# THE HAUSDORFF DIMENSION OF A GENERALIZED MORAN SET

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *THE HAUSDORFF DIMENSION OF A GENERALIZED MORAN SET* by Wyndham M. Galbraith and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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THE HAUSDORFF DIMENSION OF A GENERALIZED MORAN SET

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To compute the Hausdorff dimension of a Cantor set $E$, we define the similarity dimension $s$ as a function on the intervals $\mathcal{I}$ involved in the construction of $E$. That is, $s : \mathcal{I} \to \mathbb{R}_{\geq 0}$ where $s$ is the solution to the equation $\sum r_i^s = 1$ computed at each $I \in \mathcal{I}$. We prove $\text{Hdim} E = s$ for two classes of Cantor sets, those equipped with (1) a bounded contraction ratio, and (2) a uniform contraction ratio with a minimum gap condition.

I certify that the Abstract is a correct representation of the content of this thesis.

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Chair, Thesis Committee | Date

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# Chapter 1

# Introduction

## 1.1 Motivation

We address the problem of computing the Hausdorff dimension of Cantor sets in [0,1]. By Cantor sets we mean a nested intersection $E = \cap E_j$ where $E_0 = [0, 1]$ and each $E_j$ is a disjoint union of finitely many closed intervals. This problem, and variations thereof, have been studied extensively since the formulation of Hausdorff dimension. Traditionally these sets are described via a geometrical data given by iteration(s) of an infinite process. Given the considerable attention given to this problem it is perhaps surprising that it is not fully known under what general conditions that we may infer Hdim$E$ from local geometric data. It is our main goal to further the understanding of this problem by presenting it in a more general setting than is typically considered. As a result of this change in perspective we make progress on the stated problem while creating possibilities for deeper investigations.

## 1.2 Background

In his 1946 paper [7] P.A.P Moran showed that the dimension of a geometrically self-similar Cantor set could be simply calculated. Namely, as long as the geometric relationship between $E_{j-1}$ and $E_j$ is repeated throughout the construction, we can determine the global dimension by the solution to $\sum_{i=1}^{n} r_i^s = 1$, where the $r_i$ are the contraction ratios over $n$ intervals in level $E_j$. We will call this a constant contraction profile. A Moran fractal was introduced in 1992 by Cawley and Mauldin [2] that generalizes Moran's original construction to include Iterated Function Systems. It was shown that Moran's equation could still be applied if the intervals in $E_j$ were allowed to overlap. Since that time, most of the attention to generalize Moran's construction have been focused on overlapping constructions. Notably, progress was made in 2012 by Hochman [6] along these lines. Comparatively little attention has been paid to generalizations of the nonoverlapping kind. One remarkable exception is a structure described by Feng, Wen, and Wu (1996) [5] that allows the contraction profile to vary according to the level of the construction. We continue in this spirit by allowing our contraction profile to vary at each interval in the construction.

## 1.3 Outline

In Chapter 2, we will introduce a nonstandard formulation of a Cantor set $E$. To this end, we introduce a general notion of a nested tree of subsets and an associated limit set on a convergent tree . When we equip this tree with contraction data in $\mathbb{R}^n$, it is called a Moran structure. It is this structure that will determines a Generalized Moran set $E$. Chapter 3 introduces the similarity dimension $s$ to be a function on our tree $(T, \sigma)$ where $s : T \to \mathbb{R}_{\geq 0}$ for each $x \in T$. The similarity dimension alone will play the role of our local geometric data. We will consider the case where $s$ is constant throughout the tree. In Chapter 4, we employ a useful replacement Lemma to obtain lower bounds on Hdim$E$ which in turn give our main results. Finally, we discuss to what extent our results are sharp.

# Chapter 2

# Generalized Moran sets

This chapter lays the groundwork for introducing our set of interest, a set we call the generalized Moran set. The reader may find that some of these preliminaries are somewhat unorthodox, but we will see they are not arbitrarily so.

## 2.1 Nested tree of subsets

We formulate a tree in a topological space. Upon this foundation we develop the notion of the limit set of a convergent tree.

**Definition 2.1.** A *tree* is a triple $(T, \sigma, \pi)$ where $T$ is a set and $\sigma$ is a so-called successor function from $T$ to the set $T^*$ of all nonempty subsets of $T$, together with a surjective map $\pi : T \to \mathbb{N}$ such that the following diagram commutes

$$\begin{array}{ccc} T & \xrightarrow[\sigma]{} & T^* \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi^*}\downarrow \\ \mathbb{N} & \xrightarrow[\sigma^*]{} & \mathbb{N}^* \end{array}$$

where $\sigma^*$ is the composition of the successor function on $\mathbb{N}$ followed by inclusion into the set $\mathbb{N}^*$ of all nonempty subsets of $\mathbb{N}$ and $\pi^* : T^* \to \mathbb{N}^*$ is the map induced by $\pi$ on nonempty subsets of $T$.

