PIECEWISE ISOMETRIES, UNIFORM DISTRIBUTION AND $3 \log 2 - \pi^2/8$

YITWAH CHEUNG, AREK GOETZ AND ANTHONY QUAS

Abstract. We use analytic tools to study a simple family of piecewise isometries of the plane parameterized by an angle parameter. In previous work we showed the existence of large numbers of periodic points, each surrounded by a 'periodic island'. We also proved conservativity of the systems as infinite measure-preserving transformations. In experiments it is observed that the periodic islands fill up a large part of the phase space and it has been asked whether the periodic islands form a set of full measure. In this paper we study the periodic islands around an important family of periodic orbits and and demonstrate that for all angle parameters that are irrational multiples of π the islands have asymptotic density in the plane of $3 \log 2 - \pi^2/8 \approx 0.846$.

1. Introduction

We consider a simple family of piecewise isometries studied previously by Goetz and Quas [11]. The maps have a single parameter θ and can be conveniently expressed in terms of a complex variable:

$$T_\theta(z) = \begin{cases} e^{2\pi i\theta}(z+1) & \text{if Im } (z) \geq 0; \\ e^{2\pi i\theta}(z-1) & \text{if Im } (z) < 0. \end{cases}$$

Systematic studies of piecewise isometries began in [10] and continued in many publications by a variety of authors. One reason for studying them is that as the pieces are simply isometries, any complexity that appears in a piecewise isometric dynamical system is there as a result of the discontinuity. Piecewise isometries therefore form a test case for the study of discontinuous dynamical systems. They also have been applied as simple models for behaviour of electronic circuits [6, 7].

The family (T_θ) that we study was introduced in [9] and shown in [11] to have interesting behaviour that can be understood using elementary techniques (see Figure 1 for an example of a phase portrait). In [11] it was shown that for all values of θ the map T_θ has an infinite collection of periodic orbits that accumulate at infinity. It was also shown that even though Lebesgue measure is an infinite invariant measure under T_θ for

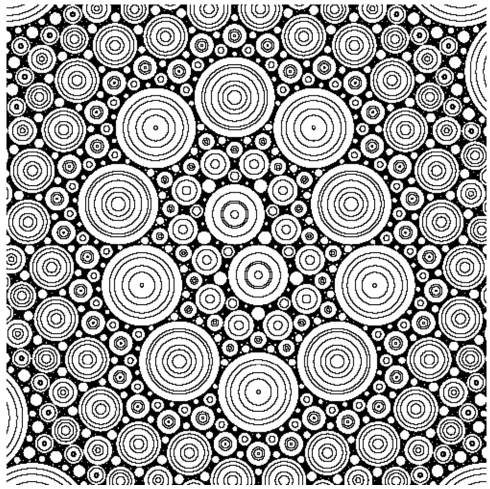

FIGURE 1. A typical phase portrait for T_θ for an irrational θ value: Many different orbits are plotted

each θ, the transformation is in fact conservative (so that a Poincaré Recurrence Theorem is satisfied).

In this paper we restrict our attention to the case where θ is irrational as this will allow us to make use of uniform distribution techniques from ergodic theory. The rational case, both for this map and others is also interesting. Since the techniques used in the rational case are different that those used in the irrational case we plan to make a comprehensive study of density in the rational case in a subsequent article. It should be mentioned that the rational case is closely related to polygonal dual billiards maps on regular polygons (see work of Vivaldi and Shaidenko [15]) as well as to some rational piecewise affine maps studied by Adler, Kitchens and Tresser in [1].

The dominant feature in the phase portrait (see Figure 1) is the large number of discs that one sees. These have a simple explanation: Suppose y is a periodic point of period q for T_θ and let $r = \min\{|\mathrm{Im}\, T_\theta^n y| \colon n \in \mathbb{N}\}$. Then an open disc of radius r about $T_\theta^j y$ is mapped rigidly by T_θ to an open disc of radius r about $T_\theta^{j+1} y$ and none of these discs intersects the discontinuity. Suppose now that z lies outside the closed disk of radius r about some point $T_\theta^j y$ on the periodic orbit. Let $R = d(z, T_\theta^j y)$. We claim that there exists an n such that $T^n z$ and $T_\theta^{j+n} y$ lie on opposite sides of the discontinuity. To see this we start by assuming that $T_\theta^j y$ is exactly at a distance r from the discontinuity. Then assuming that z and $T_\theta^j y$ lie on the same side of the discontinuity for all n, we have in particular that $T_\theta^{qn} z$ is at a distance R from $T_\theta^j y$ for each n. Since T_θ^q acts as an irrational rotation by $2\pi q\theta$ about $T_\theta^j y$ on the domain of continuity containing $T_\theta^j y$ there will eventually exist (by denseness of orbits on the circle under irrational rotations) an n such that $T_\theta^{qn} z$ and $T_\theta^j y$ lie on opposite sides of the discontinuity.

Hence we have shown that for each periodic orbit there is an r such that the open discs of radius r around each point on the orbit are rigidly permuted (with rotation) and stay for all time on the same size of the discontinuity as the periodic point whereas points outside the closed disc of size r eventually fall on the opposite side of the discontinuity. These discs will be referred to as *periodic islands*.

Given a point z in the plane we refer to its *itinerary* as the sequence of elements of the partition (into pieces on which the map acts as an isometry) that its orbit belongs to. That is we define $\mathcal{I}(z) = (s(T_\theta^n z))_{n\in\mathbb{Z}}$ where $s(z)$ is 1 if Im $(z) \geq 0$ and -1 otherwise. It is straightforward to see that a point has a periodic itinerary if and only if it belongs to a periodic island. If a point has an aperiodic itinerary we refer to it as *aperiodically coded*.

A helpful heuristic for understanding the action of T_θ is that given a point with absolute value r and argument $2\pi\phi$ its image has approximately the same radius but argument approximately $2\pi(\phi + \theta - C/r)$. To understand this notice that the map T_θ is the composition of a rotation by $2\pi\theta$ with a 'shear' sliding the two parts of the plane relative to each other. The effect of the shear part is to diminish the argument whether the point lies in the upper or lower half plane. The amount that the argument is reduced depends on the radius, as well as on the distance of the point from the discontinuity. The second dependence tends to average out and so T_θ can be thought of as approximately a twist map $(r, 2\pi\phi) \mapsto (r, 2\pi(\phi + \theta - C/r))$. If r satisfies $\theta - C/r = p/q$ that one might expect T_θ acting on a neighbourhood of a circle of radius r about the origin to behave as a rotation by $2\pi p/q$ and hence to have periodic orbits with period q. Our previous paper [11] developed this idea and showed rigorously the existence of orbits arising in this way.

A key concept in that paper is that of a *rotationally-coded* periodic orbit. Throughout the paper we will use the notation $e(\theta) = e^{2\pi i\theta}$. We say that a point z has a rotationally-coded orbit for T_θ if there exist a rational p/q (in lowest terms) and a point y whose $R_{p/q}$-orbit does not intersect the discontinuity such that $s(T_\theta^n z) = s(R_{p/q}^n y)$ for all $n \in \mathbb{Z}$, where $R_\theta(y) = e(\theta)y$ is the rotation of the plane by $2\pi\theta$ about the origin. In this case we say that orbit of z is p/q-coded and the *rotation number* of z will be p/q.

In order to avoid repeating ourselves too much we make the following natural convention: when speaking about rationals p/q we will *always* assume that they are expressed in lowest terms even if we avoid explicitly mentioning it.

Notice that for the map $R_{p/q}$, if the plane is divided up by the preimages of the real axis under $R_{p/q}$ into conical regions then two points have the same itinerary if and only if they lie in the same cone. If q is even there are q cones whereas if q is odd there are $2q$ cones (the difference arises because the negative real axis and positive real axis have disjoint orbits in the odd case). If q is even $R_{p/q}$ permutes the cones transitively whereas if q is odd the cones alternate between two families and the rotation sends each family to itself. For q even there is therefore exactly one p/q-coded itinerary up to shifts (namely $(s(R^n_{p/q}e(\frac{1}{2q})))_{n\in\mathbb{Z}}$) whereas for q odd there are exactly two p/q-coded itineraries up to shifts (namely $(s(R^n_{p/q}e(\frac{1}{4q})))_{n\in\mathbb{Z}}$ and $(s(R^n_{p/q}e(-\frac{1}{4q})))_{n\in\mathbb{Z}}$). The first of these spends $(q+1)/2$ steps per period in the upper half plane and $(q-1)/2$ steps in the lower half plane whereas for the second orbit the situation is reversed.

A key question for the family (T_θ) as well as for other piecewise isometries has been to rigorously prove quantitative theorems about the size of the set of aperiodically coded points. In our setting, letting A denote the set of aperiodically coded points (a measurable set because it is the complement of a countable family of discs), one is interested in the *density* of A in the plane:

$$\rho(A) = \lim_{R\to\infty} \frac{\mathrm{Leb}(A\cap B_R)}{\mathrm{Leb}(B_R)}.$$

If the limit in the above expression fails to exist one may instead talk about the *upper density* $\overline{\rho}(A)$ in which the limit is replaced by a limit superior and the *lower density* $\underline{\rho}(A)$ in which the limit is replaced by a limit inferior.

