lines in the upper half plane with endpoint in $\mathbb{R} \cup \{\infty\}$. The notion of Diophantine extends the property of a geodesic ray that its endpoint is a badly approximable real number.

In [10] McMullen introduced two variants of the Schmidt game, giving rise to the notions of *strong winning* and *absolute winning* sets. (We recall their definitions in the next section.) It is not hard to show that absolute winning implies strong winning while strong winning implies *Schmidt winning*, i.e. winning in the original sense of Schmidt. In particular, they also have full Hausdorff dimension. McMullen raises the question in [10] as to whether the set of Diophantine foliations $D \subset \mathcal{PMF}$ is a strong winning set. In this paper, we give an affirmative answer to this question.

Theorem 1. *The set $D \subset \mathcal{PMF}$ of Diophantine foliations is a strong winning set, hence a winning set for the Schmidt game. However, it is not absolute winning.*

We remark that the Schmidt game requires a metric on the space, but the notion of winning is invariant under bi-Lipschitz equivalence. In the case of $\mathcal{PMF}$ there are various bi-Lipschitz equivalent ways of defining a metric. The most familiar is to use train track coordinates. (See [14] for a discussion of train tracks). A fixed train track defines a local metric by pull-back of the Euclidean metric. A finite collection of train tracks can be used to parametrize all of $\mathcal{PMF}$. One can then defined a path metric on $\mathcal{PMF}$ via a finite number of locally defined metrics.

The next theorem concerns quadratic differentials (see Section 3 for the definition of quadratic differentials) that determine bounded geodesics in a stratum.

Theorem 2. *Let $Q^1(k_1, \dots, k_n, \pm)$ be any stratum of unit norm quadratic differentials. Let U be an open set with compact closure $\bar{U} \subset Q^1(k_1, \dots, k_n, \pm)$ with a metric given by the pull-back of the Euclidean metric under a local coordinate system given by the holonomy coordinates of saddle connections. Then there exists an $\alpha > 0$ depending on the smallest systole in U such that the subset $E_Q \subset \bar{U}$ consisting of those quadratic differentials q such that the Teichmüller geodesic defined by q stays in a compact set in the stratum is an α-strong winning, hence winning for the Schmidt game. It is not absolute winning.*

Again we remark that the metric is not canonical as it depends on a choice of coordinates. However different choices give bi-Lipschitz equivalent metrics and the notion of winning is well-defined. We remark that bounded has a slightly more restrictive meaning here than in the case of $\mathcal{PMF}$ in that in this case no saddle connection gets short along the geodesic, while in the case of $\mathcal{PMF}$ the condition is slightly weaker in that no simple closed curve gets short. The difference in definitions is accounted for by the fact that points in $\mathcal{PMF}$ are only defined up to equivalence by Whitehead moves ([4]) which collapse leaves of a foliation joining singularities to a higher order singularity. Thus quadratic differentials whose vertical foliations determine the same point in $\mathcal{PMF}$ may lie in different strata.

Theorem 3. *Fix a closed Riemann surface X of genus $g > 1$ and let $Q^1(X)$ denote the space of unit norm holomorphic quadratic differentials on X. Then the set of $q \in Q^1(X)$ that determine a Teichmüller geodesic that stays in a compact set of the stratum is strong winning, hence Schmidt winning. It is not absolute winning.*

Here the distance is defined by the norm; namely $d(q_1, q_2) = \|q_1 - q_2\|$.

Theorem 4. *Let Λ denote the simplex of interval exchange transformations (T, λ, π) on n intervals with a fixed irreducible permutation π defined on the unit interval $[0, 1)$. We give Λ the Euclidean metric. Let E_B consist of the* bounded *(T, λ, π). This means that $\inf_n n|T^n(p_1) - p_2| > 0$, where p_1, p_2 are discontinuities of T. Then E_B is strong winning hence Schmidt winning. It is not absolute winning.*

Because winning has nice intersection properties we obtain the following result that there are many interval exchange transformations which are bounded and any reordering of the lengths is also bounded.

Corollary 1. Let E_B be as in Theorem 4. Then the set

$$\{\lambda \in \Lambda : (\lambda_{i_1}, ...\lambda_{i_n}) \in E_B \text{ for all } \{i_1, ..., i_n\} = \{1, ..., n\}\}$$

is nonempty. In fact, it has full Hausdorff dimension.

The main theorem we prove from which the other theorems will follow is a one-dimensional version.

Theorem 5. *Let q be a holomorphic quadratic differential on a closed Riemann surface of genus $g > 1$. Then the set E of directions θ in the circle S^1 with the Euclidean metric such that the Teichmüller geodesic defined by $e^{i\theta}q$ stays in a compact set of the corresponding stratum in the moduli space of quadratic differentials is an absolute winning set; hence strong winning.*

In Theorem 5 we call the directions in E *bounded* directions. Here is an equivalent formulation of Theorem 5 (Proposition **??** establishes the equivalence).

Theorem 6. *Let*

$$S = \{(\theta, L) : \theta \text{ is the direction of a saddle connection of } q \text{ of length } L\}.$$

Then the set E of bounded directions ψ in the circle is the same as

$$\{\psi : \inf_{(\theta,L)\in S} L^2|\theta - \psi| > 0\}$$

and this is an absolute winning set.

As an immediate corollary we get the following result which was first proved by Kleinbock and Weiss [8] using quantitative non-divergence of horocycles [13].

Corollary 2. The set of directions such that the Teichmüller geodesic stays in a compact subset of the stratum has Hausdorff dimension 1.

It is well-known that a billiard in a polygon Δ whose vertex angles are rational multiples of π gives rise to a translation surface by an unfolding process. We have the following corollary to Theorem 6.

Corollary 3. Let Δ be a rational polygon. The set E of directions θ for the billiard flow in Δ with the property that there is an $\epsilon = \epsilon(\theta) > 0$, so that for all $L > 0$, the billiard path in direction θ of length smaller than L starting at any vertex, stays outside an $\frac{\epsilon}{L}$ neighborhood of all vertices of Δ, is an absolute winning set.

Since for any $0 < \alpha < 2\pi$ and $n \in \mathbb{Z}$, the set $E + n\alpha \pmod{2\pi}$ is an isometric image of E, it is also Schmidt winning with the same winning constant for each n. Therefore, the infinite intersection $\cap_{n\in\mathbb{Z}}(E + n\alpha)$ is Schmidt winning, and thus has Hausdorff dimension 1. This gives the following corollary.

Corollary 4. For any α there is a Hausdorff dimension 1 set of angles θ such that for any n, there is $\epsilon_n > 0$ such that a billiard path at angle $\theta + n\alpha(mod(2\pi))$ and length at most L from a vertex does not enter a neighborhood of radius $\frac{\epsilon_n}{L}$ of any vertex.

Another corollary uses the absolute winning property but does not follow just from Schmidt winning. Let E be the set from Corollary 3.

Corollary 5. Let $E' = \{\theta : \forall n \in \mathbb{Z}_{>0}, n\theta \in E\}$. Then E' has Hausdorff dimension 1.

The core theorems that we prove are Theorem 5 for the set of bounded directions in the disc and Theorem 2 for winning in the stratum. The former is a model for the latter although the former proves absolute winning and the latter strong winning. The fairly general Theorem 7 will reduce Theorem 1 and Theorem 4 to Theorem 2. Theorem 3 reduces to Theorem 1.

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2. Strong and absolute winning sets

2.1. **Schmidt games.** We describe the Schmidt game in $\mathbb{R}^n$. Suppose we are given a set $E \subset \mathbb{R}^n$. Suppose two players Bob and Alice take turns choosing a sequence of closed Euclidean balls

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset B_3 \dots$$

(Bob choosing the B_i and Alice the A_i) whose diameters satisfy, for fixed $0 < \alpha, \beta < 1$,

$$|A_i| = \alpha|B_i| \quad \text{and} \quad |B_{i+1}| = \beta|A_i|. \tag{1}$$

Following Schmidt

Definition 1. *We say E is an (α, β)-*winning set *if Alice has a strategy so that no matter what Bob does, $\cap_{i=1}^{\infty} B_i \in E$. It is α-*winning *if it is (α, β)-winning for all $0 < \beta < 1$. E is a* winning set for Schmidt game *if it is α-winning for some $\alpha > 0$.*

Their main properties, proved by Schmidt in [15], are:

- they have full Hausdorff dimension,
- they are preserved by bi-Lipschitz mappings (the constant α can change),
- a countable intersection of α-winning sets is α-winning.

McMullen [10] suggested two variants of the Schmidt game as follows. The first variant replaces (1) with

$$|A_i| \geq \alpha |B_i| \quad \text{and} \quad |B_{i+1}| \geq \beta |A_i|. \tag{2}$$

The notions of (α, β)-strong and α-strong winning sets are similarly defined. (Bob wins if $\cap_{i=1}^{\infty} B_i \cap E = \emptyset$; otherwise, Alice wins). A *strong winning set* refers to a set that is α-strong winning for some $\alpha > 0$.

In the second variant, the sequence of balls B_i, A_i must be chosen so that

$$B_1 \supset B_1 \setminus A_1 \supset B_2 \supset B_2 \setminus A_2 \supset B_3 \supset \dots$$

and for some fixed $0 < \beta < 1/3$,

$$|A_i| \leq \beta |B_i| \quad \text{and} \quad |B_{i+1}| \geq \beta |B_i|.$$

We say E is β-*absolute winning* if Alice has a strategy that forces $\cap_{i=1}^{\infty} B_i \cap E = \emptyset$ regardless of how Bob responds. An *absolute winning set* is one that is β-absolute winning for all $0 < \beta < 1/3$. (Remark: The condition $\beta < 1/3$ ensures Bob always has moves available to him no matter how Alice plays her moves.) It is also clear that if a set is absolute winning for some β_0 then it is absolute winning for $\beta > \beta_0$.

As noted in the introduction, absolute winning implies strong winning, which in turn implies winning in the sense of Schmidt. In particular, both types of sets have full Hausdorff dimension. These notions provide two new classes of sets that also have the countable intersection property and are not only bi-Lipschitz invariant, but preserved by the much larger class of quasi-symmetric homeomorphisms. (See [10].) As McMullen notes, most sets known to be winning in the sense of Schmidt are in fact strong winning, as is the case with the set of badly approximable vectors in $\mathbb{R}^n$. Since any subset of $\mathbb{R}^n$ that contains a line segment in its complement cannot be absolute winning (because Bob can always choose B_j centered at a point on this line segment), the set of badly approximable vectors in $\mathbb{R}^n$ ($n \geq 2$) provides a natural example of a strong winning set that is not absolutely winning. However, it is far from obvious that there are winning sets in the sense of Schmidt that are not strong winning ([10]).

2.2. **Projections and the simultaneous blocking game.** Theorem 1 and Theorem 4 will follow from Theorem 2 by use of the following fairly general statement.

Definition 2. *A surjective map $f : X \to Y$ between metric spaces is a quasi-symmetry if there exists $0 < c < 1$ such that for all $(x, r) \in X \times \mathbb{R}_+$,*

$$B_Y(f(x), cr) \subset f(B_X(x, r)) \subset B_Y(f(x), r/c).$$

Theorem 7. *Suppose $f : X \to Y$ is a surjective quasi-symmetry between complete metric spaces and $E \subset X$ is α-strong winning. Then $f(E)$ is $c^2\alpha$-strong winning.*

(Here winning means that once Bob chooses an initial ball in Y then Alice has a strategy to force the intersection point to lie in $f(E)$).

We remark that linear projection maps from $\mathbb{R}^n$ onto subspaces obviously satisfy the hypotheses and these are what will occur in the proofs of Theorem 1 and Theorem 4, but we wish to prove a more general theorem.

Proof. First we claim that if $s < r, r_0', x_0' \in X, z \in Y$, and $y' \in B_X(x_0', r_0')$ are such that $B_Y(z,s) \subset B_Y(f(y'), cr)$ then there exists $z' \in f^{-1}(z)$ such that $B_X(z', s) \subset B_X(y', r)$. We prove the claim.

Since $B_Y(z,s) \subset B_Y(f(y'), cr)$ we have $B_Y(z, cs) \subset B_Y(f(y'), cr)$ which in turn implies that $z \in B_Y(f(y'), c(r-s))$. It follows from the hypotheses of the Theorem that there is a $z' \in B_X(y', r-s)$ such that $f(z') = z$. The triangle inequality now implies that $B_X(z', s) \subset B_X(y', r)$, proving the claim.