**Remark 2.2.** We will refer to a tree simply as $(T, \sigma)$ when convenient.

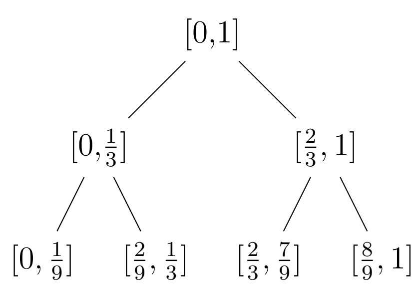
**Remark 2.3.** To see connection between definition 2.1 and the standard definition of a set-theoretic tree, we note that $(T, \prec)$ is obtained by taking the transitive closure of $\sigma$ considered as a relation on $T$ defined by $xRy$ iff $y \in \sigma(x)$ such that for all $t \in T$ $\{s \in T : s \prec t\}$ is well-ordered by $\prec$ with the property that every branch (maximal chain) has length $\omega$. If $\sigma(x)$ is finite for all $x \in T$, then $(T, \prec)$ is a $\omega$-tree.

**Definition 2.4.** Let $\mathcal{A}$ be the collection of subsets of some set $X$ that possesses the structure of tree via $\sigma : \mathcal{A} \to \mathcal{A}^*$ and $\pi : \mathcal{A} \to \mathbb{N}$. We say that $(\mathcal{A}, \sigma, \pi)$ is a *nested tree of subsets* in $X$ when for $(\mathcal{A}, \prec)$ as described in previous remark, $A \prec B$ is defined to be the containment of sets, that is $A \supset B$.

**Remark 2.5.** If $x, y \in T$ such that $y \in \sigma(x)$, we say $y$ is a *child* of $x$, or equivalently, $x$ is a *parent* of $y$.

**Definition 2.6.** A tree $(T, \sigma, \pi)$ is said to be *rooted* if there is a unique element in $T$ that maps under $\pi$ to the smallest element in $(\mathbb{N}, \prec)$.

**Example 2.7.** The Cantor ternary construction is a tree of nested subsets. The geometric construction on the Cantor ternary set is as follows. Let level 0 be $E_0 = [0, 1]$. This is our root. To obtain the intervals in level 1, $E_1$, we remove the open middle-third interval from $E_0$ giving us $E_1 = [0, 1/3] \cup [2/3, 1]$. Repeating this procedure to each of the two remaining intervals we get $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 5/9] \cup [8/9, 1]$. The first 3 levels are shown in the figure below.




In general, the $k^{th}$ level is obtained be removing the open middle-third from the intervals in the $(k-1)^{th}$ level so that each parent has two children each one-third the length of the parent.

To see that the intervals in the construction of this Cantor set are a tree $(\mathcal{I}, \sigma, \pi)$ in $\mathbb{R}$ we note that the map $\pi$ assigns to each node of the tree its level. In the specific case of the binary tree of intervals used in the construction of the middle-thirds Cantor set, $\pi([0,1]) = 0$ and

$$\pi([0,1/3]) = \pi([2/3,1]) = 1,$$

and so on: if $\pi(I) = n$ , and $I'$ and $I''$ are the two children of $I$, then $\pi(I') = \pi(I'') = n+1$. The map $\sigma^*$ takes $n \in \mathbb{N}$ to $\{n+1\}$, the singleton of the successor of $n$. To see our diagram commutes we note that when $I$ is an interval on level $n$ of the tree, so that $\pi(I) = n$, then the set of children of $I', I''$ is a subset of level $n+1$ of the tree, that is every child of $I$ will satisfy $\pi(I') = n+1$ where $I' \in \sigma(I)$. Further, to see that this tree is a nested $\omega$-tree in $\mathbb{R}$ we note that every branch $\beta \in \mathcal{I}$ is well-ordered by $\supset$ and has length $\omega$.

**Definition 2.8.** Let $T$ be a tree. Given a set $X$, we define a $T$-*tuple* of elements of $X$ to be a function

$$\mathbf{x}\colon T \to X$$

Alternatively, we sometimes refer to a $T$-tuple as a *tree of elements in* $X$. If $\mathbf{x}$ is a $T$-tuple, we often denote the value of $\mathbf{x}$ at $t$ by $x_t$. Also, the function $\mathbf{x}$ is denoted

$$(x_t)_{t \in T}.$$

**Definition 2.9.** Let $X$ be a topological space. We say a tree $(x_t)_{t\in T}$ of elements in $X$ is*convergent* if its restriction to any branch is convergent. The *limit set* of a convergent tree in $X$ is defined to be the set of all possible limits of the tree and is denoted by