Ashwin has performed computer experiments suggesting that $\bar{\rho}(A)$ is positive ([3, 4]) but other authors have conjectured that $\bar{\rho}(A)$ is 0. We do not resolve this conjecture here, but instead we study quantitative properties of P_{rot}, the set of points with rotationally-coded orbits. See also recent work of Lowenstein and Vivaldi [13] for an approach to a related problem involving density estimates for a PWI on a compact region with parameter values θ close to $1/4$.

Our main theorem in this paper is the following (see Section 5 for a precise statement).

Theorem. *Let θ be irrational. Then the set P_{rot} has upper density at least $3\log 2-\pi^2/8$. Moreover there is a naturally defined subset P_0 of P_{rot} such that $\rho(P_0)=3\log 2-\pi^2/8$.*

Further we present a conjecture that we have verified numerically in a large number of cases that would ensure that $P_0=P_{\mathrm{rot}}$.

The principal ingredients of the paper are some elementary geometric and combinatorial facts about the periodic orbits established in our previous paper [11] together with a number of deductions from uniform distribution. The result is established by first showing that the density of periodic islands is given by the sum of the limits as R approaches ∞ of $G_{\mathrm{odd}}(\theta,R)+G_{\mathrm{even}}(\theta,R)$, where these are defined by

$$G_{\mathrm{odd}}(\theta,R)=\frac{1}{2R^2}\sum_{\substack{\frac{q}{\pi^2 R}<\{q\theta\}<\frac{1}{2}\\ \gcd(q,\lfloor q\theta\rfloor)=1\\ q\ \mathrm{odd}}} q\cot^2(\pi q\theta);\ \text{and}$$

$$G_{\mathrm{even}}(\theta,R)=\frac{1}{4R^2}\sum_{\substack{\{q\theta\}>\frac{q}{\pi^2 R}\\ \gcd(q,\lfloor q\theta\rfloor)=1\\ q\ \mathrm{even}}} q\tan^2\left(\frac{\pi q\theta}{2}\right).$$

These are the approximate total densities in a disc of radius R about the origin of the periodic islands with odd period and the periodic islands with even period. Our previous paper gives exact information on the radii of the periodic islands. In this paper we use uniform distribution techniques to approximately locate the periodic islands (so that we are able to decide those terms that should be included and those that should be excluded). This allows us to compare the density $\rho(P_0)$ with $\lim_{R\to\infty}(G_{\mathrm{odd}}(\theta,R)+G_{\mathrm{even}}(\theta,R))$ in Section 5.

In computing the limits of the above quantities we use uniform distribution two further times: firstly (in Section 4) we find the density of $S=\{q\colon q \text{ and } \lfloor q\theta\rfloor \text{ are coprime}\}$. It turns out S has density $4/\pi^2$ in the even integers and $8/\pi^2$ in the odd integers. This is a mild generalization of a theorem of Estermann [8], which in turn is closely related to a well-known theorem of Mertens [14] on the density of visible points in the square lattice. Secondly (in Section 3) we develop some theorems about summation in the presence of uniform distribution that essentially allow us to replace all the terms in the summation by their

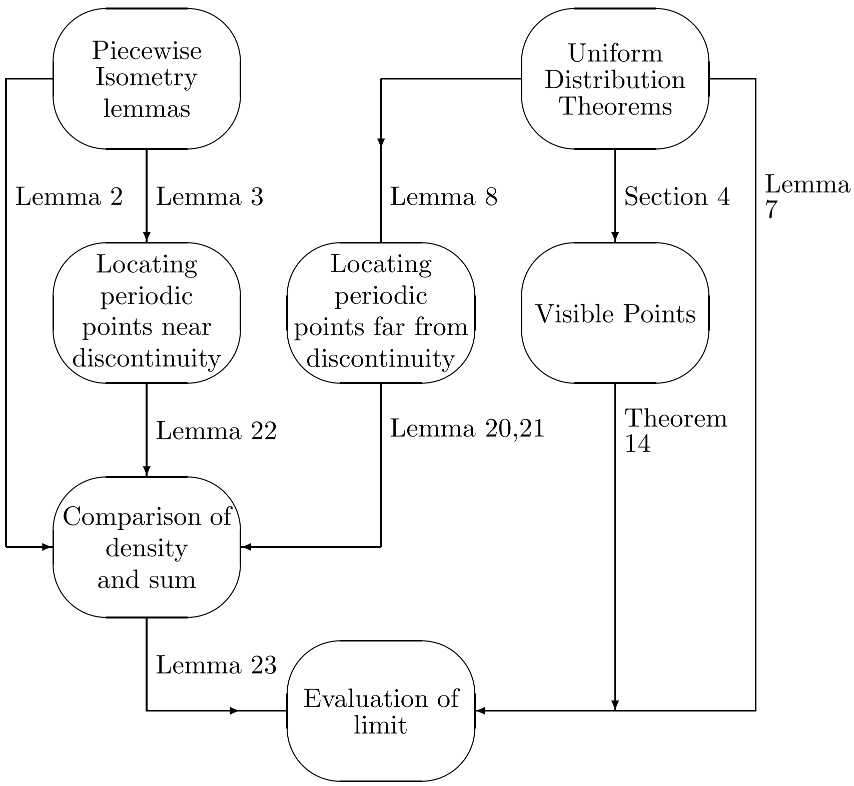

FIGURE 2. Schematic diagram of the proof

averages provided that the functions being summed are Riemann integrable.

Since in our case the functions being summed have singularities we need to deal with the contributions from the singularities separately (Section 5).

A schematic diagram of the proof is shown in Figure 2.

2. PIECEWISE ISOMETRY LEMMAS

We collect in this section a number of lemmas from our previous paper [11] concerning periodic orbits of maps of the form T_θ. We shall use the notation $e(x)$ to denote $e^{2\pi i x}$. If $\epsilon \in \{\pm 1\}^{\mathbb{Z}}$ we write $\sigma(\epsilon)$ for the shift of ϵ: $(\sigma(\epsilon))_n = \epsilon_{n+1}$.

We will write $S_\theta^{\pm 1}(z) = e(\theta)(z \pm 1)$ and let $P_{+1} = \{z\colon \operatorname{Im} z \geq 0\}$ and $P_{-1} = \{z\colon \operatorname{Im} z < 0\}$. If ϵ is a periodic ± 1-valued sequence with period q we set

$$F(\theta,\epsilon) = \frac{1}{e(-q\theta)-1}\sum_{j=0}^{q-1}\epsilon_j e(-j\theta).$$

One can check that $S_\theta^{\epsilon_0}(F(\theta,\epsilon)) = F(\theta,\sigma(\epsilon))$.

Lemma 1 (Criterion for existence of periodic points: Lemma 9 [11]). *Let ϵ be a periodic word with period q in $\{\pm 1\}^{\mathbb{Z}}$. Then T_θ has a periodic point with itinerary ϵ if and only if $F(\theta,\sigma^j\epsilon) \in P_{\epsilon_j}$ for each $0 \le j < q$.*

If this condition is satisfied then the periodic point is given by $F(\theta,\epsilon)$.

We refer to $F(\theta,\epsilon)$ as a *potential periodic orbit with itinerary θ*. If the conditions of Lemma 1 are satisfied then it is a true periodic orbit with itinerary θ.

As described above for $p/q \in \mathbb{Q}$, there are two periodic p/q-coded itineraries if q is odd and one itinerary if q is even. For q odd and let $Z_q = \{\frac{1}{4},\frac{3}{4},\dots,q-\frac{1}{4}\}$ whereas if q is even let $Z_q = \{\frac{1}{2},\frac{3}{2},\dots,q-\frac{1}{2}\}$. These elements represent the central angle of each of the $2q$ cones discussed earlier for the map $R_{p/q}$ in the case where q is odd and the central angle of each of the q cones for $R_{p/q}$ in the case where q is even. In either case we let $\left(\epsilon_{p/q}^{(a)}\right)_n = s(R_{p/q}^n(e(a/q)))$ and let $z_{p/q}^{(a)} = F(\theta,\epsilon_{p/q}^{(a)})$, so that provided the conditions of the lemma above are satisfied $z_{p/q}^{(a)}$ behaves under T_θ like the a point with angle $2\pi a/q$ under the map $R_{p/q}$.

If $\gcd(p,q)=1$ and $p/q < \theta < (p+\frac{1}{4})/q$ then we showed in [11] that the conditions of Lemma 1 are satisfied and hence there is a p/q-coded orbit (or a pair of such orbits if q is odd). In the case where there is a pair of itineraries, the orbit corresponding to one is sent to the orbit corresponding to the other by rotation by π so that they are either both present or both absent.