Now we show that $f(E)$ is $(c^2\alpha, \beta)$-strong winning in Y by winning an auxiliary $(\alpha, c^2\beta)$-strong winning game in X. We describe the inductive strategy. We are given $B_X(x_k, r)$ and $B_Y(f(x_k), cr)$ where $B_X(x_k, r)$ is part of Alice's $(\alpha, c^2\beta)$-strong winning strategy in X and $B_Y(f(x_k), cr)$ is Alice's move in the $(c^2\alpha, \beta)$-game in Y. Bob chooses $B_Y(z, s) \subset B_Y(f(x_k), cr)$ and $s \geq \beta cr$.

By the claim there exists $z' \in f^{-1}(z)$ such that

$$B_X(z', cs) \subset B_X(x_k, r) \text{ with } cs \geq c^2\beta r.$$

So $B_X(z', cs)$ is a legal move in the $(\alpha, c^2\beta)$-strong winning game (in X) given Alice's move $B_X(x_k, r)$. So Alice has a response

$$B_X(x_{k+1}, t) \subset B_X(z', cs) \text{ and } t \geq \alpha cs$$

as part of her $(\alpha, c^2\beta)$-strong winning strategy in X, which we assumed existed.

Now

$$B_Y(f(x_{k+1}), ct) \subset f(B_X(x_{k+1}, t)) \subset f(B_X(z', cs)) \subset B_Y(z, s) \text{ and } ct \geq c^2\alpha s.$$

So it is a legal move for the $(c^2\alpha, \beta)$-strong winning game given Bob's move $B_Y(z, s)$.

Because the auxiliary game is a legal $(\alpha, c^2\beta)$-game we have $\cap_{k=1}^{\infty} B_X(x_k, r_k) \in E$. Thus

$$\cap_{k=1}^{\infty} B_Y(f(x_k), cr) \subset \cap_{k=1}^{\infty} f(B_X(x_k, r)) \in f(E).$$

So $f(E)$ is $(c^2\alpha, \beta)$-strong winning. □

To prove Theorem 5, (played on the circle S^1) it will be convenient to consider a variation on the absolute winning game where Alice is permitted to simultaneously block M intervals of radii $\leq \beta|I_j|$.[1] (The condition $(2M+1)\beta < 1$ ensures that Bob will always have available moves.)

Lemma 1. *Suppose E is a winning set for the modified game where Alice is permitted to simultaneously block M intervals of length at most β^M times the length of Bob's interval. Then E is an absolute winning set with the parameter β.*

[1]The authors would like to thank Barak Weiss for bringing our attention to this variant of the absolute winning game.

Proof. Let $I_j, j \geq 1$ denote the intervals that Bob plays in the (original) absolute game. Alice will consider the subsequence $I_{1+rM}, r \geq 0$ as Bob's moves in the modified game. Given $j = 1 \mod M$ Alice considers the intervals $J_1, \dots, J_M$ she would have played in the modified game in response to Bob's choice of I_j. The strategy for her next M moves of the original game is to pick

$$U_j = J_1, \quad U_{j+1} = J_2 \cap I_{j+1}, \quad U_{j+2} = J_3 \cap I_{j+2}, \quad \dots \quad U_{j+M-1} = J_M \cap I_{j+M-1}.$$

Observe that for $i = 1, \dots, k-1$,

$$|U_{j+k-1}| \leq |J_k| \leq \beta^M |I_j| \leq \beta^{M-i} |I_{j+i}| \leq \beta |I_{j+k-1}|$$

so Alice's choice of U_{j+k-1} in response to I_{j+k-1} is valid. Note that $|I_{j+M}| \geq \beta^M |I_j|$ and that I_{j+M} is disjoint from J_k because $I_{j+M} \subset I_{j+k}$ and Bob is required to choose I_{j+k} disjoint from J_k inside I_{j+k-1}. Thus, I_{j+M} is a valid move for Bob in the modified game so that Alice can continue her next M moves by repeating the strategy just described.

Since the intervals I_j are nested, we have

$$\bigcap_{j=1}^{\infty} I_j = \bigcap_{r=1}^{\infty} I_{1+rM},$$

which has nontrivial intersection with E, by hypothesis. □

We remark that there is an obvious partial converse: *if Alice can win the absolute game then she can also win the modified game with the same parameter*. Indeed, she simply picks all her M intervals to be the same as the interval she would have chosen in the original, absolute game.

2.3. **Case of badly approximable numbers.** Recall that a real number θ is badly approximable if there exists $c > 0$ such that for all rationals $p/q \in \mathbb{Q}$

$$\left| \theta - \frac{p}{q} \right| > \frac{c}{q^2}.$$

The fact that the set of badly approximable numbers is absolute winning is a special case of Theorem 1.3 of [10]. We give a proof of this result because it serves as a motivation for the proof of Theorem 5.

Theorem 8. *The set of badly approximable real numbers is absolute winning.*

Proof. Fix $\varepsilon > 0$. Given an interval B_j chosen by Bob, let I_j be an $\varepsilon|B_j|$-neighborhood of B_j and let p_j/q_j be the rational of smallest denominator (in lowest terms) in the interval I_j. Alice's strategy is to "block p_j/q_j"; in other words, she picks

$$A_j = \left(\frac{p_j}{q_j} - \frac{\beta |B_j|}{2}, \frac{p_j}{q_j} + \frac{\beta |B_j|}{2} \right).$$

We claim that there exists $c > 0$ such that $|I_j|q_j^2 > c$ for all $j \geq 1$. Indeed, if $q_{j+1} < \beta^{-1} q_j$ then

$$|I_j| > \left|\frac{p_j}{q_j} - \frac{p_{j+1}}{q_{j+1}}\right| \geq \frac{1}{q_j q_{j+1}} > \frac{\beta}{q_j^2} \tag{3}$$

whereas if $q_{j+1} \geq \beta^{-1} q_j$ we have

$$|I_{j+1}|q_{j+1}^2 \geq \beta |I_j| q_{j+1}^2 \geq \beta^{-1}|I_j|q_j^2.$$

Now whenever the quantity $|I_j|q_j^2$ is less than β, the above inequality says it must increase by a factor of at least β^{-1} at each step until it exceeds β, after which it may decrease by a factor of at most β (since $|I_{j+1}| \geq \beta |I_j|$ and $q_{j+1} \geq q_j$) and then exceed β again at the following step. Hence, $\liminf |I_j|q_j^2 > \beta^2$ and the claim follows.

Given $x \in \cap B_j$ and $p/q \in \mathbb{Q}$, suppose first that $p/q \notin I_1$. Then

$$\left|x - \frac{p}{q}\right| \geq \varepsilon |B_1| \geq \frac{\varepsilon |B_1|}{q^2}.$$

Thus assume $p/q \in I_1$. Since our strategy guarantees that $p/q \notin \cap I_j$, there is a (unique) index j such that $p/q \in I_j \setminus I_{j+1}$ and $q_j \leq q$ (because $p/q \in I_j$) and since $p/q \notin I_{j+1}$ we have

$$\left|x - \frac{p}{q}\right| \geq \varepsilon |B_{j+1}| \geq \beta\varepsilon |B_j| > \frac{c\beta\varepsilon |I_j|}{1+2\varepsilon} > \frac{c\beta\varepsilon}{(1+2\varepsilon)q_j^2} \geq \frac{c\beta\varepsilon}{(1+2\varepsilon)q^2}$$

proving x is badly approximable. □

2.4. **Sketch of the proofs of Theorem 5 and Theorem 2.** In the game played with quadratic differentials on a higher genus surface, we have a similar criterion as for the torus, given by Proposition 1 that for a Teichmüller geodesic to lie in a compact set in the stratum, the direction is far from the direction of a saddle connection. This says that in order to show this set is winning we want to find a strategy giving us a point far from the direction of any saddle connection. Unlike the genus one case we have the major complication that directions of saddle connections in general need not be separated in the sense that the angle between them need *not* be at least a constant over the product of their lengths as it is in the case of tori. Equivalently, it may happen that on some flat surfaces there are many intersecting short saddle connections. This forces us to consider *complexes* of saddle connections that become simultaneously short under the geodesic flow. We call these complexes *shrinkable*.

An important tool is a process of combining a pair of shrinkable complexes of a certain level or complexity to build a shrinkable complex of higher level. This is given by Lemma 9 with the preliminary Lemma 7. These ideas are not really new; having appeared in several papers beginning with [6].

The main point in this paper and the strategy is given by Theorem 9. We show first that complexes of highest level are separated, as in the case of the torus, for otherwise we could combine them to build a complex of higher level that is shrinkable. This is impossible by definition of highest level. We develop a strategy for Alice where, as in the torus case, she blocks these highest level complexes. Then we consider complexes of one lower level that lie in the complement of the interval used to block highest level intervals and which are not too long in a certain sense depending on the stage of the game. We show that these are separated as well, for if not, we could combine them into a highest level complex of bounded size and these have supposedly been blocked at an earlier stage of the game. Thus there can be at most one such lower level complex (up to a certain combinatorial equivalence) and we block it. We continue this process inductively considering complexes of decreasing level one step at a time, ending by blocking single saddle connections. Then after a fixed number of steps we return to blocking highest level complexes and so forth. From this strategy, Theorem 5 will follow. For technical reasons, we need to block complexes by intervals whose length is comparable to the reciprocal of the product of their longest saddle connection and the longest saddle connection on their boundary.

In adapting the argument to the proof of Theorem 2, we need to consider complexes on distinct flat surfaces. In order to combine them so that Lemma 9 can be applied, we need to consider the problem of moving a complex on one surface to a nearby one. (See Theorem 10.) This operation is not canonical since unlike parallel transport, it does not respect the operation of concatenation along paths. However it does preserve inclusion of complexes (Proposition 2) and this is sufficient for our purposes. While the basic strategy is the same as that in the proof of Theorem 5, we caution the reader that unlike the ordinary and strong winning games, after Alice chooses A_j, the game "continues" in $B_j \setminus A_j$ rather than inside A_j. In particular, we do not have an analog of the simultaneous blocking strategy Lemma 1.

3. Quadratic differentials, complexes, and geodesic flow

3.1. **Quadratic differentials.**

A general reference here is [12]. Recall a holomorphic quadratic differential $q = \phi(z)dz^2$ on a Riemann surface X of genus $g > 1$ defines for each local holomorphic coordinate z, a holomorphic function $\phi_z(z)$ such that in overlapping coordinate neighborhoods $w = w(z)$ we have

$$\phi_w(w)\left(\frac{dw}{dz}\right)^2 = \phi_z(z).$$

On a compact surface, q has a finite set Σ of zeroes. On the complement of Σ there are *natural* local coordinates z such that $\phi_z(z) \equiv 1$ and hence q defines a flat surface. A zero of order k defines a cone singularity of angle $(k+2)\pi$. Suppose q has zeroes of orders $k_1, \ldots, k_p$ with $\sum k_i = 4g - 4$. There is a moduli space or stratum $Q = Q(k_1, \ldots, k_p, \pm)$ of quadratic differentials all of which have zeroes of orders k_i. The $+$ sign occurs if q is the square of an Abelian differential and the $-$ sign otherwise.

A quadratic differential q defines an area form $|\phi(z)||dz^2|$ and a metric $|\phi|^{1/2}|dz|$. We assume that our quadratic differentials have area one. Recall a *saddle connection* is a geodesic in the metric joining a pair of zeroes which has no zeroes in its interior. By the *systole* of q we mean the length of the shortest saddle connection.

A choice of a branch of $\phi^{1/2}(z)$ along a saddle connection β and an orientation of β determines a holonomy vector

$$hol(\beta) = \int_\beta \phi^{1/2} dz \in \mathbb{C}.$$

It is defined up to sign. Thinking of this as a vector in $\mathbb{R}^2$ gives us the horizontal and vertical components defined up to sign. We will denote by $h(\gamma)$ and $v(\gamma)$ the absolute value of these components. We will denote its length $|\gamma|$ as the maximum of $h(\gamma)$ and $v(\gamma)$. This slightly different definition will cause no difficulties in the sequel.