$$\lim_{t\in T} x_t$$

**Definition 2.10.** Let X be a topological space. We say a sequence $A_t$ of nonempty subsets of X *shrinks to a point* if there is some $x \in X$ such that for any neighborhood $U$ of $x$ there is an $N$ such that $A_t \subset U$ for all $n > N$. We say a tree $(A_t)_{t\in T}$ of nonempty subsets of $X$ is *convergent* if the subsets obtained by restricting to a branch shrinks to a point. For such convergent trees, any choice of $x_t$ from each $A_t$ gives rise to a tree in $X$ whose limit set is $\lim_{t\in T} x_t$. We refer of this common limit set as the *limit set* of $(A_t)_{t\in T}$ and denote it by

$$\lim_{t\in T} A_t$$

**Example 2.11.** The Cantor ternary set is a limit set. The Cantor ternary set $E$ is obtained by repeating the process in the previous example ad infinitum. $E$ represents the points that are not deleted in any step of the construction. We will have a nested tree of subsets in $\mathbb{R}$, denoted $(\mathcal{I}, \sigma, \pi)$, where $\mathcal{I}$ is the collection of subintervals of [0,1] that are involved in the Cantor ternary construction. In this case our $T$-tuple is

$$\mathbf{x}\colon \mathcal{I} \to \mathbb{R}$$

Our function sends $I \in \mathcal{I}$ to $x_I \in \mathbb{R}$. In this example, we will have the identity map, namely $x_I = I$. Now, the Cantor ternary construction has the property that each branch shrinks to a point in $[0, 1]$, therefore our tree is convergent. $E$ can be viewed as the collection of all such points. That is, these points are identical to the limit set $\lim_{I \in \mathcal{I}}(x_I) = E$.

## 2.2 Moran structure

In this section we focus our attention to trees $\mathbb{R}^n$ while equipping our tree with some additional information. This additional information will be determined by $\sigma$ and will describe the geometric contractions of elements in the tree.

**Definition 2.12.** Let $(T, \sigma)$ be a tree. We may think of $(T, \sigma)$ as a directed graph where $T$ is the set of vertices and $\sigma$ is the set of edges. An *unaggregated contraction profile* refers to a function $c : \sigma \to (0, 1)$ that assigns to each edge in $T$ a positive number less than one. The *aggregated contraction profile determined by $c$* refers to the function $\check{c}$ that assigns to each $x \in T$ the multiset $\check{c} = \{c(x, y) : y \in \sigma(x)\}$. We shall usually omit the overscript unless it is unclear from the context which notion of contraction profile is intended.

**Definition 2.13.** By a *Moran structure on* $\mathbb{R}^n$ *with contraction profile* $c$ $(T, \sigma, c)$ we mean an indexed family of nonempty, bounded subsets of $\mathbb{R}^n$

$$\{B_x\}_{x \in T}$$

such that for each $y \in \sigma(x)$ the sets $B_x$ and $B_y$ are related by a similarity of $\mathbb{R}^n$ that contracts lengths by the ratio $c(x, y)$. Formally, a Moran structure is a triple consisting of a tree $(T, \sigma)$, a contraction profile, and an indexed family $\{B_x\}_{x \in T}$.

**Definition 2.14.** A Moran structure is *finite branching* if $\#\sigma(x) < \infty$ for all $x \in T$.

**Definition 2.15.** A Moran structure is *nested* if $y \in \sigma(x)$ implies $B_y \subset B_x$.

**Definition 2.16.** A Moran structure is *nonoverlapping* if $y, z \in \sigma(x)$ implies $\mathrm{Int}(B_y) \cap \mathrm{Int}(B_z) = \emptyset$.

**Definition 2.17.** A Moran structure is *Cauchy* if for any branch $\{x_n\}_{n=1}^{\infty}$ of $T$

$$\lim_{N \to \infty} \operatorname{diam} \bigcup_{n \geq N} B_{x_n} = 0$$

**Remark 2.18.** The Cauchy condition implies there is a bijection between the limit set $E$ of the family $\{B_x\}_{x \in T}$ and the set of all branches in $T$. When the set $E$ arises in this way, we say that $E$ admits a Moran structure with contraction profile $c$. The reader is asked to recall what it means for a sequence of real numbers in $\mathbb{R}$ to satisfy the Cauchy criterion; our notion is an analog to this with the notable exception that our sequence consists of bounded sets in a metric space $X$.

**Definition 2.19.** A Moran structure is *homogeneous (respectively level homogeneous)* if its aggregated contraction profile is constant (respectively constant at each level).

**Definition 2.20.** A Moran structure is *nonhomogeneous* if its contraction profile is nonconstant. That is the contraction profile is allowed to vary for each $x \in T$.