When the conditions of the lemma are satisfied the points of the orbit satisfy $T_\theta(z_{p/q}^{(a)}) = z_{p/q}^{(a+p \bmod q)}$ for any a in the set of allowable a's listed above. In the case where q is odd one orbit consists of the points $\{z_{p/q}^{(n+1/4)} : 0 \le n < q\}$ and the other orbit consists of $\{z_{p/q}^{(n-1/4)} : 0 \le n < q\}$. We call the first of these the $+$-orbit and the second the $-$-orbit.

We showed in [11] that if $p/q < \theta < (p+1/4)/q$ then the p/q-coded periodic orbit(s) 'look like' orbits of the rotation by $2\pi p/q$ in the sense that the arguments of $z^{(a)}$ increase monotonically as a goes from 0 to q along the appropriate sequence and the periodic orbit(s) are mapped in an order-preserving way to itself/themselves by T_θ. The periodic orbits have another property in this case that will be of importance to us.

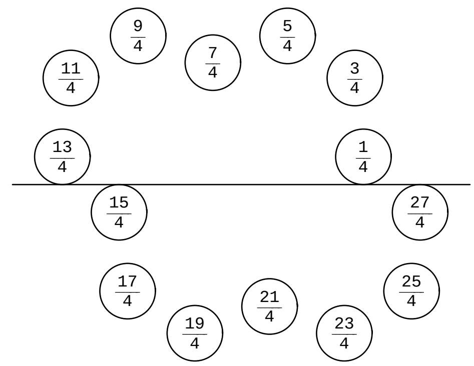

FIGURE 3. The 2/7-coded periodic islands for T_θ where $\theta = 0.3219$ labeled with their indices in Z_7.

Let θ and p/q be given. If q is odd and there are a pair of p/q-coded orbits, the orbits are said to be *well-behaved* if the closest points on the $+$-orbit to the discontinuity are $z^{(1/4)}$ and $z^{(q/2-1/4)}$. These two points are shown in [11] to be the same distance from the discontinuity. In this case of course the closest points on the $-$-orbit to the discontinuity are $z^{(-1/4)}$ and $z^{(q/2+1/4)}$. In the case where q is even the four points $z^{(\pm 1/2)}$ and $z^{(q/2\pm 1/2)}$ are all at the same distance from the discontinuity. If these points are the closest points on the orbit to the discontinuity the orbit is again said to be well-behaved. It is shown in [11] that if $p/q < \theta < (p+1/4)/q$ then the p/q-coded orbit is well-behaved. This is illustrated in Figure 3.

The significance of this condition is that it is possible using an analysis of the codes $\epsilon^{(a)}$ one can obtain exact expressions for the distance of these four points to the discontinuity. By the earlier observation this determines the radius of the periodic islands surrounding the points on the periodic orbit(s) and hence their area.

Lemma 2 (Size of periodic islands: Lemma 9 [11])**.** *Let* $\gcd(p,q) = 1$ *and* θ *be irrational.*

If q *is odd then* $z_{p/q}^{(1/4)}$ *and* $z_{p/q}^{(q/2-1/4)}$ *have imaginary component* $\cot(\pi q\theta)/2$*; while* $z_{p/q}^{(-1/4)}$ *and* $z_{p/q}^{(q/2+1/4)}$ *have imaginary component* $-\cot(\pi q\theta)/2$.

If q *is even then* $z_{p/q}^{(q/2-1/2)}$ *and* $z_{p/q}^{(1/2)}$ *have imaginary component* $-\tan(\pi q\theta/2)/2$*; while* $z_{p/q}^{(-1/2)}$ *and* $z_{p/q}^{(q/2+1/2)}$ *have imaginary component* $\tan(\pi q\theta/2)/2$.

In particular if q is odd and $\{q\theta\} \in (1/2, 1)$ then $z_{p/q}^{(1/4)}$ lies in the wrong half-plane and so there is no p/q-coded orbit. Notice that if q is even then by assumption p is odd so that if $p/q < \theta < (p+1)/q$

then $\pi q\theta/2 \in (\pi p/2, \pi(p+1)/2)$ which ensures that $-\tan(\pi q\theta/2)$ is positive.

Lemma 3 (Relative Positions of Periodic Points: Lemma 7 [11])**.** *Let $\theta = (p+h)/q$ with $0 < h < 1$ and let $(z^{(a)})_{a\in Z_q}$ be the points constructed above.*

If q is even, there is a collection of points $y^{(a)}$ (for $a \in Z_q$) lying on a circle centred at the origin with $Arg(y^{(a)}) = 2\pi\frac{a}{q}$ so that for each $a \in Z_q$, there exists an r satisfying $|r| \le q/4$ such that

$$z^{(a)} - z^{(a-1)} = e(r\tfrac{h}{q})(y^{(a)} - y^{(a-1)}). \tag{1}$$

If q is odd, there is a collection of points $y^{(n)}$ (for $a \in Z_q$) lying on a circle centred at the origin with $Arg(y^{(a)}) = 2\pi\frac{a}{q}$ so that for each $a \in Z_q$ there exists an r satisfying $|r| \le q/2$ such that

$$z^{(a)} - z^{(a-1/2)} = e(r\tfrac{h}{q})(y^{(a)} - y^{(a-1/2)}). \tag{2}$$

3. Results on Uniform Distribution

In this section we develop a number of basic results about uniform distribution modulo 1. While the results that we prove are tailored to the particular applications that we study in this paper there is a general meta-principle at work here: in the presence of uniform distribution modulo 1 one can replace each term in a sum by its average.

A sequence of numbers (x_n) is said to be *uniformly distributed modulo 1* if for each interval $[a,b] \subset [0,1]$, $|\{n \le N\colon \{x_n\} \in [a,b]\}|/N \to b-a$ as $N \to \infty$.

We shall need to make use of Weyl's Criterion [16].

Theorem 4. *(Weyl's Criterion) Let (x_n) be a sequence of real numbers. Then the following are equivalent:*

(1) *(x_n) is uniformly distributed mod 1;*
(2) *For each Riemann-integrable function f,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f;$$

(3) *For each non-zero integer k,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} = 0.$$

Lemma 5 (Uniform distribution and weighted means)**.** *Suppose that (x_n) is a sequence of numbers in $[0,1)$ and that S is a subset of $\mathbb{N}$ with* *density α. Suppose further that $\{(x_n)_{n\in S}\}$ is uniformly distributed mod 1. Let f be a Riemann integrable function. Then*

$$\frac{1}{R^2}\sum_{n\le R,\, n\in S} nf(x_n) \to \frac{\alpha}{2}\int_0^1 f \text{ as } R\to\infty.$$

Proof. Let $\epsilon > 0$. We have for all T exceeding some T_0,

$$\left|\sum_{n\le T,\, n\in S} f(x_n) - \alpha T\int_0^1 f\right| < \epsilon T.$$

Summing this for T up to R and using the triangle inequality, we obtain

$$\left|\sum_{n\le R,\, n\in S}(R-(n-1))f(x_n) - \alpha\frac{R(R+1)}{2}\int_0^1 f\right| < T_0^2\|f\| + \epsilon\frac{R(R+1)}{2}.$$

Dividing by R^2 and taking a limit superior as $R\to\infty$ we see

$$\limsup\left|\frac{1}{R^2}\sum_{n\le R,\, n\in S}(R+1-n)f(x_n) - \frac{\alpha}{2}\int_0^1 f\right| \le \epsilon.$$

Since ϵ is arbitrary we obtain

$$\lim_{R\to\infty}\frac{1}{R^2}\sum_{n\le R,\, n\in S}(R+1-n)f(x_n) = \frac{\alpha}{2}\int_0^1 f.$$

By simple algebraic manipulation (using $\lim_{R\to\infty}(R+1)/R = 1$) we have

$$\lim_{R\to\infty}\frac{1}{R^2}\sum_{n\le R,\, n\in S}(R+1)f(x_n) = \alpha\int_0^1 f.$$

Taking the difference of the last two equations gives the required result. □

Lemma 6 (Moving weighted averages and uniform distribution)**.** *Let $0<\beta<\gamma$ and let (x_n) and S be as in the previous lemma. Then*

$$\lim_{R\to\infty}\frac{1}{R^2}\sum_{\beta R\le n\le\gamma R,\, n\in S} nf(x_n) = \frac{\gamma^2-\beta^2}{2}\alpha\int_0^1 f.$$

Proof. From Lemma 5 (with R replaced by γR) we obtain

$$\lim_{R\to\infty}\frac{1}{R^2}\sum_{n\le\gamma R,\, n\in S} nf(x_n) = \frac{\gamma^2}{2}\alpha\int_0^1 f.$$

Obtaining a similar expression with β and subtracting gives the result. □

Lemma 7 (Weighted averages of slowly varying functions)**.** *Let S be a subset of the integers of density α. Let (x_n) be such that $(x_n)_{n\in S}$ is uniformly distributed on [0,1) and let f be a Riemann integrable function on $[0,1)$. Then*

$$\lim_{R\to\infty} \frac{1}{R^2} \sum_{n\le R,\, n\in S} n f(x_n) \mathbf{1}_{[n/R,1)}(x_n) = \frac{\alpha}{2}\int_0^1 x^2 f(x)\,dx.$$

Proof. We note that it is sufficient to prove the lemma in the case where f is a continuous function, as a Riemann integrable function may be approximated above and below by continuous functions whose integrals differ by an arbitrarily small amount.