Given $\epsilon > 0$, let Q^1_ϵ denote the compact set of unit area quadratic differentials in the stratum such that the shortest saddle connection has length at least ϵ. The group $SL(2,\mathbb{R})$ acts on Q^1 and on saddle connections. (In the action we will suppress the underlying Riemann surface). Let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

denote the Teichmüller flow acting on Q^1 and

$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

denote the rotation subgroup.

The Teichmüller flow acts by expanding the horizontal component of saddle connections by a factor of e^t and contracting the vertical components by e^t. For σ a saddle connection we will also use the notation $g_t r_\theta \sigma$ for the action on saddle connections. The action of $SL(2,\mathbb{R})$ is linear on holonomy of saddle connections.

Definition 3. *We say a direction θ is* bounded *if there exists ϵ such that $g_t r_\theta q \in Q^1_\epsilon$ for all $t \geq 0$.*

3.2. **Conditions for β-absolute winning.**

Definition 4. *Given a saddle connection γ on q we denote by θ_γ the angle such that γ is vertical with respect to $r_{\theta_\gamma} q$.*

We can think of the set of saddle connections as a subset of $S^1 \times \mathbb{R}$ by associating to each γ the pair $(\theta_\gamma, |\gamma|)$. The following proposition gives the equivalence of Theorem 5 and Theorem 6 and will be the motivation for what follows.

Proposition 1. *Let*

$$S = \{(\theta, L) : \theta \text{ is the vertical direction of a saddle connection of length } L\}.$$

Then $\inf_{(\theta,L)\in S} L^2|\theta - \psi| > 0$*, if and only if ψ determines a bounded direction.*

Proof. If $(\theta, L) \in S$, let $c = |\theta - \psi|$. We can assume $c \leq \frac{\pi}{4}$. Then the length of the saddle connection in $g_t r_\psi q$ coming from (θ, L) is $\max\{e^t \sin(c)L, e^{-t}L\cos(c)\}$ and minimized in t when equality of the two terms holds; that is when $e^{-t} = \sqrt{\tan(c)}$. At this time the length is

$$L\sqrt{\sin(c)\cos(c)} = L\sqrt{\frac{\sin(2c)}{2}} \geq L\sqrt{\frac{c}{2}}.$$

So if $c > \frac{\delta}{L^2}$, for some $\delta > 0$, then

$$\max\{e^t \sin(c)L, e^{-t}L\cos(c)\} > \sqrt{\frac{\delta}{2}}.$$

Conversely, if the minimum length $L\sqrt{\frac{\sin(2c)}{2}}$ is bounded below by some ℓ_0 then the difference in angles c satisfies

$$2c \geq \sin(2c) \geq \frac{2\ell_0^2}{L^2}.$$

□

3.3. **Complexes.** In this section we fix a quadratic differential q. Let Γ be a collection of saddle connections of q, any two of which are disjoint except possibly at a common zero. Let K be the simplicial complex having Γ as its set of 1-simplices and whose 2-simplices consist of all triangles that have all three edges in Γ. By a *complex* we mean any simplicial complex that arises in this manner. We shall also use the same term to mean the closed subset K of the surface given by the union of all simplices; in this case, we call Γ a *triangulation* of K. We shall often leave it to the context to determine which sense of the term is intended. For example, "a saddle connection in K" refers to an element of Γ, whereas "the interior of K" refers to the largest open subset contained in K, which may be empty. An edge e is in the topological boundary ∂K of K if any neighborhood of an interior point of e intersects the complement of K. We note that the boundary of a complex may fail to satisfy the requirement that a triangle with edges in the complex is also included in the complex, as happens when K is simply a triangle.

We distinguish between *internal* saddle connections in ∂K, which lie on the boundary of a 2-simplex in K and *external* ones, which do not. Note that a triangulation Γ of K may contain both internal and external saddle connections. The remaining saddle connections are on the boundary of two 2-simplices, and we refer to them as *interior* saddle connections, which, of course, depend on the choice of Γ. Each internal saddle connection comes with a *transverse orientation*, which is determined by the choice of an inward normal vector at any interior point of the segment. The interior of K is determined by the data consisting of ∂K, the subdivision into internal and external saddle connections, together with the choice of transverse orientation for each internal saddle connection. Simplicial homeomorphisms respect these notions in the obvious sense, while simplicial maps generally do not.

Definition 5. *The* level *of a complex is the number of edges in any triangulation.*

An easy Euler characteristic argument says that the level M is well defined and is bounded by $6g-6+3n$, where n is the number of zeroes.

We say two complexes are *topologically equivalent* if they determine the same closed subset of the surface. Otherwise, they are *topologically distinct*.

Lemma 2. *If K_1 and K_2 are topologically distinct complexes of the same level, then any triangulation of K_2 contains a saddle connection $\gamma \in K_2$ that intersects the exterior of K_1, i.e. $\gamma \not\subset K_1$.*

Proof. Arguing by contradiction, we suppose that the conclusion does not hold. Then there is a triangulation of K_2 such that every edge is contained in K_1. It would follow that $K_2 \subset K_1$, and properly so, since they are topologically distinct. By repeatedly adding saddle connections $\sigma \subset K_1$ that are disjoint from those in K_2, we can extend the triangulation of K_2 to one of K_1 to obtain one where the number of edges is strictly greater than the level of K_1. This contradicts the fact that any two triangulations of a complex contains the same number of edges. □

A *path in* Γ refers to a sequence of edges in Γ such that the terminal endpoint of the previous edge coincides with the initial endpoint of the next edge. We may also think of it as a map of the unit interval into X. The *combinatorial length* of a path in Γ refers to the number of edges in the sequence, including repetitions. For a homotopy class of paths with endpoints fixed at the zeroes of q we define the *combinatorial length* to be the minimum combinatorial length of a path in Γ in the homotopy class. We denote the combinatorial length of a saddle connection by $|\gamma|_\Gamma$.

To show that combinatorial and flat lengths are comparable, we first need a lemma.

Lemma 3. *For any saddle connection σ there is $\delta = \delta(\sigma) > 0$ such that the length of a geodesic segment with endpoints in σ but otherwise not contained in σ is at least δ.*

Proof. For any small $\delta > 0$ take the δ neighborhood of σ. This is simply connected if σ has distinct endpoints and is an annulus if the endpoints coincide. Then any geodesic starting and ending on σ must leave the neighborhood; otherwise the geodesic and a segment of σ would bound a disc, which is impossible. □

Definition 6. *Given q, let $L_0 = L_0(q)$ denote the systole and for Γ a triangulation of a complex K let $L_1 = L_1(\Gamma, q)$ denote the length of the longest edge of Γ.*

Lemma 4. *Let $\delta = \delta(\Gamma)$ denote the minimum of the constants given by Lemma 3 associated to each saddle connection in Γ. There are constants $\lambda_2 > \lambda_1 > 0$ depending on L_0, L_1 and δ such that for any saddle connection $\gamma \subset \Gamma$, $\lambda_1|\gamma| \le |\gamma|_\Gamma \le \lambda_2|\gamma|$.*

Proof. Since γ is the geodesic in its homotopy class, its length is bounded above by $L_1|\gamma|_\Gamma$. Hence, we may take $\lambda_1 = 1/L_1$. For the other inequality,

$$|\gamma|_\Gamma \le N + 2$$

where N is the number of times γ crosses a saddle connection in Γ. Since one of these saddle connections is crossed at least $[N/e]$ times, where e is the number of elements in Γ, we have

$$|\gamma| \ge ([N/e] - 1)\delta.$$

First, if $N \geq 4e$ then $|\gamma| \geq \frac{N}{2e}\delta \geq 2\delta$ so that

$$|\gamma|_\Gamma \leq \frac{2e}{\delta}|\gamma| + 2 \leq \frac{2e+1}{\delta}|\gamma|.$$

On the other hand, if $N < 4e$ then $|\gamma|_\Gamma < 4e+2$ whereas $|\gamma|$ is bounded below by the L_0. Hence, the lemma holds with

$$\lambda_2 = \max\left(\frac{2e+1}{\delta}, \frac{4e+2}{L_0}\right).$$

□

Lemma 5. *There exists ϵ_0 such that a $3\epsilon_0$-complex must have strictly fewer than $6g-6+3n$ saddle connections.*

Proof. We can triangulate the surface by disjoint saddle connections. If the surface can be triangulated by edges of length ϵ_0 then there is a bound in terms of ϵ_0 for the area. However we are assuming that the area of q is one. □

Lemma 6. *Given q, there is a number $M < 6g-6+3n$ such that for any $\epsilon \leq \epsilon_0$, M is the maximum level of any ϵ-complex for any $g_t r_\theta q$.*

The following will be applied to complexes on the surface $g_t r_\theta q$ for some suitable choice of t and θ.

Lemma 7. *Let K be an ε-complex and $\gamma \not\subset K$, i.e. a saddle connection that intersects the exterior of K. Then there exists a complex $K' = K \cup \{\sigma\}$ formed by adding a disjoint saddle connection σ satisfying $h(\sigma) \leq h(\gamma) + 3\varepsilon$ and $v(\sigma) \leq v(\gamma) + 3\varepsilon$.*

Proof. We have that γ must be either disjoint from K or cross the boundary of K.

Case I. γ is disjoint from K. Add γ to K to form $\tilde{K}$. It is clear that the estimate on lengths holds.

Case II γ intersects ∂K crossing $\beta \subset \partial K$ at a point p dividing β into segments β_1, β_2.

Case IIa One endpoint p' of γ lies in the exterior of K. Let $\hat{\gamma}$ be the segment of γ that goes from p' to p. We consider the homotopy class of paths $\hat{\gamma} * \beta_i$ which is the segment $\hat{\gamma}$ followed by β_i. Together with β they bound a simply connected domain Δ. Replace each path by the geodesic ω_i joining the endpoints in the homotopy class. Then $\partial\Delta$ is made up of at most M saddle connections σ all of which have their horizontal and vertical lengths bounded by the sum of the horizontal and vertical lengths of γ and β. If some $\sigma \notin K$ we add it to form K'. It is clear that the estimate on lengths holds.

The other possibility is that $\Delta \subset \partial K$. It cannot be the case that Δ is a triangle, since then Δ would be a subset of K, contradicting the assumption on γ. Since the edges of Δ all have length at most ε we can find a diagonal σ in Δ of length at most 2ε and add it to form K'. See Figure 1.

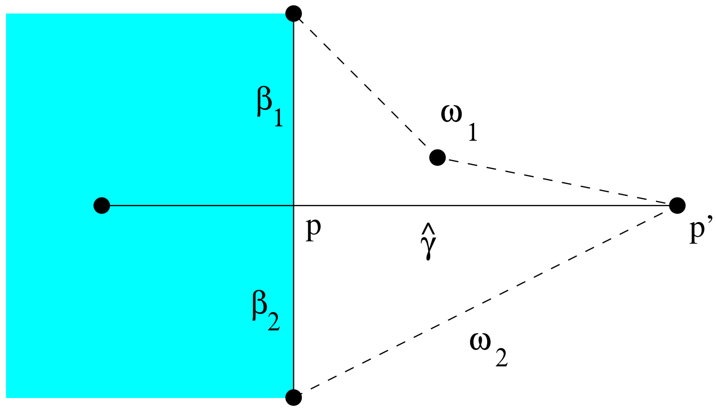

FIGURE 1. Case IIa.

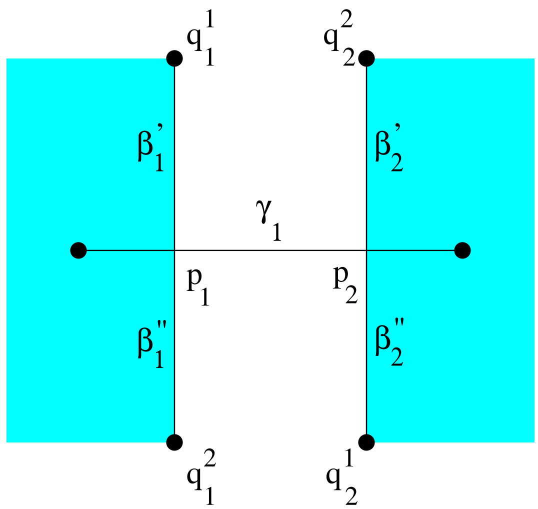

FIGURE 2. First subcase of Case IIb.

Case IIb Both endpoints of γ lie in K. Let γ successively cross ∂K at p_1, p_2, and let γ_1 be the segment of γ lying in the exterior of K between p_1 and p_2.