**Definition 2.21.** A *generalized Moran set* is a limit set $E$ that arises from a Moran structure.

# Chapter 3

# Similarity dimension

**Definition 3.1.** Let $(T, \sigma)$ be a tree. Given a function $c : \sigma \to (0, 1)$ that assigns a positive real number in (0,1) to every pair $(x, y)$ such that $y \in \sigma(x)$ where $\sigma \subset T \times T$ and $\sum_{y \in \sigma(x)} c(x, y) \leq 1$ , we define the *similarity dimension* $s$ as the function $s : T \to \mathbb{R}_{\geq 0}$ where

$$s(x) = \inf \left\{ t \geq 0 : \sum_{y \in \sigma(x)} c(x, y)^t \leq 1 \right\} \tag{3.1}$$

**Remark 3.2.** In the case that $\#\sigma(x) < \infty$ our definition has the following consequences:

(1) $\#\sigma(x) \geq 2$ if and only if $s(x) > 0$

(2) $\#\sigma(x) < \infty$ implies $s(x) < \infty$

(3) $\#\sigma(x) < \infty$ implies the similarity dimension is the unique solution to the equation $c(x,y)^t = 1$. See Lemma below.

**Remark 3.3.** To see why can not define similarity dimension as in condition (3) above in general, we note that when $\#\sigma(x) = \infty$ there may be no such $t \geq 0$ where $c(x,y)^t = 1$. However, we can be sure that there exists a $0 < t \leq 1$ such that $\sum_{y\in\sigma(x)} c(x,y)^t \leq 1$.

**Lemma 3.4.** Let $\{r_i\}_{i=1}^n$ be a nonempty finite sequence of positive real numbers satisfying the following

(1) $0 < r_i < 1$

(2) $\sum_{i=1}^n r_i < 1$.

(3) $2 \leq \#\sigma(x) < \infty$

Then there exists a unique solution to the equation $r_1^s + r_2^s + ... + r_n^s = 1$

*Proof.* Let $f(s) = \sum_{i=1}^n r_i^s$. We will employ the intermediate value theorem to show that there exists an $s$ such that $f(s) = 1$. First we note that $f$ is continuous over $[0,\infty)$ since it represents the sum of continuous functions. We have $f(0) = n$ where $n \geq 1$ by assumption, and we have $f(1) < 1$. Hence we have that $f(s) = 1$ for some $s \in (0,1)$. Finally, to see that the solution is unique we show that $f$ is strictly decreasing over $[0,\infty)$.

$$
\begin{aligned}
f(s) &= \sum_{i=1}^{n} r_i^s \\
f'(s) &= \sum_{i=1}^{n} (\log r_i) r_i^s < 0
\end{aligned}
$$

□

# Chapter 4

# Hausdorff dimension of generalized Moran sets

This chapter contains the two main results of this paper. These results give a lower bound on the Hausdorff dimension of a generalized Moran set $E \in \mathbb{R}^2$ arising from a nonoverlapping Moran structure with a bounded contraction assumption (section 4.1), and under the assumption of uniform contraction assumption plus a gap condition (section 4.2).

For convenience of the reader we provide the definitions of Hausdorff measure and Hausdorff dimension [4].

**Definition 4.1.** For any non-empty subset $U \subset \mathbb{R}^n$, we call the greatest distance between any pair of points the *diameter* of $U$. More precisely

$$|U| = \sup\{|x - y| : x, y \in U\}$$

**Definition 4.2.** Let $\{U\}_{i\in J}$ be a collection of nonempty subsets of $\mathbb{R}^n$ indexed by a countable set $J$. We say $\{U\}$ is an *$\epsilon$-cover of* $E \subset \mathbb{R}^n$, if $E \subset \cup U_i$, with $|U_i| \leq \epsilon$ for each $i \in J$. Let $s > 0$ and $\epsilon > 0$, then define

$$\mathcal{H}^s_\epsilon(E) = \inf\left\{\sum_{U_i\in\mathcal{U}} |U_i|^s : \{U_i\} \text{ is an } \epsilon\text{-cover of } E\right\}.$$

As $\epsilon$ decreases we note that the number of possible covers is reduced. Hence, $\mathcal{H}^s_\epsilon(E)$, because it represents an infimum, increases and approaches a limit as $\epsilon \to 0$. We now define the *s-dimensional Hausdorff measure* as

$$\mathcal{H}^s(E) = \lim_{\epsilon\to 0} \mathcal{H}^s_\epsilon(E).$$

The $s$-dimensional Hausdorff measure exists for any $E \subset \mathbb{R}^n$, and for $\epsilon < 1$, $\mathcal{H}^s_\epsilon(E)$ is nonincreasing as $s$ increases. This implies that the Hausdorff $s$-dimensional measure is nonincreasing as well. The Hausdorff $s$-dimensional measure is usually 0 or $\infty$, but there exists a critical value for $s$ at which $\mathcal{H}^s(E) < \infty$. This value we call the *Hausdorff dimension*

**Definition 4.3.** The *Hausdorff dimension*

$$\text{Hdim}(E) = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}$$

Equivalently

$$\mathcal{H}^s(E) = \begin{cases} \infty & \text{if } 0 \leq s < \text{Hdim}E \\ 0 & \text{if } s > \text{Hdim}E \end{cases}$$

**Remark 4.4.** It is known [3] that the Hdim$E \leq s$ when $E$ is a generalized Moran set in $\mathbb{R}^n$. We find sufficient conditions in $\mathbb{R}^2$ for the lower bound of Hdim$E$ to agree with the upper bound result.