We assume without loss of generality that f is a non-negative function. Let $\epsilon > 0$ be given and let $M > \|f\|/\epsilon$ satisfy $|f(x) - f(y)| \le \epsilon$ whenever $|x - y| \le 1/M$.

Let $f_0, f_1, \dots, f_M$ be the sequence of Riemann integrable functions defined by

$$f_i(x) = \begin{cases} f(x) & \text{if } x \ge i/M; \\ 0 & \text{if } x < i/M. \end{cases}$$

Using Lemma 6 we can choose R_0 such that for all $R \ge R_0$, all i and j,

$$\left| \frac{1}{R^2} \sum_{\substack{(i-1)R/M<n\le iR/M \\ n\in S}} n f_j(x_n) - \frac{(i^2-(i-1)^2)}{2M^2}\alpha \int_0^1 f_j \right| \le \epsilon/M.$$

For n between $(i-1)R/M$ and iR/M, we have the inequality

$$n f_i(x_n) \le n f(x_n)\mathbf{1}_{[n/R,1)}(x_n) \le n f_{i-1}(x_n).$$

Summing we have

$$\begin{aligned} \alpha\frac{i^2-(i-1)^2}{2M^2}\int_0^1 f_i - \epsilon/M &\le \frac{1}{R^2} \sum_{\substack{(i-1)R/M<n\le iR/M \\ n\in S}} n f(x_n)\mathbf{1}_{[n/R,1)}(x_n) \\ &\le \alpha\frac{i^2-(i-1)^2}{2M^2}\int_0^1 f_{i-1} + \epsilon/M. \end{aligned}$$

Note that $\int_0^1 f_i = \int_{i/M}^1 f$ for each i.

Summing this over i shows that

$$\sum_{i=1}^{M} \alpha \frac{i^2-(i-1)^2}{2M^2} \int_{i/M}^{1} f - \epsilon \le \frac{1}{R^2} \sum_{n\le R,\, n\in S} n f(x_n) \mathbf{1}_{[n/R,1)}(x_n)$$
$$\le \sum_{i=1}^{M} \alpha \frac{i^2-(i-1)^2}{2M^2} \int_{(i-1)/M}^{1} f + \epsilon.$$

These lower and upper bounds may be rewritten as

$$\frac{\alpha}{2M^2}\left(1^2\int_{1/M}^{2/M} f + 2^2 \int_{2/M}^{3/M} f + \ldots + (M-1)^2 \int_{(M-1)/M}^{1} f\right) - \epsilon; \text{ and}$$
$$\frac{\alpha}{2M^2}\left(1^2\int_{0}^{1/M} f + 2^2 \int_{1/M}^{2/M} f + \ldots + M^2 \int_{(M-1)/M}^{1} f\right) + \epsilon.$$

The upper and lower bounds differ by at most $\alpha(2M+1)\|f\|/(2M^2)+2\epsilon < 5\epsilon$.

Since $\int_{i/M}^{(i+1)/M} f$ differs from $f(i/M)/M$ by at most ϵ/M, the upper and lower bounds are within constant multiples of ϵ of the quantity

$$\frac{\alpha}{2}\sum_{i=1}^{M}(i/M)^2 f(i/M)/M.$$

This is a Riemann sum approximation of $(\alpha/2)\int_0^1 x^2 f(x)\,dx$. As ϵ is shrunk to 0 and M grows to ∞, the two bounds converge giving the required limiting value.

□

Remark. To see that Lemma 7 is an instance of the meta-principle stated at the beginning of the section, applying this principle to the quantity in Lemma 7, we expect the left hand side to be replaced by

$$\lim_{R\to\infty}(1/R^2)\,\mathrm{dens}(S)\sum_{n\le R} n \int_{n/R}^{1} f = \alpha \lim_{R\to\infty}\sum_{n=1}^{R}(1/R)(n/R)\int_{n/R}^{1} f$$
$$= \alpha\int_0^1 \left(x\int_x^1 f(t)\,dt\right)dx = \alpha\int_0^1\left(f(t)\int_0^t x\,dx\right)dt$$
$$= \frac{\alpha}{2}\int_0^1 t^2 f(t)\,dt$$

which is confirmed by Lemma 7.

For the next lemma we regard $[0,1)$ as the unit circle, so that addition and subtraction are interpreted modulo 1. We will write T_a for the translation operator on functions so that $T_a f(x) = f(x-a)$.

Lemma 8 (Averages of slowly varying functions)**.** *Let f and g be Riemann integrable functions on the circle. Let $A > 0$ and let (x_n) be uniformly distributed modulo 1. Then the quantity*

$$\frac{1}{N}\sum_{n=0}^{N-1} f(x_n + an/N)g(x_n) - \frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 f(t+an/N)g(t)\,dt$$

converges to 0 uniformly for $a \in [-A, A]$ as $N \to \infty$.

Proof. Clearly it is sufficient to prove the lemma in the case when f and g are non-negative functions. We further claim that it is sufficient to prove the lemma in the case where f and g are continuous functions. To see this we note that if f and g are Riemann integrable then we may approximate them arbitrarily closely above and below by continuous functions. This then gives arbitrarily close lower and upper bounds for $(1/N)\sum_{n\le N} f(x_n + an/N)g(x_n)$ completing the proof of the lemma.

We make one further simplifying assumption for the proof, namely that $x_n = n\theta \bmod 1$ for a fixed irrational θ. This is not essential for the proof, but provides a simplification at one point. Further the lemma will only be applied in this case in the sequel.

Now let $\epsilon > 0$. There exists $\delta > 0$ such that if $|c-d| \le \delta$ then $\|T_c f - T_d f\| < \epsilon/(4\|g\|)$. Let $M = \lceil A/\delta \rceil$ and for $-M \le j \le M$ let $h_j(x) = (T_{j\delta} f)(x) g(x)$. By unique ergodicity of the rotation by θ applied to each of the continuous functions h_j, there exists a $K > 0$ such that for any $N \ge K$, any $x \in [0,1)$ and any $|j| \le M$, we have $|(1/N)\sum_{n\le N} h_j(x+n\theta) - \int_0^1 h_j| < \epsilon/4$.

Now let N be such that $\lfloor N\delta \rfloor > K$ and $K\|f\|\|g\|/N < \epsilon/4$. Assume without loss of generality that $a > 0$ and let $n_i = \min(\lfloor Ni\delta/a \rfloor, N)$. Then for $n_{j-1} \le n < n_j$ we have $|an/N - j\delta| < \delta$. Let $s = \lceil a/\delta \rceil$. We then have

$$\begin{aligned}
&\frac{1}{N}\left|\sum_{n=0}^{N-1} f(n\theta + an/N)g(n\theta) - \sum_{n=1}^{N}\int_0^1 f(t+an/N)g(t)\,dt\right| \\
&\le \frac{1}{N}\sum_{j=1}^{s}\left|\sum_{n=n_{j-1}}^{n_j-1}\left(f(n\theta + an/N)g(n\theta) - \int_0^1 f(t+an/N)g(t)\,dt\right)\right| \\
&\le \frac{\epsilon}{2} + \frac{1}{N}\sum_{j=1}^{s}\left|\sum_{n=n_{j-1}}^{n_j-1}\left(h_j(n\theta) - \int_0^1 h_j(t)\,dt\right)\right| \\
&\le \frac{\epsilon}{2} + \frac{N\epsilon/4 + K\|f\|\|g\|}{N} \le \epsilon,
\end{aligned}$$

where the $K\|f\|\|g\|$ is an upper bound for the error that may arise if the final block is of length less than K. □

4. Visible Points

We will in later sections be summing over rotationally coded orbits, that is over pairs (p,q) of coprime integers. More specifically given an irrational slope θ we need some analytic information about the density of the points (p,q) with coprime coordinates in the strip $\{(m,n)\colon \theta m - 1 < n \le \theta m\}$. We collect in this section the results that we shall need.

It is well-known that the set of visible points in the plane (those points in $\mathbb{Z}^2$ satisfying $\gcd(p,q)=1$) is of density $6/\pi^2$. That is, in a disc of size R about the origin the ratio of the number of visible points to the area of the disc tends to $6/\pi^2$ as R approaches ∞. Indeed this result was originally proved by Mertens [14] in the 19th century.

A generalization of this result was obtained by Estermann [8] using techniques of analytic number theory where the density $6/\pi^2$ of visible points was shown to hold for any infinite strip in the plane with irrational slope. More precisely

Theorem 9 (Estermann)**.** *Let θ be irrational and let $a<b$. Then we have*

(3)
$$\lim_{N\to\infty}\frac{1}{N}\#\{(q,r)\colon\ \gcd(q,r)=1;\ q\le N,\ \theta q-b\le r<\theta q-a\}=\frac{6(b-a)}{\pi^2}$$

We make use of the following convention: We identify the circle with $[0,1)$ and if g is a function defined on the unit circle when we write $g(x)$, we mean $g(x \bmod 1)$. A corollary of Estermann's theorem, convenient for our purposes, is the following.