The first case is where p_1, p_2 lie on different β_1, β_2 which have endpoints q_1^1, q_1^2 and q_2^1, q_2^2. Then p_1, p_2 divide β_i into segments β_i', β_i''. We can form a homotopy class $\beta_1' * \gamma_1 * \beta_2'$ joining q_1^1 to q_2^2 and a homotopy class $\beta_1'' * \gamma_1 * \beta_2''$ joining q_1^2 to q_2^1. cWe replace these with their geodesics with the same endpoints and then together with β_1, β_2 they bound a simply connected domain. We are then in a situation similar to Case IIa. See Figure 2.

The last case is that p_1, p_2 lie on the same saddle connections β of ∂K. Let $\hat{\beta}$ be the segment between p_2 and p_1. Let β_1 and β_2 be the segments joining the endpoints q_1, q_2 of β to p_1, p_2. Find the geodesic in the homotopy class of $\beta_1 * \gamma_1 * \beta_2$ joining

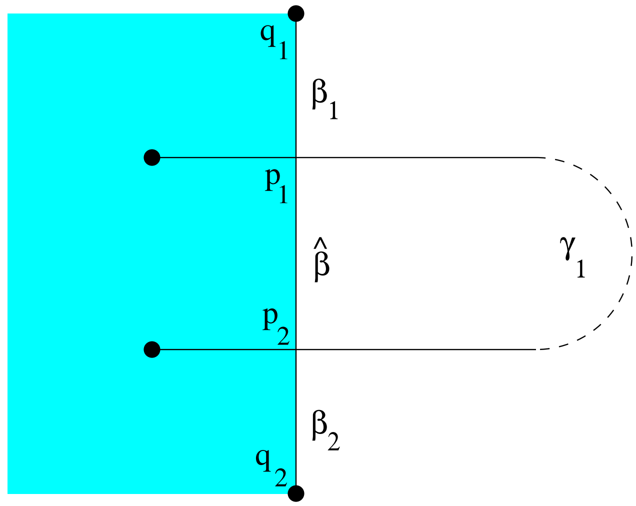

FIGURE 3. Second subcase of Case IIb.

q_1 to q_2 and the geodesic in the class of the loop $\beta_1 * \gamma_1 * \beta_2 * \beta^{-1}$ from q_1 to itself. These two geodesics together with β bound a simply connected domain. The analysis is similar to the previous cases. See Figure 3.

□

Now fix the base surface q. All angles and lengths will be measured on the base surface. We shall often let K denote a complex equipped with a triangulation without explicit mention the choice of triangulation, as in the statement and proof of Lemma 7.

Definition 7. *Denote by $L(K)$ the length of the longest saddle connection in K. Let $\theta(K)$ the angle that makes the longest saddle connection vertical.*

We assume that the complexes considered now have the property that for any saddle connection $\gamma \in K$ we have $|\theta_\gamma - \theta(K)| \leq \frac{\pi}{4}$. This implies that measured with respect to the angle $\theta(K)$ we have $|\gamma| = v(\gamma)$. In other words the vertical component is larger than the horizontal component. This will exclude at most finitely many complexes from our game and these will be excluded in any case by our choice of c_{M+1} in Theorem 9.

Definition 8. *We say a complex K is* ϵ-shrinkable *if for all saddle connections β of K if we let $h(\beta)$ be the component of the holonomy vector in the direction perpendicular to the direction $\theta(K)$ of the longest saddle connection, then $h(\beta) \leq \frac{\epsilon^2}{L(K)}$.*

We note that this condition could equally well be stated as follows. For any saddle connection β of K we have $|\theta_\beta - \theta(K)| \leq \frac{\epsilon^2}{|\beta| L(K)}$.

The following is immediate.

Lemma 8. *If $\epsilon_1 < \epsilon_2$ and K is ϵ_1-shrinkable, then it is ϵ_2-shrinkable.*

Definition 9. *A complex K and a saddle connection γ that intersects the exterior of K are* jointly ϵ-shrinkable *if K is ϵ-shrinkable and*

- *if $|\gamma| \leq L(K)$ then $|\theta(K) - \theta_\gamma| \leq \frac{\epsilon^2}{|\gamma| L(K)}$.*
- *if $L(K) < |\gamma|$ then $|\theta_\gamma - \theta_\omega| \leq \frac{\epsilon^2}{|\gamma||\omega|}$ for all $\omega \in K$.*

The next lemma says that if the longest saddle connections of each of two complexes have comparable lengths and the angles between these saddle connections is not too large, then the complexes can be combined to form another shrinkable complex.

Lemma 9. *Let K_1 and K_2 be ε-shrinkable complexes of level i satisfying*

$$|\theta(K_1) - \theta(K_2)| < \frac{\rho_1 \varepsilon^2}{L(K_1)L(K_2)} \quad \text{and} \quad L(K_1) \leq L(K_2) < \rho_2 L(K_1)$$

for some $\rho_1 > 3$ and $\rho_2 > 3$ and assume they are topologically distinct. Then there is an ε'-shrinkable complex K' of one level higher satisfying $L(K') < \rho_2' L(K_1)$ where

$$\varepsilon' = (16\rho_1\rho_2')^{1/2}\varepsilon \quad \text{and} \quad \rho_2' = \sqrt{4\rho_2^2 + 9\rho_1^2\varepsilon^4/L(K_1)^4}. \tag{4}$$

Proof. By Lemma 2 there is a saddle connection $\gamma \in K_2$ such that $\gamma \not\subset K_1$. Let $\theta = \theta(K_1)$ and $t = \log(L(K_1)/\varepsilon)$. Then

$$|\theta(K_1) - \theta_\gamma| < \frac{\rho_1 \varepsilon^2}{L(K_1)L(K_2)} + \frac{\varepsilon^2}{|\gamma| L(K_2)} < \frac{2\rho_1\varepsilon^2}{|\gamma| L(K_1)}$$

so that

$$h_\theta(\gamma) \cdot \frac{L(K_1)}{\varepsilon} < 2\rho_1\varepsilon \quad \text{and} \quad v_\theta(\gamma) \cdot \frac{\varepsilon}{L(K_1)} < \rho_2\varepsilon.$$

We apply Lemma 7 to produce a new saddle connection σ. On $g_t r_\theta X$ we have $\sigma_{\theta,t} = g_t r_\theta \sigma$ satisfies

$$h(\sigma_{\theta,t}) < (2\rho_1 + 3)\varepsilon < 3\rho_1\varepsilon \quad \text{and} \quad v(\sigma_{\theta,t}) < (\rho_2 + 3)\varepsilon < 2\rho_2\varepsilon$$

which implies that

$$\begin{aligned} |\sigma| &= \sqrt{\left(h(\sigma_{\theta,t}) \cdot \frac{\varepsilon}{L(K_1)}\right)^2 + \left(v(\sigma_{\theta,t}) \cdot \frac{L(K_1)}{\varepsilon}\right)^2} \\ &< \sqrt{(3\rho_1\varepsilon^2/L(K_1))^2 + 4\rho_2^2 L(K_1)^2} = \rho_2' L(K_1) \end{aligned}$$

so that $K' = K \cup \{\sigma\}$ satisfies

$$L(K') = \max(L(K_1), |\sigma|) < \rho_2' L(K_1). \tag{5}$$

If $|\sigma| < L(K_1)$ we have K' is ε'-shrinkable because

$$|\theta(K_1) - \theta_\sigma| \leq 2\frac{h_\theta(\sigma)}{|\sigma|} \leq \frac{6\rho_1\varepsilon^2}{|\sigma| L(K_1)}$$

while if $|\sigma| \geq L(K_1)$ it follows that K' is ε'-shrinkable because $\theta(K') = \theta_\sigma$ and for every $\xi \in K_1$

$$|\theta(K') - \theta_\xi| \leq |\theta_\sigma - \theta(K_1)| + |\theta(K_1) - \theta_\xi| < \frac{6\rho_1\varepsilon^2}{|\sigma|L(K_1)} + \frac{\varepsilon^2}{|\xi|L(K_1)} < \frac{6\rho_1\varepsilon^2}{|\xi|L(K_1)} < \frac{6\rho_1\rho_2'\varepsilon^2}{|\xi|L(K')}$$

where $|\sigma| \geq L(K_1) \geq |\xi|$ and (5) were used in last two inequalities. □

The following symmetric version allows us to bypass Lemma 2.

Lemma 10. *Let K_1 and K_2 be ε-shrinkable complexes of level i satisfying*

$$|\theta(K_1) - \theta(K_2)| < \frac{\rho_1\varepsilon^2}{L(K_1)L(K_2)} \quad \textit{and} \quad \rho_2^{-1}L(K_1) \leq L(K_2) < \rho_2 L(K_1)$$

for some $\rho_1 > 3$ and $\rho_2 > 3$ and assume they are topologically distinct. Then there is an ε'-shrinkable complex K' of one level higher satisfying $L(K') < \rho_2' L(K_1)$ where

$$\varepsilon' = (8\rho_*\rho_2')^{1/2}\varepsilon, \quad \rho_2' = \sqrt{4\rho_2^2 + 9\rho_*^2\varepsilon^4/L(K_1)^4} \quad \textit{and} \quad \rho_* = \max(\rho_1, \rho_2). \tag{6}$$

Proof. The only place where $L(K_1) \leq L(K_2)$ was used in the previous proof was at the last inequality in the first displayed line. The entire proof goes through if every occurrence of ρ_1 is replaced with ρ_*. □

In what follows we will be considering shrinkable complexes. In each combinatorial equivalence class of such shrinkable complexes we will consider the complex K which minimizes $L(\cdot)$ and the corresponding angle $\theta(\cdot)$. We note that this complex is perhaps not unique and so there is ambiguity in $\theta(K)$ but this will not matter.

We let $L(\partial K)$ denote the length of the longest saddle connection on the boundary of K. We also let $\theta(\partial K)$ the angle that makes the longest vertical.

4. Proofs of Theorem 5 and Theorem 9

We are now ready to begin the proof of Theorem 5. It is based on Theorem 9 whose statement and proof were suggested in the outline.

Theorem 9. *For all β sufficiently small and given Bob's first move I_1 in the game, there exist positive constants c_i, $i = 1, \ldots, M+1$, and a strategy for Alice such that regardless of the choices I_j made by Bob, the following will hold. For all βc_i-shrinkable level i-complexes K if*

$$L(K)L(\partial K)|I_j| < c_i^2,$$

then

$$d(\theta(K), I_j) > \frac{\beta c_i^2}{4L(K)L(\partial K)}.$$

Proof. We assume $\beta < 1/12$. (The value $1/12$ is chosen so that some inequalities that appear in the proof are satisfied). Let L_0 denote the length of the shortest saddle connection on (X, q). Let $0 < c_1 < \cdots < c_{M+1}$ be given by

$$c_i = \beta^{N_i} c_{i+1}$$

and

$$c_{M+1} = \min(L_0 \beta^{N_M}, L_0 |I_1|^{1/2} \varepsilon_0),$$

where N_i are defined recursively by $N_1 = 6$ and $N_{i+1} = 6 + 2(N_1 + \cdots + N_i)$. (Again the value 6 is chosen only so that a particular inequality is satisfied) . These N_i are chosen so that

$$\frac{c_{i+1}}{c_i} = \beta^{-6} \left(\frac{c_i}{c_1} \right)^2. \tag{7}$$

We remark that this last equation will be used in Step 2 of the proof. All that is needed is an inequality, but to simplify matters we present it as an equality.

Let $\mathcal{E}_i$ be the set of all marked βc_i-shrinkable complexes of level i. Given I_j, we let

$$\mathcal{A}_i(j) := \left\{ K \in \mathcal{E}_i : d(\theta(K), I_j) > \frac{\beta c_i^2}{4L(K)L(\partial K)} \right\}$$

and

$$\Omega_i(j) := \left\{ K \in \mathcal{E}_i \setminus \mathcal{A}_i(j) : \frac{c_i^2}{L(K)L(\partial K)} \leq |I_j| < \frac{\beta^{-1} c_i^2}{L(K)L(\partial K)} \right\}$$

and

$$z_i(j) := \frac{\inf \Theta_i(j) + \sup \Theta_i(j)}{2} \quad \text{where} \quad \Theta_i(j) = \{\theta(K) : K \in \Omega_i(j)\}.$$

Alice chooses M intervals of length $\beta|I_j|$ centered at the points $z_i(j)$, $i = 1, \ldots, M$.