To assist in finding a lower estimate for Hausdorff dimension we will employ the following Lemma which will allow us to replace an arbitrary cover $\mathcal{U}$ of $E$ with a covering consisting of intervals.

**Lemma 4.5.** Let $E$ be the limit set arising from the nested Moran structure $(\mathcal{I}, \sigma, c)$ where $\mathcal{I}$ consists of subintervals in [0,1]. Suppose there is a real number $s > 0$ such that

$$\sum_{I' \in \sigma(I)} |I'|^s \geq |I|^s$$

for all $I \in \mathcal{I}$.

Further suppose there is a constant $C > 0$ such that for any interval $U$ there is a collection $\mathcal{J} \subset \mathcal{I}$ such that $\sum_{J \in \mathcal{J}} |J|^s \leq C|U|^s$ and $U \cap E \subset \cup_{J \in \mathcal{J}} J$. Then Hdim $E \geq s$.

*Proof.* Let $\mathcal{U} = \{U\}_{k=1}^{\infty}$ be an arbitrary $\epsilon$-cover of $E$. Since we can replace each $U \in \mathcal{U}$ with an interval of the same diameter that contains $U$, we may assume the $\mathcal{U}$ is a covering by intervals. For each interval $U_k \in \mathcal{U}$, we define a collection of intervals

$$\mathcal{J}_k = \{J \in \mathcal{I} : U_k \cap E \subset \bigcup_{J \in \mathcal{J}_k} J \text{ and } \sum_{J \in \mathcal{J}_k} |J|^s \leq C|U_k|^s\}.$$

Further, we define $\mathcal{J} = \cup_k \mathcal{J}_k$. So we have

$$\begin{aligned}
C|U_k|^s &\geq \sum_{J \in \mathcal{J}_k} |J|^s \\
C\sum_k |U_k|^s &\geq \sum_k \sum_{J \in \mathcal{J}_k} |J|^s \\
\sum_k |U_k|^s &\geq \frac{1}{C} \sum_k \sum_{J \in \mathcal{J}_k} |J|^s \\
&\geq \frac{1}{C} \sum_{J \in \mathcal{J}} |J|^s \\
&\geq \frac{1}{C} |I_0|^s.
\end{aligned}$$

So $\mathcal{H}^s_\epsilon(E) > 0$ and since $\epsilon > 0$ was arbitrary, it follows that $\mathcal{H}^s(E) > 0$. Hence, Hdim $E \geq s$. □

## 4.1 Bounded contraction

**Definition 4.6.** Let $(\mathcal{I}, \sigma, c)$ be a nested Moran structure where $\mathcal{I}$ consists of subintervals in [0,1]. We say that the Moran structure is said to have the *bounded contraction condition* if there exists a $\rho > 0$ such that

$$\rho(\mathcal{I}) = \inf_{I \in \mathcal{I}} \rho(I)$$

all $I \in \mathcal{I}$ where

$$\rho(I) = \inf_{I' \in \sigma(I)} c(I, I')$$

all $I' \in \sigma(I)$.

Note: The bounded contraction condition is equivalent to the notion of a bounded aggregated contraction profile.

**Remark 4.7.** A Moran structure with the bounded contraction condition implies the structure has finite branching.

**Theorem 4.8.** Let $s > 0$ be a real number and let $E$ be the limit set arising from a nested, nonoverlapping, Cauchy Moran structure $(\mathcal{I}, \sigma, c)$ having the bounded contraction condition where $\mathcal{I}$ consists of subintervals of [0,1]. Suppose further that for all $I \in \mathcal{I}$.

$$\sum_{I' \in \sigma(I)} |I'|^s \geq |I|^s.$$

Then Hdim $E \geq s$

*Proof.* Let $U$ be an interval. And let $I$ be minimal in the sense that $I = \bigcap_{I_k \in \mathcal{I}} I_k$ such that $I_k \supset U \cap E$. We note that this intersection is nonempty, since it contains $U \cap E$. $I$ must be an interval in $\mathcal{I}$ since $(\mathcal{I}, \sigma)$ is a nested tree of intervals.