Theorem 10. *Let θ be irrational and let g be a Riemann integrable function on the unit circle. Then we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{q\le N\\ \gcd(q,\lfloor q\theta\rfloor)=1}} g(q\theta)=\frac{6}{\pi^2}\int_0^1 g. \tag{4}$$

We shall give a proof of this theorem using uniform distribution and supplement it with an extension that we need in the remainder of the paper. We initially consider the strip $S_\theta=\{(x,y)\colon \theta x-1<y\le\theta x\}$ of height 1 and slope θ. For an irrational number θ, define $f_\theta\colon\mathbb{N}\to\{0,1\}$ by

$$f_\theta(q)=\begin{cases}1 & \text{if } \gcd(q,\lfloor q\theta\rfloor)=1;\\ 0 & \text{otherwise.}\end{cases}$$

This function records whether or not S_θ contains a primitive lattice point with x-coordinate q.

We remark that Theorem 10 can be informally described as saying that the value of the fractional part of $q\theta$ is independent of whether q and $\lfloor q\theta \rfloor$ are coprime.

Our approach to proving Theorem 10 will be to split f_θ up into a regular part and a small part. The two lemmas that follow then control the average values of these two parts. For $r \geq 1$, define the regular part, $f_\theta^{(r)}$ as follows:

$$f_\theta^{(r)}(q) = \begin{cases} 0 & \text{if there exists } p < 2^r \text{ such that } p|q \text{ and } p|\lfloor q\theta \rfloor; \\ 1 & \text{otherwise.} \end{cases}$$

Notice that $f_\theta^{(r)}$ dominates f_θ and the sequence $(f_\theta^{(r)})_r$ decreases pointwise to f_θ.

Lemma 11. *Let θ be irrational and let $k \in \mathbb{Z}$. Then*

$$\lim_{Q\to\infty} \frac{1}{Q} \sum_{q=1}^{Q} f_\theta^{(r)}(q) e(kq\theta) = \begin{cases} \prod_{p<2^r} \left(1 - \frac{1}{p^2}\right) & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(In the above formula the product is taken over primes p less than 2^r.)

Lemma 12. *Let θ be irrational. Then for all $\epsilon > 0$ there exists an r_0 such that for all $r \geq r_0$ we have*

$$\limsup_{Q\to\infty} \frac{1}{Q} \sum_{q=1}^{Q} (f_\theta^{(r)}(q) - f_\theta(q)) < \epsilon. \tag{5}$$

We now give a proof of Theorem 10 assuming the lemmas.

Proof of Theorem 10. Fix an irrational θ and let $\epsilon > 0$. There exists an r_0 satisfying the conditions of Lemma 12. Since $\prod_{p \text{ prime}}(1 - 1/p^2) = 6/\pi^2$ (see Hardy and Wright [12] for a proof that $\prod_{p \text{ prime}}(1 - 1/p^2)^{-1} = \zeta(2)$ and Apostol [2] for a proof that $\zeta(2) = \pi^2/6$), there exists $r \geq r_0$ such that $\prod_{p<2^r}(1 - 1/p^2) < 6/\pi^2 + \epsilon$.

By Weyl's criterion it is sufficient to show that for any $k \in \mathbb{Z}$,

$$\lim_{Q\to\infty} \frac{1}{Q} \sum_{q=1}^{Q} f_\theta(q) e(kq\theta) = \begin{cases} 6/\pi^2 & \text{if } k = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Let $k \in \mathbb{Z}$ be fixed and let A be $6/\pi^2$ if $k = 0$ and 0 otherwise. We have

$$\begin{aligned}&\limsup_{Q\to\infty}\left|\frac{1}{Q}\sum_{q=1}^{Q} f_\theta(q)e(kq\theta) - \frac{1}{Q}\sum_{q=1}^{Q} f^{(r)}(q)e(kq\theta)\right| \\ &\le \limsup_{Q\to\infty}\frac{1}{Q}\sum_{q=1}^{Q}(f_\theta^{(r)}(q) - f_\theta(q)) < \epsilon,\end{aligned} \tag{6}$$

where the last inequality follows from Lemma 12.

By Lemma 11 we have

$$\left|\lim_{Q\to\infty}\frac{1}{Q}\sum_{q=1}^{Q} f_\theta^{(r)}(q)e(kq\theta) - A\right| < \epsilon. \tag{7}$$

Combining (6) and (7) we get

$$\limsup_{Q\to\infty}\left|\frac{1}{Q}\sum_{q=1}^{Q} f_\theta(q)e(kq\theta) - A\right| < 2\epsilon.$$

Since ϵ is arbitrary the proof is complete. □

Proof of Lemma 11. List the primes less than 2^r as $p_1, p_2, \ldots, p_k$ and let $P = p_1 p_2 \cdots p_k$.

We then define a dynamical system on $\{0, 1, \ldots, P-1\} \times [0, P)$ by $T(n, x) = (n + 1 \bmod P, x + \theta \bmod P)$. Define the function $g\colon X \to \mathbb{C}$ by

$$g(n, x) = \begin{cases} 0 & \text{if there exists } p < 2^r \text{ such that } p|n \text{ and } p|\lfloor x\rfloor; \\ e(kx) & \text{otherwise.}\end{cases}$$

We then claim that $f_\theta^{(r)}(q)e(kq\theta)$ can be dynamically defined, namely that $f_\theta^{(r)}(q)e(kq\theta) = g(T^q(0, 0))$. To see this, note that the following statements are pairwise equivalent

- $f_\theta^{(r)}(q) = 0$.
- There exists $p < 2^r$ such that $p|q$ and $p|\lfloor q\theta\rfloor$.
- There exists $p < 2^r$ such that $p|(q \bmod P)$ and $p|\lfloor q\theta \bmod P\rfloor$.
- $g(q \bmod P, q\theta \bmod P) = 0$.
- $g(T^q(0, 0)) = 0$.

If the functions do not take the value 0, then they both take the value $e(kq\theta)$.

$$\frac{1}{Q}\sum_{q=1}^{Q} f_\theta^{(r)}(q) = \frac{1}{Q}\sum_{q=1}^{Q} g(T^q(0,0)).$$

By unique ergodicity the right hand side converges to $\int g\,d\mu$. If $k \neq 0$, the integral is clearly 0. To evaluate this integral when $k = 0$, we use the Chinese Remainder Theorem which states that there is a ring isomorphism between $\mathbb{Z}_P$ and $\mathbb{Z}_{p_1} \times \dots \mathbb{Z}_{p_k}$ given by $n \to (n \bmod p_1, \dots, n \bmod p_k)$. Under this isomorphism the elements divisible by p_i correspond to those elements of the right side with a 0 in the $\mathbb{Z}_{p_i}$ coordinate.

Since $[0, P)$ is in measure-preserving bijection with $\{0, \dots, P-1\} \times [0, 1)$ (by $t \mapsto (\lfloor t \rfloor, \{t\})$), we see that $\{0, 1, \dots, P-1\} \times [0, P)$ is in measure-preserving bijection with $\prod_{i \le k} \mathbb{Z}_{p_i}^2 \times [0, 1)$. Under this bijection the support of g is mapped to $\prod_{i \le k} (\mathbb{Z}_{p_i}^2 \setminus \{(0,0)\}) \times [0, 1)$. Since the measure of this set is $\prod_{i=1}^{k}(1 - 1/p_i^2)$ the lemma is proven. □

Lemma 13 (Counting visible points in parallelograms)**.** *Consider a parallelogram of the form $S = \{(x, y) \colon a \le x < 2a,\ \theta x - c < y \le \theta x\}$. Then S contains at most $\lceil 4Area(S) \rceil$ primitive lattice points.*

Proof. We consider the slopes of the lines joining the origin to the primitive lattice points in S. These are all distinct and have denominators bounded above by $2a$. From the simple inequality $|p/q - p'/q'| \ge 1/(qq')$ when p/q and p'/q' are distinct, we see that each pair of slopes differs by at least $1/(4a^2)$. Since the slopes all lie between $\theta - c/a$ and θ, the number of distinct slopes (and hence the number of primitive lattice points) is bounded above by $\lceil (c/a)/(1/4a^2) \rceil = \lceil 4ac \rceil$. □

Proof of Lemma 12. We start the proof by making two simplifications.

Firstly given $\epsilon > 0$ it is sufficient to show the existence of a single $r > 0$ such that $\limsup_{Q\to\infty}(1/Q)\sum_{q \le Q}(f_\theta^{(r)}(q) - f_\theta(q)) < \epsilon$ since the functions $f_\theta^{(r)}$ decrease pointwise to f_θ.