The restatement of theorem then is that for every $j \geq 1$:

$$(P_j) \qquad \forall i \in \{1, \ldots, M\}\ \forall K \in \mathcal{E}_i \qquad L(K)L(\partial K)|I_j| < c_i^2 \implies K \in \mathcal{A}_i(j)$$

Note that (P_j) holds for all $j < j_0 := \min\{k : |I_k| < \beta^{2N_M}\}$ because

$$L(K)L(\partial K)|I_j| \geq L_0^2 \beta^{2N_M} \geq c_{M+1}^2 > c_i^2$$

while if $j_0 = 1$, we note that $L(K)L(\partial K)|I_1| \geq L_0^2 |I_1| \geq c_{M+1}^2 > c_i^2$.

We proceed by induction and suppose that $j \geq j_0$ and that (P_{j-1}) holds.

Step 1. For any $K \in \Omega_i(j)$,

$$\frac{L(K)}{L(\partial K)} < \beta^{-2} \left(\frac{c_i}{c_1} \right)^2. \tag{8}$$

Consider first the case

$$L(\partial K)^2 |I_j| < \beta c_1^2. \tag{9}$$

Then $L(\partial K)^2|I_{j-1}| < c_1^2$ so that (P_{j-1}) implies the longest saddle connection on ∂K belongs to $A_1(j-1)$, meaning

$$d(\theta(\partial K), I_{j-1}) > \frac{\beta c_1^2}{4L(\partial K)^2}. \tag{10}$$

But since K is βc_i-shrinkable and not in $\mathcal{A}_i(j)$, the triangle inequality implies

$$\begin{aligned} d(\theta(\partial K), I_{j-1}) &\leq |\theta(\partial K) - \theta(K)| + d(\theta(K), I_j) \\ &\leq \frac{\beta^2 c_i^2}{L(K)L(\partial K)} + \frac{\beta c_i^2}{4L(K)L(\partial K)} \leq \frac{\beta c_i^2}{L(K)L(\partial K)} \end{aligned}$$

which together with (10) and the fact that $\beta < \frac{1}{3}$ implies (8). Suppose then that (9) does not hold. Then we have

$$\frac{L(K)}{L(\partial K)} = \frac{L(K)L(\partial K)|I_j|}{L(\partial K)^2|I_j|} < \frac{\beta^{-1}c_i^2}{\beta c_1^2}$$

so that (8) holds in this case as well.

Step 2. Any pair $K_1, K_2 \in \Omega_i(j)$ are topologically equivalent.

For any $K_1, K_2 \in \Omega_i(j)$, we have

$$\frac{L(K_2)}{L(\partial K_1)} < \beta^{-1}\frac{L(K_1)}{L(\partial K_2)}. \tag{11}$$

Multiplying (11) by $L(K_2)/L(\partial K_1)$ and invoking (8), we get

$$\frac{L(K_2)}{L(\partial K_1)} < \beta^{-5/2}\left(\frac{c_i}{c_1}\right)^2$$

so that, exploiting $2 < \beta^{-1/2}$, and the above inequality we have

$$|\theta(K_1) - \theta(K_2)| \leq 2|I_j| < \frac{2\beta^{-1}c_i^2}{L(K_1)L(\partial K_1)} < \frac{\beta^{-4}(c_i/c_1)^2 c_i^2}{L(K_1)L(K_2)}.$$

On the other hand again multiplying (11) by $L(K_2)/L(K_1)$ and applying (8), we obtain

$$\left(\frac{L(K_2)}{L(K_1)}\right)^2 < \beta^{-1}\frac{L(K_2)L(\partial K_1)}{L(\partial K_2)L(K_1)} < \beta^{-3}\left(\frac{c_i}{c_1}\right)^2.$$

Now suppose K_1 and K_2 are topologically distinct. We shall derive a contradiction. Without loss of generality, we may assume $L(K_1) \leq L(K_2)$. The hypotheses of Lemma 9 are then satisfied with the parameters

$$\varepsilon = c_i, \qquad \rho_1 = \beta^{-4}\left(\frac{c_i}{c_1}\right)^2, \qquad \text{and} \qquad \rho_2 = \beta^{-2}\left(\frac{c_i}{c_1}\right).$$

Consider the two terms under the radical in the expression for ρ_2' in (4). Note that the second being dominated by the first is equivalent to $L(K_1)^2 > \dfrac{3\rho_1}{2\rho_2}c_i^2$. we have

$$|I_j| \leq |I_{j_0}| < \beta^{2N_M} = \left(\frac{c_M}{c_{M+1}}\right)^2 < \left(\frac{c_1}{c_i}\right)^4 < \frac{2\rho_2}{3\rho_1},$$

the first inequality a conseqeunce of (7). Since $L(K_1)^2 \geq L(K_1)L(\partial K_1) \geq c_i^2/|I_j|$, it now follows that

$$\rho_2' < 3\rho_2.$$

Also, since $\rho_1 = \rho_2^2$ and $\beta^2 c_{i+1} = \rho_2^2 c_i$, we have

$$\varepsilon' = (8\rho_1\rho_2')^{1/2}c_i < 2\sqrt{6}\rho_2^{3/2}c_i < \frac{\beta c_{i+1}}{\sqrt{6\rho_2}} < \beta c_{i+1} \tag{12}$$

where we used $\beta < 1/12$ in the middle inequality.

By Lemma 9, there exists a βc_{i+1}-shrinkable complex K' of level $i+1$ satisfying

$$L(K') < 3\rho_2 L(K_1).$$

Since there are no c_{M+1}-shrinkable complexes of level $M+1$, we have our desired contradiction when $i = M$. For $i < M$, we have

$$L(K')L(\partial K')|I_{j-1}| < \frac{9\rho_2^2 L(K_1)^2|I_j|}{\beta} < \frac{9\rho_2^2 c_i^2 L(K_1)}{\beta^2 L(\partial K_1)} < 9\beta^{-8}c_i^2\left(\frac{c_i}{c_1}\right)^4 < c_{i+1}^2,$$

where in the last inequality we have used (7) and (8). The induction hypothesis (P_{j-1}) implies $K' \in \mathcal{A}_{i+1}(j-1)$, meaning

$$d(\theta(K'), I_{j-1}) > \frac{\beta c_{i+1}^2}{4L(K')L(\partial K')} \geq \frac{\beta c_{i+1}^2}{4L(K')^2}.$$

On the other hand by (12), K' is in fact $\dfrac{\beta c_{i+1}}{\sqrt{6\rho_2}}$-shrinkable, and since $c_{i+1} > \sqrt{6\rho_2}c_i$, we have

$$\begin{aligned} d(\theta(K', I_{j-1}) &\leq |\theta(K') - \theta(K_1)| + d(\theta(K_1), I_j) \\ &< \frac{\beta^2 c_{i+1}^2}{6\rho_2 L(K')L(K_1)} + \frac{\beta c_i^2}{L(K_1)L(\partial K_1)} < \frac{\beta c_{i+1}^2}{12\rho_2 L(K')L(K_1)} < \frac{\beta c_{i+1}^2}{4L(K')^2} \end{aligned}$$

which contradicts the previously displayed inequality. This finishes the proof of Step 2.

Step 3. For each $i = 1, \ldots, M$, we have $\operatorname{diam}\Theta_i(j) < \frac{\beta}{2}|I_j|$.
Assume $\Omega_i(j) \neq \emptyset$ and fix a complex K_0 in it. The previous step implies for any $K \in \Omega_i(j)$, we have $\partial K = \partial K_0$. Moreover, the longest saddle connection on ∂K_0 belongs to K so that since K is βc_i-shrinkable, we have (using $\beta < 1/5$)

$$|\theta(K) - \theta(\partial K_0)| < \frac{\beta^2 c_i^2}{L(K)L(\partial K_0)} = \frac{\beta^2 c_i^2}{L(K)L(\partial K)} < \frac{\beta}{5}|I_j|.$$

Thus, $\Theta_i(j)$ is inside a ball of radius $\frac{\beta}{5}|I_j|$, so that its diameter is $\leq \frac{2\beta}{5}|I_j| < \frac{\beta}{2}|I_j|$.

Step 4. We now show that (P_j) holds.
Suppose $K \in \mathcal{E}_i$ is such that $L(K)L(\partial K)|I_j| < c_i^2$. Since $|I_j| \geq \beta|I_{j-1}|$, we have $L(K)L(\partial K)|I_{j-1}| < \beta^{-1}c_i^2$. There are two cases. If $L(K)L(\partial K)|I_{j-1}| < c_i^2$, then (P_{j-1}) implies $K \in \mathcal{A}_i(j-1) \subset \mathcal{A}_i(j)$ and we are done. Otherwise, we conclude that $K \in \Omega_i(j-1)$ so that, by the previous step, $\theta(K)$ lies in an interval of length $< \frac{\beta}{2}|I_{j-1}|$

centered about $z_i(j-1)$. Since Bob's interval I_j must be disjoint from the interval of length $\beta|I_{j-1}|$ centered at $z_i(j-1)$ chosen by Alice, we have

$$d(\theta(K), I_j) > \frac{\beta}{4}|I_{j-1}| > \frac{\beta c_i^2}{4L(K)L(\partial K)}.$$

In any case, we have $K \in \mathcal{A}_i(j)$. □

Proof of Theorem 5. By Theorem 9 we are able to ensure that for any level i complex K_i, we have

$$\max\{L(\partial K) \cdot L(K) \cdot |I_j|, L(\partial K) \cdot L(K) \cdot d(\theta(K), I_j)\} > \frac{\beta c_i^2}{4}.$$

In particular this holds when $i = 1$. Since there is only one saddle connection in a 1-complex, and since for any fixed saddle connection γ, $|\gamma|^2|I_j| \to 0$ as $j \to \infty$, we conclude that for all but finitely many intervals I_j we have

$$|\gamma|^2 d(\theta_\gamma, I_j) = \max\{|\gamma|^2|I_j|, |\gamma|^2 d(\theta_\gamma, I_j)\} > \frac{\beta c_1^2}{4}.$$

Thus if $\phi = \cap_{l=-1}^{\infty} I_l$ is the point we are left with at the end of the game, and γ is a saddle connection, then $|\gamma|^2|\theta_\gamma - \phi| > \frac{\beta c_1^2}{4}$, which by Proposition **??** establishes Theorem 5. □

5. Playing the Game in the Stratum

In this section we prove a theorem that as a corollary will imply Theorem 2, Theorem 3, Theorem 4 and Theorem 1. In the general situation we will be playing the game in a subset of a stratum $Q^1(k_1, \ldots, k_n, \pm)$. In the case of Theorem 2 it will be the entire stratum. In the case of Theorem 3 and Theorem 1 it is the entire space of quadratic differentials on a fixed Riemann surface, and in the case of Theorem 4 a subset of the space of Abelian differentials on a compact Riemann surface. What these examples have in common is that there is a S^1 action on the space given by $q \to e^{i\theta}q$. This will allow us to use the ideas of the previous section.

5.1. **Product structure and metric.** Given a quadratic differential q_0 that belongs to a stratum $Q^1(k_1, \ldots, k_n, \pm)$ and a triangulation $\Gamma = \{e_i\}_{i=1}^{6g-6+3n}$ of it, we have a chart $\varphi : U \to \mathbb{C}^{6g-6+3n}$ on a neighborhood U of q_0 in the stratum where the triangulation remains defined, i.e. none of the triangles are degenerate.

For sufficiently small U, using holonomy coordinates, we obtain an embedding

$$\varphi_\Gamma : U \to \mathbb{C}^{6g-6+3n}$$

whose image is a convex subset of a linear subspace. Equip U with the metric induced by the norm $\|\boldsymbol{z}\|_\Gamma = \max(|z_1|, \ldots, |z_{6g-6+3n}|)$. Note that the notation $\|q_1 - q_2\|_\Gamma$ for the distance between q_1 and q_2 in these holonomy coordinates should not be confused with the possibility that q_1, q_2 are quadratic differentials on the same Riemann surface in which case $q_1 - q_2$ will refer to vector space subtraction and $\|q_1 - q_2\|$ the area.

We note that U and the induced metric depend only on the $6g-6+3n$ homotopy classes relative to the zeroes of the saddle connections in the triangulation. However a change in homotopy classes will induce a bi-Lipschitz map of metrics and since winning is invariant under bi-Lipschitz maps, we are free to choose any triangulation.