**Case 1**: $U$ contains at least one $I' \in \sigma(I)$. Let $\mathcal{J} = I$

$$|I| \leq \frac{1}{\rho}|I'|$$
$$|I|^s \leq \frac{1}{\rho^s}|I'|^s$$
$$|I|^s \leq \frac{1}{\rho^s}|U|^s$$

Hence, for case 1, we have met the conditions for Lemma 4.5. Namely, $I \supset U \cap E$ by assumption, and the inequality $\sum_{J \in \mathcal{J}} |J|^s \leq C|U|^s$ is satisfied by $\mathcal{J} = I$ and $C = 1/\rho^2$.

**Case 2**: $U$ does not contain any child of $I$. Let $\mathcal{A} = \{I' \in \sigma(I) : I' \cap (U \cap E) \neq \emptyset\}$.

*Claim 1*: $\#\mathcal{A} = 2$.

First we see that $\mathcal{A} \geq 2$ by definition of $I$. Namely, $\mathcal{A} \neq 0$ because at least one child of $I$ must also contain $U \cap E$, and $\mathcal{A} \neq 1$ because that would imply that $I$ was not minimal. Furthermore $\#\mathcal{A} \leq 2$ because if $\#\sigma(I)$ were more than 2 then at least one child would have to be contained in $U$ and we would revert to case 1. Thus we justify the claim.

Now let $I_1', I_2' \in \sigma(I)$ where $\sup I_1' \leq \inf I_2'$. We set $\mathcal{J} = \{J_1, J_2\}$ where $J_1$ is minimal such that $J_1 \supset I_1' \cap (U \cap E)$. Define $J_2$ similarly. Define $J_1'$ as the rightmost child of $J_1$, and $J_2'$ is the leftmost child of $J_2$. Using a similar argument used to justify claim 1, we see that $\#\{J_1' \in \sigma(J) : J_1' \cap (U \cap E)\} \geq 2$. Define a leftmost child $J_1'' \in \sigma(I_1)$.

*Claim 2*: There exists $J_1' \in \sigma(J_1)$ such that $J_1' \subset U$ and a $J_2' \in \sigma(J_2)$ such that $J_2' \subset U$.

Since $U \cap J_2 \neq \emptyset$, we know that $U$ contains a right endpoint of $J_1$. Now we note that $J_1'' \cap (U \cap E) \neq \emptyset$. Pick $x_L \in J"_1 \cap U \cap E$. Let $x_R$ be the right endpoint of $J_1$.

Then, $J_1' \subset [x_L, x_R] \subset U$. A similar argument to can be applied to see that $J_2' \subset U$. Hence claim 2 is justified. So we have

$$|J_1| \leq \frac{1}{\rho}|J_1'|$$
$$|J_1|^s \leq \frac{1}{\rho^s}|J_1'|^s$$
$$|J_1|^s \leq \frac{1}{\rho^s}|U|^s$$

We can define $J_2$ similarly and obtain $|J_2|^s \leq \dfrac{1}{\rho^s}|U|^s$. So we have

$$|J_1|^s + |J_2|^s \leq \tfrac{2}{\rho^s}|U|^s.$$

hence we have $\sum_{J \in \mathcal{J}} |J|^s \leq C|U|^s$ with $C = \frac{2}{\rho^s}$.

Finally, we need to confirm $J_1 \cup J_2 \supset U \cap E$. Recalling the definition of $J_1$ and $J_2$ we get

$$J_1 \cup J_2 \supset (I_1' \cap (U \cap E) \cup (I_2' \cap (U \cap E)) = (U \cap E).$$

We justify the last equality by noting that all of $U \cap E$ must be contained in the children of $I$ since $I \subset U \cap E$. Having satisfied the conditions for Lemma 4.5 for case 2, and hence both cases, we can conclude that $\text{Hdim}(E) \geq s$.

□

Synthesizing the known upper bound result with the results from Theorem 4.8 we can conclude that Hdim$E = s$. We observe that the placement of the intervals in the tree has no effect on the Hausdorff Dimension of the resulting limit set $E$ under the assumptions of Theorem 4.8.

## 4.2 Uniform contraction plus gap condition in dimension s

**Definition 4.9.** Let $(\mathcal{I}, \sigma, c)$ be a nested Moran structure. We say $(\mathcal{I}, \sigma, c)$ has the *uniform contraction condition* if there exists $\kappa > 1$ such that for any parent $I \in \mathcal{I}$ and for any children $I', I'' \in \sigma(I)$, we have

$$\kappa(\mathcal{I}) = \sup_{I \in \mathcal{I}} \kappa(I)$$

where

$$\kappa(I) = \sup_{I', I'' \in \sigma(I)} \frac{c(I, I')}{c(I, I'')}.$$

**Definition 4.10.** Let $(\mathcal{I}, \sigma)$ be a tree of subintervals of [0,1]. A *gap* is an excluded interval between consecutive $I' \in \sigma(I)$.