Secondly we observe that in order to prove the lemma it is sufficient to ensure that for any ϵ, there is an $r > 0$ such that the following condition holds for all sufficiently large t

$$\frac{1}{2^t}\sum_{q=2^t}^{2^{t+1}-1}(f_\theta^{(r)}(q) - f_\theta(q)) < \epsilon. \tag{8}$$

To see the sufficiency, suppose that for a fixed value of r, (8) is satisfied for all $t \ge t_0$ with ϵ replaced by $\epsilon/4$. Given Q, let s satisfy

$2^s \le Q < 2^{s+1}$. We then have

$$\begin{aligned}\sum_{q\le Q}(f_\theta^{(r)}(q) - f_\theta(q)) &\le \sum_{t\le s}\sum_{q=2^t}^{2^{t+1}-1}(f_\theta^{(r)}(q) - f_\theta(q))\\ &\le 2^{t_0} + \sum_{t\le s}(\epsilon/4)2^t\\ &\le 2^{t_0} + \epsilon 2^{s-1}.\end{aligned}$$

Since $Q \ge 2^s$ we see

$$\frac{1}{Q}\sum_{q\le Q}(f_\theta^{(r)}(q) - f_\theta(q)) \le 2^{t_0}/Q + \epsilon/2$$

So that the desired conclusion (5) follows.

Since $\sum_{p \text{ prime}} 1/p = \infty$, we can pick an r such that $\prod_{p<2^r}(1-1/p) < \epsilon$. By increasing r if necessary we may assume that $2^{-r} < \epsilon$. Fix a t such that $2^{t/2} > \prod_{p<2^r} p$.

We let A^r denote the set of integers greater than 1 that can be formed as products of primes greater than 2^r. Notice that for $n > 1$, $n \in A^r$ if and only if $n \bmod p \ne 0$ for each prime $p < 2^r$. Using the Chinese Remainder Theorem, we see that in each block of integers of length $P = \prod_{p<2^r} p$, A^r has exactly $\prod_{p<2^r}(p-1)$ elements so that A^r has density $\prod_{p<2^r}(1-1/p)$ in any such block. Now for $s \ge t/2$, have 2^{s+1} exceeds P so that we have

$$|A^r \cap [0, 2^{s+1})| < 2^{s+2}\epsilon. \tag{9}$$

We note that $f_\theta^{(r)}(q) - f_\theta(q)$ takes the value 1 if and only if there exists $p > 2^r$ such that p divides both q and $\lfloor q\theta\rfloor$ but no $p < 2^r$ with this property. This is the case if and only if $\gcd(q, \lfloor q\theta\rfloor) \in A^r$. In other words we have $f_\theta^{(r)}(q) - f_\theta(q)$ takes the value 1 if and only if $(q, \lfloor q\theta\rfloor) \in B = \bigcup_{n\in A^r} nV$ where V is the set of primitive lattice points in $\mathbb{Z}^2$.

Let S denote the strip in $\mathbb{R}^2$ given by

$$S = \{(x, y)\colon 2^t \le x < 2^{t+1}, \theta x - 1 < y \le \theta x\}.$$

By the above, we have

$$\frac{1}{2^t}\sum_{q=2^t}^{2^{t+1}-1}(f_\theta^{(r)}(q) - f_\theta(q)) = |B \cap S|/2^t. \tag{10}$$

In order to prove the lemma we need an upper bound on $|B \cap S|$.

If $n \in A^r$, $(u, v) \in V$ and $n(u, v) \in S$, then we see that $2^t/n \le u < 2^{t+1}/n$ and $\theta u - 1/n < v \le \theta u$. If $2^{t-k} \le u < 2^{t-k+1}$ then

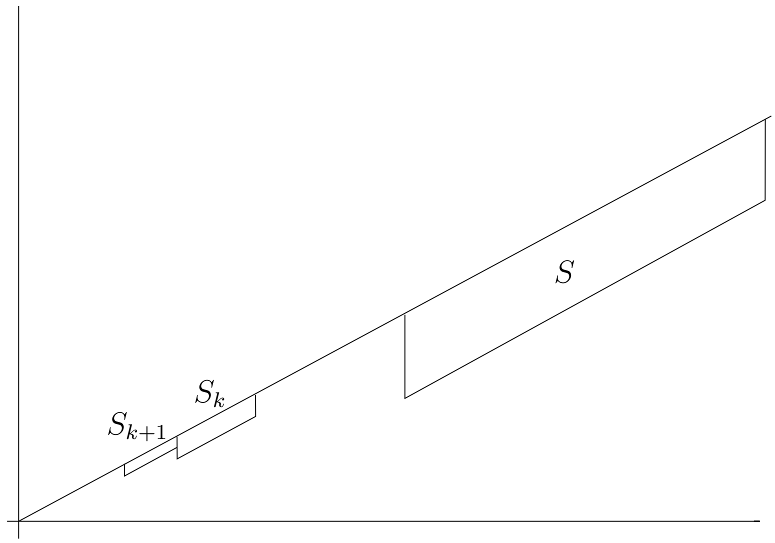

FIGURE 4. The parallelograms S_j

$2^t/n \le u < 2^{t-k+1}$ so that $1/n < 2^{-k+1}$. The point (u,v) then satisfies the inequalities

$$2^{t-k} \le u < 2^{t-k+1}$$
$$\theta u - 2^{-k+1} < v \le \theta u.$$

We define smaller parallelograms as follows:

$$S_k = \{(x,y)\colon 2^{t-k} \le x < 2^{t-k+1},\ \theta x - 1/2^{k-1} < y \le \theta x\}.$$

If $n\cdot(u,v) \in B\cap S$ for $n \in A^r$, then since $n > 2^r$, we have $1 \le u < 2^{t+1-r}$. The above shows that $(u,v) \in \bigcup_{k=r}^t S_k \cap V$.

We have

$$B\cap S \subset \bigcup_{k=r}^{t} (A^r \cap [2^{k-1},2^{k+1}))\cdot(S_k\cap V).$$

so that in particular

$$|B\cap S| \le \sum_{k=r}^{t} |A^r\cap[2^{k-1},2^{k+1})|\cdot|S_k\cap V|. \tag{11}$$

For $k \le t/2$ we use the trivial bound $|A^r\cap[2^{k-1},2^{k+1})| \le 2^{k+1}$. Using Lemma 13, we have $|S_k\cap V| \le 1+4(2^{t+1-2k}) < 2^{t-2k+4}$ so that we see

$$\sum_{k=r}^{\lfloor t/2\rfloor} |A^r\cap[2^{k-1},2^{k+1})|\cdot|S_k\cap V| \le \sum_{k=r}^{\lfloor t/2\rfloor} 2^{t-k+5} \le 2^{t-r+6}. \tag{12}$$

Notice that the rays joining the origin to the primitive lattice points in $\bigcup_{t/2<k\leq t} S_k$ have distinct slopes between $\theta - 2^{-(t-1)}$ and θ and denominators not exceeding $2^{t/2+1}$. As in the proof of Lemma 13, the difference between the slopes of any two primitive points in the region differs by at least $2^{-(t+2)}$. It follows that there can be at most 9 points in $\bigcup_{k>t/2} S_k \cap V$. Given a point in $S_k \cap V$, it gives rise to at most $|A^r \cap [2^{k-1}, 2^{k+1})|$ points of $B \cap S$. Since $k > t/2$, this is at most $2^{k+2}\epsilon$ by (9). Since there are at most 9 points it follows that

$$\sum_{k=\lfloor t/2\rfloor+1}^{t} |A^r \cap [2^{k-1}, 2^{k+1})| \cdot |S_k \cap V| \leq 36\epsilon 2^t.$$

Combining this with (12) we see that $|B \cap S| \leq 100\epsilon 2^t$. Since ϵ was arbitrary, this is sufficient to complete the proof by (10). $\square$

The following is an extension of Theorem 10 that we shall need later in the paper.

Theorem 14 (Estermann along an arithmetic progression)**.** *Let θ be irrational and let $0 \leq a < d$ be integers. Let g be a Riemann integrable function on the unit circle. We then have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{q\leq N \\ \gcd(q,\lfloor q\theta\rfloor)=1 \\ q\equiv a \bmod d}} g(q\theta) = \frac{1}{d} \prod_{p|\gcd(a,d)} \left(1 - \frac{1}{p}\right) \prod_{p\nmid d} \left(1 - \frac{1}{p^2}\right) \int_0^1 g.$$

In fact all we shall need from this extension is the following corollary.