Note that multiplication by $e^{i\theta}$ defines an S^1-action that is equivariant with respect to (U,φ). Let $\pi_a : \varphi(U) \to S^1$ be the map that gives the argument of e_1. Let $Z = \pi_a^{-1}(\theta_0)$ where $\theta_0 = \pi_a(q_0)$ and let $\pi_Z : \varphi(U) \to Z$ be the map that sends q to the unique point of Z that is contained in the S^1-orbit of q. Then

$$\varphi(U) \simeq Z \times S^1$$

with projections given by π_Z and π_a.

The metric on Z is the ambient metric:

$$d_Z(q_1,q_2) = \|q_1 - q_2\|_\Gamma \quad \text{for} \quad q_1, q_2 \in Z$$

The metric on U is given by

$$d_U(q_1,q_2) = \max(d_Z(\pi_Z(q_1),\pi_Z(q_2)), d_a(\pi_a(q_1),\pi_a(q_2)))$$

where $d_a(\cdot,\cdot)$ is the distance on S^1 measuring difference in angles. This metric has the property that a ball in the metric d_U is a ball in each factor.

Definition 10. *By an ε-*perturbation *of q we mean any flat surface in U whose distance from q is at most ε.*

We now show that the holonomy of any *any* saddle connection of q is not changed much by an ε-perturbation. Recall the constant λ_2, given in Lemma 4 that depends on the and the choice of triangulation.

Lemma 11. *Let q' be an ε-perturbation of q and suppose that the homotopy class specified by a saddle connection γ in q is represented on q' by a union of saddle connections $\cup_1^k \gamma_i'$. Then the total holonomy vector $hol(\cup\gamma_i')$ makes an angle at most $2\lambda_2\varepsilon$ with the direction of γ and and its length differs from that of γ by a factor between $1 \pm \lambda_2\varepsilon$. Also, the direction of the individual γ_i' also lie within $2\lambda_2\epsilon$ of γ.*

Proof. Represent γ as a path in the triangulation Γ on q. After perturbation, the total holonomy vectors $hol(\gamma), hol(\gamma')$ satisfy

$$hol(\gamma') - hol(\gamma) \le |\gamma|_\Gamma \varepsilon \le \lambda_2 |\gamma| \varepsilon,$$

by Lemma 4. Hence the difference in angle is at most

$$\arcsin\left(\frac{\lambda_2\varepsilon|\gamma|}{|\gamma|}\right) \le 2\lambda_2\varepsilon,$$

proving the first statement.

For the individual γ_i', fix a linear parametrization $q_t, 0 \le t \le 1$, so that $q_0 = q$ and $q_1 = q'$. Then there are times $0 = t_0 < t_1 < \cdots < t_n = 1$ and saddle connections γ_j on q_{t_j} (that are parallel to other saddle connections on q_{t_j}) such that $\gamma_0 = \gamma$, $\gamma_n = \gamma_i'$, and, by the first part of the lemma, the angle between γ_j and γ_{j+1} is at most

$$2\lambda_2\varepsilon(t_{j+1} - t_j).$$

The triangle inequality now implies the angle between the holonomies of γ and γ_i' is at most $2\lambda_2\varepsilon$. □

5.2. **Moving complexes.** In the proof of the theorems we will need to move triangulations from one quadratic differential to another in order to play the games. In such a move, vertices of the triangulation may hit other edges forcing degenerations. The following theorem is the mechanism for keeping track of complexes as they move. We first note that about each point in the stratum there is a neighborhood where the homotopy class of a saddle connection can be consistently defined.

Theorem 10. *Suppose $q_t; 0 \leq t \leq 1$ is a smooth path of quadratic differentials in a given stratum. Suppose K is a complex on q_0 (with triangulation Γ). Then there is a complex K' on q_1 with triangulation, denoted Γ' and a piecewise linear map $F : K \to K'$ such that*

1. *the homotopy class of every saddle connection of K is mapped by F to a union of saddle connections on K'. These saddle connections have the same homotopy class.*
2. *the closed subset K' depends only on K and the path of quadratic differentials; in particular it does not depend on the choice of triangulation of Γ of K.*

Proof. Let T be the set of $t \geq 0$ such that the geodesic representative on q_t of the homotopy class of each saddle connection in Γ is realized by a single saddle connection in q_t. Let $A(0)$ be the connected component of T containing 0. For each $t \in A(0)$, let Γ_t be the collection of saddle collections in q_t representing these homotopy classes. It is easy to see that Γ_t is a pairwise disjoint collection and that three saddle connections in Γ bound a triangle if and only if the corresponding saddle connections in Γ_t bound a triangle. Let K_t be the complex determined by Γ_t. The obvious piecewise linear map $f_t : K \to K_t$ is a homeomorphism onto its image.

Let $t_1 = \sup A(0)$. We claim that the closed set K_t is independent of the choice of triangulation Γ for $0 < t < t_1$. Indeed, suppose $\tilde{\Gamma}$ is another triangulation of K such that for $0 < t < t_1$ the geodesic representative on q_t of the homotopy class of each saddle connection in $\tilde{\Gamma}$ is realized by a single saddle connection on q_t. Let $\tilde{f}_t : K \to \tilde{K}_t$ be the simplicial homeomorphism between K and the complex K_t determined by the corresponding collection $\tilde{\Gamma}_t$ of saddle connections on q_t. Then $\tilde{f}_t$ and f_t agree on ∂K, so that $\partial K_t = f_t(\partial K) = \tilde{f}_t(\partial K) = \partial \tilde{K}_t$. Note that f_t and $\tilde{f}_t$ induce the same transverse orientation on any saddle connection in ∂K_t. Since $\tilde{f}_t \circ f_t^{-1}$ restricts to the identity on ∂K_t, it maps each connected component of the interior of K_t to itself. Hence, the interiors of K_t and $\tilde{K}_t$ coincide, and therefore $K_t = \tilde{K}_t$, proving the claim.

Let $(E, \pi : E \to [0, t_1])$ be the pull-back of the tautological bundle so that each fiber $\pi^{-1}(t)$ is a copy of q_t for each $t \in [0, t_1]$. Let $\Omega \subset E$ be the subset that intersects each fiber in K_t. Define $K_{t_1} = \bar{\Omega} \setminus \Omega$ and note that it is a closed set contained in the fiber over $t = t_1$. Let $f_{t_1} : K \to K_{t_1}$ be the pointwise limit of the maps f_t as $t \to t_1^-$. Each saddle connection γ of Γ is mapped by f_{t_1} to a union of parallel saddle connections $\gamma_i', i = 1, \ldots, r = r(\gamma)$. A triangle Δ determined by Γ may collapse under f_{t_1} to a union of parallel saddle connections; otherwise, $f_{t_1}(\Delta)$ has $n = n(\Delta)$ saddle connections on its boundary, possibly with $n > 3$. If $n > 3$, then $n-3$ zeroes of q_t hit the interior of an edge of Δ at $t = t_1$. In this case we triangulate $f_{t_1}(\Delta)$ by adding $n-3$ "extra" saddle connections. Let Γ_{t_1} be the collection of saddle connections γ_i' associated to $\gamma \in \Gamma$ together with the "extra" saddle connections needed to triangulate $f_{t_1}(\Delta)$ for Δ that do not collapse and have $n(\Delta) > 3$. Let K_{t_1}' be the complex determined by Γ_{t_1} and let $f_{t_1}' : K \to K_{t_1}'$ be the composition of f_{t_1} with the inclusion of K_{t_1} into K_{t_1}'. It is easy to see that $F_{t_1} := f_{t_1}'$ maps saddle connections to unions of saddle connections and that K_{t_1}' does not depend on the choice of Γ.

If $t_1 = 1$ we are done and we set $K' = K_{t_1}'$. Thus assume $t_1 < 1$. We repeat the construction above starting with K_{t_1}' and form the maximal set $A(t_1)$ of times $t_1 \le t < t_2$ such that the homotopy class of each saddle connection of K_{t_1}' is realized by a single saddle connection on q_t. We repeat the procedure, building a new complex K_{t_2}' and finding a map $f_2' : K_{t_1}' \to K_{t_2}'$. We then let $F_{t_2} = F_1 \circ f_{t_1}'$. The compactness of $[0,1]$ implies that this procedure only need be repeated a finite number of times $t_1 < t_2 < \cdots < t_N = 1$. We inductively find F_{t_N} and set $F = F_{t_N}$ and $K' = K_{t_N}'$. □

Definition 11. *Let q' be an ε-perturbation of q and K a complex in q. Let K' be the complex obtained by applying Theorem 10 using the linear path in the stratum joining q and q'. We call K' the* moved *complex.*

Corollary 6. Let q' be an ε-perturbation of q and suppose that K is a complex on q that moves to K' on q' by F. Let $\gamma \in K$. Let $\cup_1^k \gamma_i' = F(\gamma)$. Then $|\theta_\gamma - \theta_{\gamma_i'}| < 2\lambda_2\varepsilon$.

Proof. Each $f_{t_k}', i = 1, \ldots, N$ in the proof of Theorem 10 is a piecewise linear map between complexes on q_k that are ε_k-perturbations of one another, where $\sum_{k=1}^N \varepsilon_k = \varepsilon$. Let $\sigma_k, k = 0, 1, \ldots, N$ be the saddle connections such that $\sigma_0 = \gamma$, $f_{t_k}'(\sigma_{k-1}) \supset \sigma_k$, and $\sigma_N = \gamma_i'$. Lemma 11 implies $|\theta_{k-1} - \theta_k| < 2\lambda_2\varepsilon_k$ where θ_k denotes the direction of σ_k. The conclusion of the corollary now follows from the triangle inequality. □

Proposition 2. *Suppose $K_1 \subset K_2$ as closed subsets and each is a complexes on q_0. They are both moved to q_1 to become complexes K_1', K_2'. Then again viewed as closed subsets, we have $K_1' \subset K_2'$.*

Proof. We can extend the triangulation of K_1 to a triangulation $\hat{K}_1$ of the same closed subset as is defined by K_2 and such that $\hat{K}_1$ and K_2 coincide on the boundary. We move both K_2 and $\hat{K}_1$ to q_1 obtaining triangulations K_2' and $\hat{K}_1'$. Theorem 10 says that K_2' and $\hat{K}_1'$ are triangulations of the same closed set. Clearly $K_1' \subset \hat{K}_1'$ and we are done. □

We adopt the notation (K, q) to refer to a complex on the flat surface defined by q.

Definition 12. *Suppose (K_1, q_1) and (K_2, q_2) are complexes of distinct flat surfaces. We say (K_1, q_1) and (K_2, q_2) are* not combinable *if K_1 moved to q_2 satisfies $K_1' \subseteq K_2$ and K_2 moved to q_1 satisfies $K_2' \subseteq K_1$. Otherwise they are said to be* combinable.

5.3. **Proof of Theorem 2.** We are given the compact set $\bar{U}$ in the stratum. For any α, β we can play the game a finite number of steps so that we are allowed to assume that U is a ball with center q_0 which has a triangulation Γ which remains defined for all $q \in U$. We are therefore able to talk about ϵ perturbations in U. Furthermore since $\bar{U}$ is compact, the constant λ_2 given by Lemma 4 which depends only on the and the triangulation can be taken to be uniform in U. Furthermore because our choice of metric is the sup metric, each ball B_j will be of the form

$$B_j = Z_j \times I_j,$$

where Z_j is a ball in the space Z and $I_j \subset S^1$.