**Definition 4.11.** Let $(\mathcal{I}, \sigma)$ be a nested tree where $\mathcal{I}$ consists of subintervals of [0,1] and let $s > 0$ be a real number. We say $(\mathcal{I}, \sigma)$ satisfies a *gap condition in dimension* $s$ if there is a constant $c > 0$ such that for any parent $I \in \mathcal{I}$, for any gap $\Gamma$ between consecutive children $I' \in \sigma(I)$

$$|\Gamma| \geq c|I'|^s|I|^{1-s}$$

Equivalently, we can take $\Gamma$ to be the smallest gap and $I'$ the longest child.

**Theorem 4.12.** Let $s > 0$ be a real number and $E$ the limit set of a nested Moran structure $(\mathcal{I}, \sigma, c)$ where $\mathcal{I}$ consists of subintervals of [0,1] having the uniform contraction condition and satisfying the gap condition in dimension s. Suppose further that for all $I \in \mathcal{I}$

$$\sum_{I' \in \sigma(I)} |I'|^s \geq |I|^s.$$

Then Hdim$E \geq s$.

*Proof.* Given an interval $U$, let $I \in \mathcal{I}$ be minimal in the sense that $I = \cap_{I_k \in \mathcal{I}} I_k$ such that $I_k \supset (U \cap E)$. Let $\mathcal{J} = \{I' \in \sigma(I) : I' \cap (U \cap E) \neq \emptyset\}$ and let $\mathcal{G}$ be the collection of gaps between consecutive children in $\mathcal{J}$. We define $\Gamma_{min}$ to be the minimal gap, that is $|\Gamma_{min}| \leq |\Gamma|$ for all $\Gamma \in \mathcal{G}$. Define $I'_{max}$ to be the maximal interval in $\mathcal{J}$ in the sense that $|I'_{max}| \geq |I'|$ for all $I' \in \mathcal{J}$.

$$
\begin{aligned}
|U \cap I| &\geq \#\mathcal{G}|\Gamma_{min}| \\
&\geq c\left(\frac{\#\mathcal{J}}{2}\right)|I'_{max}|^s|I|^{1-s} \\
&\geq \frac{c}{2}\sum_{I'\in\mathcal{J}}|I'|^s|I|^{1-s}
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{c}{2}\sum_{I'\in\mathcal{J}}|I'|^s &\leq \frac{|U \cap I|}{|I|^{1-s}} \\
&\leq \frac{|U \cap I|}{|U \cap I|^{1-s}} \\
&= |U \cap I|^s \\
&\leq |U|^s
\end{aligned}
$$

We meet the conditions of Lemma 4.5, with $\cup_{I'\in\mathcal{J}} I' \supset U \cap E$ and $C = \frac{2}{c}$, hence we have Hdim$E \geq s$ as desired. □

Synthesizing the known upper bound result with the results from 4.12 we can conclude that $\text{Hdim}E = s$.

## 4.3 Sharpness

To highlight the necessity for our assumptions in Theorems 4.8 and 4.12, we introduce a special type of Moran structure so that we can investigate the potential effects that interval spacing and contraction profiles can have on the dimension of generalized Moran sets.

**Definition 4.13.** Consider a finite branching nonoverlapping Moran structure $\mathcal{M}$ with a level homogeneous aggregated contraction profile that gives rise to a limit set in $\mathbb{R}^2$. Namely, each level $k$ in the construction of the limit set arising from $\mathcal{M}$ determines both the contraction ratio $c_k$ where $c_k = c(I_{k-1}, I')$, $0 < c_k < \frac{1}{2}$ for all $I' \in \sigma(I_{k-1})$. The branching factor $n_k = \#\{I'_k \in \sigma(I_{k-1})\} \geq 2$ for all $I' \in \sigma(I_{k-1})$. If the gaps between any consecutive children, $I'$ and $I''$ are equal, and the left endpoint of the leftmost child of $I$ is the same as the left endpoint of $I$ and the right endpoint of the rightmost child of $I$ is the same as the rightmost endpoint of $I$ then the resulting limit set $C$ is called a *homogeneous Cantor set.* If the left endpoint leftmost child of $I$ is the same as the left endpoint of $I$ and the gap between consecutive children is equal to 0 then $C^*$ is called a *partial homogeneous Cantor set.*

**Lemma 4.14.** [5] Suppose $C(I_0, \{n_k\}, \{c_k\})$ and $C^*(I_0, \{n_k\}, \{c_k\})$ are a homogeneous Cantor set and a partial homogeneous Cantor set respectively. Then

$$s_1\text{=Hdim}C = \liminf_{k\to\infty} \frac{\log n_1 n_2 ... n_k}{-\log c_1 c_2 ... c_k}$$

and

$$s_2\text{=Hdim}C^* = \liminf_{k\to\infty} \frac{\log n_1 ... n_k}{-\log c_1 c_2 ... c_{k+1} n_{k+1}}$$

The next example helps us investigate under what conditions will we see $s_2 < s1$.