Corollary 15 (Estermann along the even and odd subsequences)**.** *Let θ be irrational and let g be a Riemann integrable function on the unit circle. We have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{q\leq N \\ \gcd(q,\lfloor q\theta\rfloor)=1 \\ q \text{ odd}}} g(q\theta) = \frac{4}{\pi^2} \int_0^1 g.$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{q\leq N \\ \gcd(q,\lfloor q\theta\rfloor)=1 \\ q \text{ even}}} g(q\theta) = \frac{2}{\pi^2} \int_0^1 g.$$

Remark 16. *In other words the corollary states that the set of q's such that q is odd and $\gcd(q, \lfloor q\theta\rfloor) = 1$ has density $4/\pi^2$ and the fractional parts of the $q\theta$ along this sequence are uniformly distributed. Similarly* *the set of q's such that q is even and* $\gcd(q, \lfloor q\theta \rfloor) = 1$ *has density* $2/\pi^2$ *and the fractional parts of the* $q\theta$ *are uniformly distributed along this sequence.*

The proof of Theorem 14 is entirely analogous to the proof of Theorem 10 except that Lemma 11 has to be replaced by the following lemma.

Lemma 17. *Let* θ *be irrational and let* $k \in \mathbb{Z}$*. Let* $d > 0$ *be given and let* $0 \le a < d$*. Further let* r *be such that all prime factors of* d *are less than* 2^r*. Then we have the following:*

$$\lim_{Q\to\infty} \frac{1}{Q} \sum_{\substack{q=1 \\ q \equiv a \bmod d}}^{Q} f_\theta^{(r)}(q) = \frac{1}{d} \prod_{\substack{p<2^r \\ p\nmid d}} \left(1 - \frac{1}{p^2}\right) \prod_{p \mid \gcd(a,d)} \left(1 - \frac{1}{p}\right).$$

For $k \neq 0$ *we have*

$$\lim_{Q\to\infty} \frac{1}{Q} \sum_{\substack{q=1 \\ q \equiv a \bmod d}}^{Q} f_\theta^{(r)}(q) e(kq\theta) = 0.$$

The proof of this lemma is essentially the same as the proof of Lemma 11. The point is to show that the quantity being summed is again equal to a dynamical sequence given by evaluating a Riemann integrable function along the orbit of a dense rotation. Defining $R = d \cdot \prod_{p\nmid d; p<2^r} p$ and $P = \prod_{p<2^r} p$, the dynamics are defined on the group $\big(\{0, 1, \dots, R-1\} \bmod R\big) \times \big([0, P) \bmod P\big)$.

We remark that the right hand side is very intuitive: we include only every dth term, accounting for the factor of $1/d$. Given that $q \equiv a \bmod d$, if p divides a and d then there is a $1 - 1/p$ probability that q and $\lfloor q\theta \rfloor$ do not have p as a common factor whereas if p divides d but not a then the probability that q and $\lfloor q\theta \rfloor$ always have no common factor of p. If p does not divide d then there is a $1 - 1/p^2$ probability that q and $\lfloor q\theta \rfloor$ have a common factor of p.

5. Application to Piecewise Isometries

For $\theta > 0$ fixed we shall study p/q-coded orbits with $p/q < \theta < (p+1)/q$.

According to the following conjecture we believe that this exhausts the full set of rotationally-coded orbits but we have not yet been able to give a complete proof of this.

Conjecture 18 (Exact Ranges of Parameters for existence of each rotationally-coded periodic point)**.** *Let θ be irrational and let* $\gcd(p,q) = 1$.

If q is odd then T_θ has a pair of p/q-coded orbits if and only if $p/q < \theta < (p+\frac{1}{2})/q$.

If q is even then T_θ has a p/q-coded orbit if and only if $p/q < \theta < (p+1)/q$.

In the case where these orbits exist, they are well-behaved.

Theorem 19 (Main Theorem: Explicit density of nice rotationally coded points)**.** *Let θ be irrational and let R be the set of points in $\mathbb{C}$ with rotationally-coded orbits with rotation numbers p/q satisfying $p/q < \theta < (p+1)/q$. Then $\rho(R) = 3\log 2 - \pi^2/8$.*

Remark. Notice that if Conjecture 18 holds then the conclusion of Theorem 19 applies to all rotationally-coded orbits; not just those satisfying the condition $p/q < \theta < (p+1)/q$.

Recall from before that for a periodic sequence $\epsilon \in \{\pm 1\}^{\mathbb{Z}}$, $F(\epsilon,\theta)$ is the location of a potential periodic orbit with itinerary ϵ. Now let θ be a fixed irrational. The following lemma will be used to estimate $F(\theta,\epsilon)$ for various rotationally-coded itineraries ϵ.

We define the periodic itinerary $\epsilon^{(a)}_{p/q}$ by $\left(\epsilon^{(a)}_{p/q}\right)_j = s(e((a+jp)/q))$ (where the variable a will run over the set $Z_q = \{1/4,\ldots,q-1/4\}$ if q is odd or $Z_q = \{1/2,\ldots,q-1/2\}$ if q is even). We then define $z^{(a)}_{p/q}(\theta) = F(\theta,\epsilon^{(a)}_{p/q})$ so that

$$z^{(a)}_{p/q}(\theta) = \frac{1}{e(-q\theta)-1}\sum_{j=0}^{q-1}\left(\epsilon^{(a)}_{p/q}\right)_j e(-j\theta).$$

Lemma 20 (Control of location of periodic points away from the discontinuity (q odd))**.** *Let θ be a fixed irrational. Then for q satisfying $\{q\theta\} < 1/2$, let $p = \lfloor q\theta \rfloor$. Provided that* $\gcd(p,q) = 1$ *we have*

$$z^{(a)}_{p/q} = \frac{qe(a/q)}{\pi^2\{q\theta\}}(1+o(1)),$$

as $q \to \infty$ where the convergence to 0 is uniform in a running over Z_q.

Proof. Let θ be as in the statement of the lemma. Now given a q such that $\gcd(\lfloor q\theta \rfloor, q) = 1$, let $p = \lfloor q\theta \rfloor$ and let $h = q\theta - p$. Assume as in the statement of the lemma that $h < 1/2$. Let $a \in Z_q$. We then have

$$\begin{aligned}
z_{p/q}^{(a)} &= F(\theta, \epsilon_{p/q}^{(a)}) \\
&= \frac{1}{e(-q\theta) - 1} \sum_{j=0}^{q-1} \left(\epsilon_{p/q}^{(a)}\right)_j e(-j\theta) \\
&= \frac{1}{e(-h) - 1} \sum_{j=0}^{q-1} s(e((a + jp)/q))e(-j\theta) \\
&= \frac{1}{e(-h) - 1} \sum_{j=0}^{q-1} s(e(\tfrac{a}{q} + j\theta - j\tfrac{h}{q}))e(-j\theta).
\end{aligned}$$

Now we have from Lemma 8 and writing $g(t)$ for $s(e(t))$:

$$\begin{aligned}
&\sum_{j=0}^{q-1} s(e(\tfrac{a}{q} + j\theta - j\tfrac{h}{q}))e(-j\theta) \\
&= \sum_{j=0}^{q-1} g(\tfrac{a}{q} + j\theta - j\tfrac{h}{q})e(-j\theta) \\
&= \sum_{j=0}^{q-1} \int_0^1 g(\tfrac{a}{q} + t - j\tfrac{h}{q})e(-t)\,dt + o(q) \\
&= \sum_{j=0}^{q-1} \int_0^1 g(t)e(-t + \tfrac{a}{q} - j\tfrac{h}{q})\,dt + o(q) \\
&= e(\tfrac{a}{q}) \left(\sum_{j=0}^{q-1} e(-j\tfrac{h}{q}) \right) \int_0^1 g(t)e(-t)\,dt + o(q) \\
&= e(\tfrac{a}{q}) \frac{e(-h) - 1}{e(-\frac{h}{q}) - 1} \cdot \frac{-2i}{\pi} + o(q).
\end{aligned}$$

Hence using the fact that $h \leq 1/2$ we obtain

$$\begin{aligned}
z_{p/q}^{(a)} &= \frac{-2ie(\frac{a}{q})}{\pi(e(-\frac{h}{q}) - 1)} + o(\frac{q}{h}) \\
&= \frac{qe(\frac{a}{q})}{\pi^2 h} + o(\tfrac{q}{h}) = \frac{qe(\frac{a}{q})}{\pi^2 h}(1 + o(1)).
\end{aligned}$$

$\square$

The next lemma has a similar proof. It gives information on the location of p/q-coded periodic orbits when q is even and $(p+(1-\delta))/q < \theta < (p+1)/q$, this being the remaining case not covered by Lemma 20.

Lemma 21 (Control of location of periodic points far from the discontinuity (q even))**.** *Let θ be a fixed irrational. Then for even q satisfying $0 < \{q\theta\} < 1$, let $p = \lfloor q\theta \rfloor$. Provided that* $\gcd(p, q) = 1$ *we have*

$$z_{p/q}^{(a)} = \frac{qe(a/q)}{\pi^2\{q\theta\}}(1 + o(1)),$$

where the convergence to 0 is uniform in a running over Z_q.

The difference between the proof of this lemma and the previous one is that it is necessary to regroup the summation prior to applying the uniform distribution results as there can be substantial cancellation in the summation which could otherwise result in a situation where the errors dominate the quantity being bounded.