Now choose

$$\alpha < \min\{\frac{1}{8}, \frac{1}{720(6g-6+n)^2\lambda_2}\}, \tag{13}$$

where n is the number of zeroes. (Again the significance of the choice of constants 8, 720 is only to make certain inequalities hold.) Let L_0 denote the length of the shortest saddle connection on q_0. Let $0 < c_1 < \cdots < c_{M+1}$ be given by $c_i = (\alpha\beta)^{N_i} c_{i+1}$ and

$$c_{M+1} = \min(L_0 \beta^{N_M}, L_0 |I_1|^{1/2} \varepsilon_0)$$

where N_i are defined by $N_1 = 4M + 1$ and $N_{i+1} = 4M + 4(N_1 + \cdots + N_i)$ so that

$$\frac{c_{i+1}}{c_i} = (\alpha\beta)^{-4M-1} \left(\frac{c_i}{c_1}\right)^4 \geq 100(\alpha\beta)^{-4M} \left(\frac{c_i}{c_1}\right)^4. \tag{14}$$

Now inductively, given a ball $B_j = Z_j \times I_j$, where Z_j is centered at q_j, let $\mathcal{E}_i(B_j) := \mathcal{E}_i$ be the set of all marked $(\alpha\beta)^{3+M} c_i$-shrinkable complexes (K, q) of level i where $q \in B_j$. Given I_j with $j \equiv M - i + 1 \bmod M$, we let

$$\mathcal{A}_i(j) := \left\{(K, q) \in \mathcal{E}_i : d(\theta(K), I_j) > \frac{(\alpha\beta)^M c_i^2}{24 L(K) L(\partial K)}\right\}.$$

We shall prove the following statement for every $j \geq 1$ and every $i \in \{1, \dots, M\}$, where $j \equiv M - i + 1 \bmod M$.

$$\forall K \in \mathcal{E}_i \qquad L(K)L(\partial K)|I_j| < c_i^2 \implies K \in \mathcal{A}_i(j) \tag{P_j}$$

Note that (P_j) holds automatically for all $j < j_0 := \min\{k : |I_k| < \beta^{2N_M}\}$ because

$$L(K)L(\partial K)|I_j| \geq L_0^2 \beta^{2N_M} \geq c_{M+1}^2 > c_i^2$$

while if $j_0 = 1$, we note that $L(K)L(\partial K)|I_1| \geq L_0^2 |I_1| \geq c_{M+1}^2 > c_i^2$.

We proceed by induction and suppose that Alice is given a ball $B_j = Z_j \times I_j$, $j \geq j_0$, where $Z_j \subset Z$ is a ball and $I_j \subset I$ is an interval. Suppose inductively that (P_k) holds for $k \leq j$. We will show that Alice has a choice of a ball $A_j \subset B_j$ to ensure (P_{j+M}) will hold.

Define

$$\Omega_i(j) := \left\{(K, q) \in \mathcal{E}_i \setminus \mathcal{A}_i(j) : \frac{c_i^2}{L(K)L(\partial K)} \leq |I_j| < \frac{(\alpha\beta)^{-M} c_i^2}{L(K)L(\partial K)}\right\}.$$

We summarize our strategy. We will show that for any $q_1, q_2 \in B_j$ such that $d_U(q_1, q_2) \leq \alpha|B_j|$, no two complexes $(K_1, q), (K_2, q)$ are combinable (step 2). Then we will show that if $d_U(q', q) \leq \alpha|Z_j|$, and $(K, q), (K', q') \in \Omega_i(j)$ are complexes which are not pairwise combinable, then $|\theta(K) - \theta(K')|$ is small (step 3). Then choosing some (K, q), Alice can choose an angle ϕ where $d(\phi, \theta(K)) > \frac{1}{3}|I_j|$, and an interval I'_j centered at ϕ of radius $\alpha|I_j|$. There will be no (K', q') with $\theta(K') \in I'_j$ and where K' is combinable with K and $d_U(q, q') \leq \alpha|Z_j|$ (step 4). As in the proof of Theorem 9, step 1 is a technical result controlling the ratio of $L(K)$ and $L(\partial K)$, which is necessary for step 2.

Step 1. We show that for any $(K, q) \in \Omega_i(j)$,

$$\frac{L(K)}{L(\partial K)} < (\alpha\beta)^{-2M} \left(\frac{c_i}{c_1}\right)^2. \tag{15}$$

This is essentially the same as the proof of Step 1 in the proof of Theorem 9. We provide the details. Let k be the previous stage for dealing with 1-complexes. Consider first the case that

$$L(\partial K)^2|I_j| < (\alpha\beta)^M c_1^2. \tag{16}$$

Then $L(\partial K)^2|I_k| < c_1^2$ so that (P_k) implies the longest saddle connection on ∂K belongs to $A_1(k)$, meaning

$$d(\theta(\partial K), I_k) > \frac{(\alpha\beta)^M c_1^2}{24L(\partial K)^2} > c_1^2|I_k|.$$

But since K is $(\alpha\beta)^{3+M}c_i$-shrinkable and not in $\mathcal{A}_i(j)$, the triangle inequality implies

$$\begin{aligned} d(\theta(\partial K), I_j) &\leq |\theta(\partial K) - \theta(K)| + d(\theta(K), I_j) \\ &\leq \frac{(\alpha\beta)^{6+2M}c_i^2}{L(K)L(\partial K)} + \frac{(\alpha\beta)^M c_i^2}{24L(K)L(\partial K)} \leq \frac{(\alpha\beta)^M c_i^2}{L(K)L(\partial K)} \end{aligned}$$

which implies (15) using the fact that $\alpha\beta < 1$. Suppose now that (16) does not hold. Since $(K, q) \in \Omega_i(j)$, we have

$$\frac{L(K)}{L(\partial K)} = \frac{L(K)L(\partial K)|I_j|}{L(\partial K)^2|I_j|} < \frac{(\alpha\beta)^{-M}c_i^2}{(\alpha\beta)^M c_1^2}$$

so that (15) holds in this case as well.

Step 2. Now we show that if (K_1, q_1) and (K_2, q_2) are in $\Omega_i(j)$ and

$$d_U(q_1, q_2) \leq \alpha|B_j|,$$

then K_1 and K_2 are not combinable. Assume on the contrary that they are combinable. So without loss of generality assume there exists $\gamma \in K_2$ so that the moved $\gamma' \not\subset K_1$.

Choose the following constants.

Let $\varepsilon = c_i(\alpha\beta)^{3+M}$, $\rho_1 = 2(\alpha\beta)^{-4M}\left(\frac{c_i}{c_1}\right)^2$, $\rho_2 = (\alpha\beta)^{-2M}\left(\frac{c_i}{c_1}\right)$ and $\rho_3 = 12\rho_2(\rho_1 + \rho_2)$.

We now show that we can combine γ' to K_1 to make a $\sqrt{\rho_3}\epsilon$-shrinkable complex. Observe that similarly to the proof of Step 2 in Theorem 9 we have

$$|\theta(K_1)-\theta(K_2)| \leq (1+\frac{1}{12})|B_j| \leq \frac{\rho_1\epsilon^2}{L(K_1)L(K_2)} \text{ and } \frac{L(K_1)}{L(K_2)} \leq \rho_2.$$

Therefore

$$(17)\quad |\theta_{\gamma'}-\theta(K_1)| \leq |\theta_\gamma-\theta_{\gamma'}|+|\theta_\gamma-\theta(K_2)|+|\theta(K_1)-\theta(K_2)| \leq$$
$$\lambda_2\|q_1-q_2\|+\frac{3\epsilon^2}{2|\gamma'|L(K_2)}+\frac{\rho_1\epsilon^2}{L(K_1)L(K_2)} \leq \frac{2\epsilon^2}{|\gamma'|L(K_2)}+\frac{2\rho_1\epsilon^2}{L(K_1)L(K_2)} \leq \frac{(2\rho_1+2\rho_2)\epsilon^2}{|\gamma'|L(K_1)}$$

Let $\theta=\theta(K_1)$ and $t=\log\frac{(L(K_1)}{\epsilon}$. There is a saddle connection σ disjoint from K_1 such that on $g_t r_\theta X$ the saddle connection $\sigma_{\theta,t}=g_t r_\theta\sigma$ satisfies

$$h_\theta(\sigma_{\theta,t}) \leq (2\rho_2+2\rho_1+3)\epsilon \text{ and } v_\theta(\sigma_{\theta,t}) \leq \frac{3}{2}\rho_2\epsilon.$$

It follows that

$$|\sigma| \leq \sqrt{\left(\frac{(2\rho_2+2\rho_1+3)\epsilon}{L(K_1)}\right)^2+\left(\frac{3\rho_2L(K_1)}{2}\right)^2} \leq 2\rho_2L(K_1).$$

Now we show $\hat{K}=K_1\cup\sigma$ is $\epsilon\sqrt{\rho_3}$ shrinkable. We have $L(\hat{K})=\max\{L(K_1),|\sigma|\}$. If $L(\hat{K})=L(K_1)$ then $\hat{K}$ is $\epsilon\sqrt{\rho_3}$ shrinkable because

$$|\theta_\sigma-\theta(K_1)| \leq 2\frac{h_\theta(\sigma)}{|\sigma|} \leq \frac{(5\rho_1+5\rho_2)\epsilon^2}{|\sigma|L(K_1)}.$$

If $L(\hat{K})=|\sigma|$ for every $\xi\in K_1$ we have

$$\left|\theta(\hat{K})-\theta_\xi\right| \leq |\theta_\sigma-\theta(K_1)|+|\theta(K_1)-\theta_\xi|$$
$$< \frac{(5\rho_1+5\rho_2)\varepsilon^2}{|\sigma|L(K_1)}+\frac{\varepsilon^2}{|\xi|L(K_1)} < \frac{(6\rho_1+6\rho_2)\varepsilon^2}{|\xi|L(K_1)} < \frac{(12\rho_1+12\rho_2)\rho_2\varepsilon^2}{|\xi|L(\hat{K})}$$

where $2\rho_2L(K_1)\geq|\sigma|\geq L(K_1)\geq|\xi|$ was used in last two inequalities. This shows that $\hat{K}$ is $\sqrt{\rho_3}\epsilon$-shrinkable.

Now we derive a contradiction. Notice that $c_{i+1}(\alpha\beta)^{3+M} > \sqrt{\rho_3}\epsilon$ and so $\hat{K}$ is $c_{i+1}(\alpha\beta)^{3+M}$ shrinkable.

Since there are no c_{M+1}-shrinkable complexes of level $M+1$, we have our desired contradiction when $i=M$. For $i<M$, we have

$$L(\hat{K})L(\partial\hat{K})|B_{j-1}| < \frac{4\rho_2^2L(K_1)^2|B_j|}{\alpha\beta} < \frac{4\rho_2^2c_i^2L(K_1)}{(\alpha\beta)^{1+M}L(\partial K_1)} < 4(\alpha\beta)^{-7M-1}c_i^2\left(\frac{c_i}{c_1}\right)^4 < c_{i+1}^2.$$

The first inequality uses the bound on $L(\hat{K})$ in terms of $L(K_1)$ and the definition of the game. The third inequality uses Step 1. We conclude that the induction hypothesis

(P_{j-1}) implies $\hat{K} \in \mathcal{A}_{i+1}(j-1)$, meaning

$$d_a(\theta(\hat{K}), I_{j-1}) > \frac{(\alpha\beta)^M c_{i+1}^2}{24L(\hat{K})L(\partial\hat{K})} \geq \frac{(\alpha\beta)^M c_{i+1}^2}{24L(\hat{K})^2}. \tag{18}$$

Since $\hat{K}$ is $\sqrt{\rho_3}\epsilon$ shrinkable by the choice of ϵ, it is in fact $c_i\sqrt{\rho_3}(\alpha\beta)^M$-shrinkable, and since $c_{i+1} > c_i\sqrt{\rho_3}\sqrt{4(\alpha\beta)^{-M}\rho_2}$, we have

$$\begin{aligned} d_a(\theta(\hat{K}), I_{j-1}) \leq \left|\theta(\hat{K}) - \theta(K_1)\right| + d_a(\theta(K_1), I_j) &\leq \frac{\rho_3(\alpha\beta)^M c_i^2}{L(K_1)L(\hat{K})} + \frac{(\alpha\beta)^M}{24L(K_1)L(\partial K_1)} \\ &< \frac{2\rho_3\rho_2(\alpha\beta)^M c_i^2}{L(\hat{K})^2} + \frac{(\alpha\beta)^M 2\rho_2\rho_2\rho_1 c_i^2}{L(\hat{K})^2} < \frac{(\alpha\beta)^M c_{i+1}^2}{24L(\hat{K})^2} \end{aligned}$$

giving us the desired contradiction to (18).

Step 3. We show that if (K_1, q_1) is not combinable with (K_2, q_2), each belongs to $\Omega_i(j)$, and $d_U(q_1, q_2) \leq 2\alpha|B_j|$, then $|\theta(K_1) - \theta(K_2)| \leq \frac{1}{3}|B_j|$.