**Example 4.15.** Let $C$ and $C^*$ be be a homogeneous Cantor set and a partial homogeneous Cantor set respectively. Let $s \in [0, 1]$ and let $c_k = n_k^{-1/s}$.

$$\begin{aligned} s_1 &= \liminf_{k\to\infty} \frac{\log n_1 ... n_k}{-\log n_1 ... n_k^{\frac{-1}{s}}} \\ &= s \end{aligned}$$

Manipulating our formula for $s_2$ we get

$$\begin{aligned}
s_2 &= \liminf_{k\to\infty} \frac{\log n_1...n_k}{-\log c_1...c_k + \log c_{k+1}n_{k+1}} \\
&= \liminf_{k\to\infty} \frac{\log n_1...n_k}{-\log(n_1...n_k)^{\frac{-1}{s}} - \log n_{k+1}^{1-\frac{1}{s}}} \\
&= \liminf_{k\to\infty} \frac{\log n_1...n_k}{-\log(n_1...n_k)^{\frac{-1}{s}} - \log n_{k+1}^{\frac{s-1}{s}}} \\
&= s \liminf_{k\to\infty} \frac{\log n_1...n_k}{-\log n_1...n_k - \log n_{k+1}^{s-1}} \\
&= s \liminf_{k\to\infty} \frac{1}{1 + (1-s)\frac{\log n_{k+1}}{\log n_1...n_k}}.
\end{aligned}$$

We can now see that question concerning the agreement of $s_1$ and $s_2$ is reduced to the following statement. If we have

$$\limsup_{k\to\infty} \frac{\log n_{k+1}}{\log n_1...n_k} < \delta$$

for all $\delta > 0$ then $s_2 = s$. Otherwise $s_2 < s$.

We look at two examples that illustrate how our choices of $n_k$ can effect our calculation. Both examples are cases where $\inf c_k = 0$. However, we will see that the rate at which these contractions are going to zero can make a difference.

**Example 4.16.** Let $C$ and $C^*$ be be a homogeneous Cantor set and a partial homogeneous Cantor set respectively. Let $s \in [0,1]$ and choose $n_k = 2^{2^k}$ and let

$c_k = n_k^{-1/s}$. By formula, we get $s_1 = s$ and

$$s_2 = \limsup_{k\to\infty} \frac{\log 2^{2^{k+1}}}{\log 2^{2^{k+1}-2}} = 1$$

This gives us

$$s_2 = \frac{s}{2-s} < s_1 \qquad \text{when} \;\; 0 < s < 1.$$

**Example 4.17.** Let $C$ and $C^*$ be be a homogeneous Cantor set and a partial homogeneous Cantor set respectively. Let $s \in [0,1]$ and choose $n_k = 2^k$ and let $c_k = n_k^{-1/s}$. By formula we get $s_1 = s$ and

$$s_2 = \limsup_{k\to\infty} \frac{\log 2^{k+1}}{\log 2^{\frac{k(k+1)}{2}}} = 0 < \delta$$

So $s = s_2$.

These examples show us that when we choose an $n_k$ such that $\inf c_k = 0$ it is not true that $s_1 = s_2$ in general. This justifies our bounded contraction condition. Further, we note that our examples satisfied the uniform contraction condition but $s_2 < s_1$ in example 4.15. This shows us that uniform contraction alone is not sufficient without adding a spacing condition.

# Chapter 5

# Conclusion and Future Research

While these results provide a practical tool for a painless calculation for Hdim$E$ for sets arising from large class of interval trees, we feel the philosophical implications deserve to be highlighted. Where it may have been thought that the question of the dimension of Cantor sets arising from nonoverlapping interval trees was somewhat exhausted, we are hopeful that our alternative formulation can serve to revitalize the discussion. Moving forward, we can view Moran structures possessing a constant similarity dimension as a basic case upon which we may begin to explore to what degree the behavior of local dimension can help us infer Hdim $E$. That is, we would like to know how well-behaved $s$ must be before our lower and upper estimates fail to agree. Some natural areas of further exploration include the case of infinite children, overlapping children, and of course $n$-dimensional trees. Additionally, we would like to see some refinement of the bounded contraction condition. The discrepancy in the results from examples 4.15 and 4.16 imply that some relaxation of this condition may be possible. More interestingly perhaps, are the generalizations that would allow $s$ to vary throughout the structure. For example, if we consider a notion of a limit on a tree in the sense of Ahmadieh [1], we would venture that if $\lim_{x \in T} s(x) = s$ then Hdim$E = s$, or $\limsup s(x) =$Hdim$(E)$ if such a limit does not exist.

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