Since (K_1, q_1) and (K_2, q_2) are not combinable, when we move K_1 to q_2 which we denote by K_1', we have $K_1' \subseteq K_2$. Similarly $K_2' \subset K_1$. By Proposition 2 when we move K_1' back to q_1, denoted by K_1'', we have

$$K_1'' \subset K_2' \subset K_1.$$

But since each saddle connection on ∂K_1 is homotopic to a union of saddle connections of K_1'' with common endpoints, K_1 and K_1'' bound a union of (possibly degenerate) simply connected domains each of which has a segment of ∂K_1 as a side. Let γ be the longest saddle connection on ∂K_1 and let Δ be the corresponding simply connected domain. Since $K_1'' \subset K_2' \subset K_1$, Δ contains a union of saddle connections $\hat{\kappa}' \subset \partial K_2'$ that join the endpoints of γ. Let $p \leq 6g - 6 + n$ denote the cardinality of $\hat{\kappa}'$.

Since $d_U(q_1, q_2) \leq 2\alpha|B_j|$, each is a $2\alpha|B_j|$ perturbation of the other. Since the angle the saddle connections of ∂K_2 make with each other goes to π as the length of the segments goes to ∞, and these angles change by a small factor, by Corollary 6 we have for some $\kappa' \in \hat{\kappa}'$,

$$L(\partial K_1) \leq 2p|\kappa'|. \tag{19}$$

and for all $\kappa' \in \hat{\kappa}'$,

$$|\kappa'| \leq pL(\partial K_1). \tag{20}$$

Since lengths change by a factor of at most $1 + \lambda_1 \leq \frac{3}{2}$ in moving, and since κ' arises from a saddle connection $\kappa \subset \partial K_2$, we have that

$$L(\partial K_1) \leq 3p|\kappa| \leq 3pL(\partial K_2),$$

and by symmetry

$$L(\partial K_2) \leq 3rL(\partial K_1),$$

for some constant $r \leq 6g-6+n$, so that

$$|\kappa'| \geq \frac{L(\partial K_2)}{9rp} \geq \frac{L(\partial K_2)}{9(6g-6+3n)^2}. \tag{21}$$

We also claim that

$$|\theta_{\kappa'} - \theta(K_1)| < \frac{1}{20}|B_j|.$$

To see this, by Corollary 6 applied twice, first to the moved K_1' and then to K_1'', and by the choice of α, in (13), we have that for all $\gamma' \in \partial\Delta$

$$|\theta_{\gamma'} - \theta_\gamma| \leq \frac{1}{40}|B_j|,$$

which implies since κ' is a union of saddle connections in Δ that

$$|\theta_\gamma - \theta_{\kappa'}| \leq \frac{1}{40}|B_j|.$$

The shrinkability of K_1 implies that

$$|\theta(K_1) - \theta_\gamma| \leq \frac{c_i^2(\alpha\beta)^{2M}}{L(K_1)|\gamma|} \leq \frac{1}{40}|B_j|,$$

so the claim follows from the last two inequalities.

By Corollary 6, the triangle inequality, and the above claim we have

$$|\theta(K_1)-\theta(K_2)| \leq |\theta(K_1)-\theta_{\kappa'}|+|\theta_{\kappa'}-\theta_\kappa|+|\theta_\kappa-\theta(K_2)| \leq \frac{1}{20}|B_j|+\frac{1}{40}|B_j|+|\theta(K_2)-\theta_\kappa|.$$

By our shrinkability assumption on K_2,

$$|\theta_\kappa - \theta(K_2)| \leq \frac{c_i^2(\alpha\beta)^{2M}}{L(K_2)|\kappa|} \leq \frac{1}{20}|B_j|,$$

where the second inequality follows from (21), the definition of $\Omega_i(j)$, and the choice of α given in (13). Step 3 follows.

Step 4.

Bob presents Alice with a ball $B_j = Z_j \times I_j$ where $j \equiv M-i+1 \mod M$. If there is no (K,q) in $\Omega_i(j)$, Alice makes an arbitrary move. Otherwise, pick a $(K,q) \in \Omega_i(j)$. Alice chooses a ball $A_j = Z_j' \times I_j'$ of diameter $\alpha|B_j|$, whose center has first coordinate q and second coordinate is as far from $\theta(K,q)$ as possible. Observe that

$$d_a(\theta(K,q), I_j)' \geq (\frac{1}{2} - 2\alpha)|B_j|.$$

By Step 2 if $\hat{q} \in Z_j$ and $(\hat{K},\hat{q}) \in \Omega_i(j)$ then $\hat{K}$ and K are not combinable. By Step 3

$$d(\theta(\hat{K},\hat{q}), I_j') \geq (\frac{1}{2} - 2\alpha)|B_j| - \frac{1}{3}|B_j| \geq (\frac{1}{6} - 2\alpha)|B_j|.$$

Now because $(\hat{K},\hat{q}) \in \Omega_i(j)$, using the left hand inequality in the definition and that $\alpha\beta < 1$, we conclude that

$$d_a(\theta(\hat{K},\hat{q}), I_j') \geq (\frac{1}{6} - 2\alpha)\frac{c_i^2(\alpha\beta)^M}{L(\hat{K})L(\partial\hat{K})}.$$

Because $d_a(\theta, I'_j) \leq d_a(\theta, I'_{j+M})$ we know then that (P_{j+M}) holds. This finishes the inductive proof of (P_j).

We finish the proof of Theorem 2. By (P_j) we are able to ensure that for any level i complex K on a surface q, we have

$$\max\{L(\partial K) \cdot L(K) \cdot |B_j|, L(\partial K) \cdot L(K) \cdot d_\theta((q, K), B_j\} > \frac{(\alpha\beta)^M c_i^2}{4}.$$

In particular this holds when $i = 1$. Since there is only one saddle connection in a 1-complex, and since for any fixed saddle connection γ on a surface q, $|\gamma|^2 |B_j| \to 0$ as $j \to \infty$, we conclude that for all but finitely many balls B_j we have

$$|\gamma|^2 d_\theta((q, \gamma), B_j) = \max\{|\gamma|^2 |B_j|, |\gamma|^2 d_\theta((q, \gamma), B_j\} > \frac{(\alpha\beta)^M c_1^2}{4}.$$

Thus if $(q, \phi) = \cap_{l=-1}^{\infty} B_l$ is the point we are left with at the end of the game, and γ is a saddle connection on q, then $|\gamma|^2 |\theta_\gamma - \phi| > \frac{(\alpha\beta)^M c_1^2}{4}$, which by Proposition **??** establishes the statement of strong winning in Theorem 2.

The set cannot be absolute winning for the following reason. Bob begins by choosing a ball I_1 centered at some quadratic differential which has a vertical saddle connection γ. The set X_γ consisting of quadratic differentials with a vertical saddle connection γ is a closed subset of codimension one and such quadratic differentials are clearly not bounded. Then whatever Alice's move of a ball $J_1 \subset I_1$, Bob can find a next ball $I_2 \subset I_1 \setminus J_1$ centered at some new point in X_γ which shows that bounded quadratic differentials are not absolute winning.

An identical proof allows us the same theorem in the case of marked points.

Theorem 11. *Let Q be a stratum of quadratic differential with k marked points. Let $U \subset Q$ be an open set with compact closure in Q where the metric given by local coordinates is well defined. The set $E \subset \bar{U}$ consisting of those quadratic differentials q such that the Teichmüller geodesic defined by q stays in a compact set in the stratum is α winning for Schmidt's game. In fact it is α-strong winning.*

In fact with a similar proof we have the following

Theorem 12. *Let P be a rotation invariant subset of the stratum of quadratic differentials with k marked points where a metric given by local coordinates is well defined. Assume P has compact closure in the stratum. The set $E \subset P$ consisting of those quadratic differentials q such that the Teichmüller geodesic defined by q stays in a compact set in the stratum is α winning for Schmidt's game. In fact it is α-strong winning.*

5.4. **Proof of Theorem 1.** Again as before the Diophantine foliations are not absolute winning since for any closed curve γ the set of foliations F such that $i(F, \gamma) = 0$ is a codimension one subset.

We now show strong winning. We can assume we start with a fixed train track τ, and a small ball $B(F_0, r)$ of foliations carried by τ. Indeed, if the initial ball Bob chooses contains points on the boundary of two or more charts, then Alice can use the strategy of choosing her balls furthest away from these boundary points, so that in a finite number of steps her choice will be contained in a single chart.

By choosing transverse foliations, we can insure that there a ball $B'(q_0, r') \subset Q^1(1, \dots, 1, -)$ of quadratic differentials contained in the principal stratum so that

- the vertical foliation of each $q \in B'(q_0, r')$ is in $B(F_0, r)$.
- each vertical foliation in $B(F_0, r)$ is the vertical foliation of some $q \in B(q_0, r')$.
- $B(F_0, r)$ and $B'(q_0, r')$ are small enough so that the holonomies of a fixed set of saddle connections serve as local coordinates.
- There is a fixed constant so that the holonomy of any $q \in B'(q_0, r')$ is bounded away from 0 by that constant.

In holonomy coordinates the map that sends $q \in B'(q_0, r')$ to its vertical foliation is just projection onto the horizontal coordinates. This map clearly satisfies the hypotheses of Theorem 7. Since the bounded geodesics form a strong winning set in $Q^1(1, \dots, 1, -)$ by Theorem 2 they are strong winning in $\mathcal{PMF}$.

5.5. **Proof of Theorem 4.** If the condition $\inf_n n|T^n(p_1) - p_2| > 0$ holds for any pair of discontinuities p_1, p_2 of T, we say T is *badly approximable.* The following lemma connects the badly approximated condition for interval exchanges with the bounded condition for geodesics.

Lemma 12. *(Boshernitzan* [1, Pages 748-750]*) T is badly approximable if and only if the Teichmüller geodesic corresponding to vertical direction is bounded for any zippered rectangle such that T arises as the first return of the vertical flow to a transversal.*

See in particular the first equation on page 750, which relates the size of smallest interval bounded by discontinuities of T^n and closeness to a saddle connection direction. That is, let T be an IET that arises from first return to a transversal of a flow on a flat surface q. Assume that the smallest interval of continuity of T^n is less than $\frac{\epsilon}{n}$ then

$$|\theta(v) - \theta| < \frac{C\epsilon}{nL(v)},$$

where v is a saddle connection on q with length $O(n)$. Recall Theorem 6 relates closeness to saddle connection directions to boundedness of the Teichmüller geodesics.

We now give the proof of Theorem 4. For the same reason as above the set of bounded interval exchanges is not absolute winning. For strong winning, the proof is identical to the one for $\mathcal{PMF}$ except that now, using for example the the zippered rectangle construction, we can assume we have a small ball B in the space of interval exchange transformations [16], a corresponding ball in some stratum $B' \subset Q^1(k_1, \dots, k_n, +)$, such that each interval exchange transformation in B arises from the first return to a horizontal transversal of some $\omega \in B'$, and conversely for each $\omega \in B'$, the first return to a horizontal transversal gives rise to a point in B. We can assume these transversals vary continuously. Again the map from holonomy coordinates in B' to lengths in B is given by projection onto horizontal coordinates. We now apply Lemma 12, Theorem 2 and Theorem 7.

If we mark points a, b in the interval then we have

Theorem 13. *Given any irreducible permutation π there exists $\alpha > 0$ such that for any pair of points (a, b) we have*

$$\{T = T_{L,\pi} : \inf_{n>0}\{nd(T^n a, b)\} > 0\}$$

is an α-strong winning set.

Proof. As in the last theorem we find a ball in the stratum such that first return to transversals give the interval exchange. Now mark the points along each transversal at distances a, b to obtain a set B' of marked translation surfaces. It is not a ball but it is invariant under rotations lying in a small interval about the identity. Then by Theorem 12 the set of bounded trajectories in it is strong winning. By Theorem 7 the image of this set is strong winning in the space of marked interval exchange transformations. □

5.6. **Proof of Theorem 3.** Let U be the intersection of the principle stratum with $Q^1(X)$. Since the complement of U is contained in a finite union of smooth submanifolds, then for any sufficiently small $\alpha > 0$ and for any sufficiently small ball chosen by Bob, Alice can respond with a ball contained entirely in U with the bounded away from zero. Thus, we may assume Bob's initial ball B_1 is contained in U. By the main theorem of [5], the homeomorphism from $Q^1(X) \to \mathcal{PMF}$ sending a quadratic differential to the projective class of its vertical foliation is smooth when restricted to U. We can now apply Theorem 1.

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