Proof. The case $0 < \{q\theta\} \le 1/2$ is covered by the previous lemma so we assume that $1/2 < \{q\theta\} < 1$. Since the proof is very similar to the proof of the previous lemma, we abbreviate, emphasizing the few significant differences. Let θ, p and q be as in the statement of the lemma. As before, let $g(t) = s(e(t))$. Let $h = q\theta - p$ so that $1/2 < h < 1$.

We have

$$\begin{aligned} F(\theta, \epsilon_{p/q}^{(a)}) &= \frac{1}{e(-q\theta) - 1} \sum_{j=0}^{q-1} g((a + jp)/q)e(-j\theta) \\ &= \frac{1}{e(-q\theta) - 1}(1 - e(-\tfrac{q}{2}\theta)) \sum_{j=0}^{\frac{q}{2}-1} g(\tfrac{a}{q} + j\tfrac{p}{q})e(-j\theta) \\ &= \frac{-1}{1 + e(-\frac{q}{2}\theta)} \sum_{j=0}^{\frac{q}{2}-1} g(a + j\tfrac{p}{q})e(-j\theta), \end{aligned}$$

where for the second equality we used the fact that $g(a+(j+\frac{q}{2})\frac{p}{q}) = -g(a+j\frac{p}{q})$. As before we have

$$\begin{aligned}\sum_{j=0}^{\frac{q}{2}-1} g(\tfrac{a}{q}+j\tfrac{p}{q})e(-j\theta) &= \sum_{j=0}^{\frac{q}{2}-1} g(\tfrac{a}{q}+j\theta-j\tfrac{h}{q})e(-j\theta)+o(q)\\ &= e(\tfrac{a}{q})(-2i/\pi)\sum_{j=0}^{\frac{q}{2}-1} e(-j\tfrac{h}{q})+o(q)\\ &= e(\tfrac{a}{q})(-2i/\pi)\frac{1+e(-\frac{q}{2}\theta)}{1-e(-\frac{h}{q})}+o(q).\end{aligned}$$

Combining the two and using the fact that $1+e(-\frac{q}{2}\theta) = \Omega(1)$ we have

$$F(\theta,\epsilon^{(a)}_{p/q}) = \frac{e(\frac{a}{q})q}{\pi^2\{q\theta\}}(1+o(1))$$

as required.

□

The following lemma states that provided $\{q\theta\}$ is bounded away from the endpoints of the range in which there are $\lfloor q\theta\rfloor/q$-coded orbits then the periodic orbit is 'locally well-behaved' in the sense that the $z^{(a)}_{p/q}$ lie above $z^{(1/4)}_{p/q}$ or $z^{(1/2)}_{p/q}$ for small values of a with similar results near the other periodic points just above and below the discontinuity.

Lemma 22 (Control of locations of periodic points near the discontinuity)**.** *Let $\epsilon > 0$ and let θ be irrational. Let $p = \lfloor q\theta\rfloor$, let $h = q\theta - p$ and suppose that* $\gcd(p,q) = 1$. *If q is odd let $b = Im\ z^{(1/4)}_{p/q}$ otherwise let $b = Im\ z^{(1/2)}_{p/q}$.*

If q is odd and $h < \frac{1}{2}-\epsilon$ or if q is even and $h < 1-\epsilon$ then $Im\ (z^{(a)}_{p/q}) \geq b$ for $a \in Z_q \cap ((0,\epsilon q/2) \cup (q/2-\epsilon q/2, q/2)$. Similarly $Im\ (z^{(a)}_{p/q}) \leq -b$ for $a \in Z_q \cap ((q/2, q/2+\epsilon q/2) \cup (q-\epsilon q/2, q))$.

Proof. We give the proof only for the odd case in the first quadrant, the even case and the other quadrants being precisely analogous. If q is odd then we have by Lemma 3, $\arg(z^{(a+1/4)}_{p/q} - z^{(a-1/4)}_{p/q}) = 2\pi(a/q+1/4+rh/q)$ with $|r| \leq q/2$. It follows that $|\arg(z^{(a+1/4)}_{p/q} - z^{(a-1/4)}_{p/q}) - \pi/2| \leq \pi/2(1-2\epsilon+4a/q)$. Provided $a < \epsilon q/2$ the difference is less than $\pi/2$ which ensures that $z^{(a+1/4)}_{p/q} - z^{(a-1/4)}_{p/q}$ has positive imaginary part. The result follows. □

For θ irrational, let $P_{\text{odd}}(\theta)$ denote the set of z such that z is rotationally coded with rotation number some p/q (in lowest terms) with q odd and $p/q < \theta < (p+1)/q$. In fact by the remark following Lemma 2 this can only happen if $p/q < \theta < (p+\frac{1}{2})/q$. Also let $P_{\text{even}}(\theta)$ denote the set of z such that z has rotation number some p/q with q even and $p/q < \theta < (p+1)/q$.

Define

$$G_{\text{odd}}(\theta, R) = \frac{1}{2R^2} \sum_{\substack{\frac{q}{\pi^2 R} < \{q\theta\} < \frac{1}{2} \\ \gcd(q, \lfloor q\theta \rfloor)=1 \\ q \text{ odd}}} q \cot^2(\pi q \theta); \text{ and}$$

$$G_{\text{even}}(\theta, R) = \frac{1}{4R^2} \sum_{\substack{\{q\theta\} > \frac{q}{\pi^2 R} \\ \gcd(q, \lfloor q\theta \rfloor)=1 \\ q \text{ even}}} q \tan^2\left(\frac{\pi q \theta}{2}\right)$$

Similarly let $H_{\text{odd}}(\theta, R) = |D_R \cap P_{\text{odd}}(\theta)|/|D_R|$ and $H_{\text{even}}(\theta, R) = |D_R \cap P_{\text{even}}(\theta)|/|D_R|$.

Lemma 23 (Comparison of density with averages)**.** *Let θ be irrational. Then $G_{odd}(\theta, R)$ and $H_{odd}(\theta, R)$ have identical limits superior and inferior as $R \to \infty$. Similarly for the even versions.*

Proof. We deal with the odd case first. Let an irrational θ be fixed and suppose $\epsilon > 0$ is given. We may suppose that ϵ is small enough so that $\tan(\pi\epsilon) < 4\epsilon$. By Lemma 20 let q_0 satisfy the following conditions:

- $1/q_0 < \epsilon/(2\pi)$;
- If $q \geq q_0$, let $p = \lfloor q\theta \rfloor$. If $\{q\theta\} < 1/2$ and $\gcd(p, q) = 1$ one has $|z_{p/q}^{(a)} - qe(a/q)/(\pi^2\{q\theta\})| < \epsilon q/(4\pi^2\{q\theta\})$.

Let R be so large that D_R contains all periodic islands surrounding periodic points with rotation number whose denominator is less than q_0.

We first give an upper bound for $|D_R \cap P_{\text{odd}}(\theta)|$. Suppose the intersection meets a periodic island of a periodic point $z_{p/q}^{(a)}$ with q odd. If $q \geq q_0$ then the periodic point is at a distance at least $(1-\epsilon/4)q/(\pi^2\{q\theta\})$ from the origin. The periodic island has size $\cot(\pi q\theta)/2 < 1/(2\pi\{q\theta\}) < (\epsilon/2)q/(\pi^2\{q\theta\})$. In particular the entire periodic island is at a distance at least $(1-\epsilon)q/(\pi^2\{q\theta\})$ from the origin. If the island intersects D_R we must have $(1-\epsilon)q/(\pi^2\{q\theta\}) < R$ so that $(1-\epsilon)q/(\pi^2 R) < \{q\theta\}$. In order to exist we must have $\{q\theta\} < 1/2$.

This establishes the inequality

$$|D_R \cap P_{\text{odd}}(\theta)| \leq \sum_{\substack{(1-\epsilon)q/(\pi^2 R)<\{q\theta\}<\frac{1}{2} \\ q \text{ odd} \\ \gcd(q,\lfloor q\theta\rfloor)=1}} (2q)\pi\cot^2(\pi q\theta)/4.$$

We then see that

$$H_{\text{odd}}(\theta, R) \leq (1-\epsilon)^2 G_{\text{odd}}(\theta, R/(1-\epsilon)). \tag{13}$$

For a comparison in the opposite direction we will make a comparison between terms appearing in $G_{\text{odd}}(\theta, R)$ and areas of periodic islands in $D_{(1+\epsilon)R}$. Suppose $q \geq q_0$ is odd and let $p = \lfloor q\theta\rfloor$. Suppose further that $\gcd(p, q) = 1$ and that $q/(\pi^2 R) < \{q\theta\} < \frac{1}{2}$ for some R. We will separate the cases where $\frac{1}{2} - \epsilon \leq \{q\theta\} < \frac{1}{2}$ and those where $q/(\pi^2 R) < \{q\theta\} < \frac{1}{2} - \epsilon$. In the latter case we will show that there is a p/q-coded pair of orbits; that they are well-behaved and that they lie completely inside $D_{(1+\epsilon)R}$ so that the p/q-coded periodic islands in $D_{(1+\epsilon)R}$ have a combined area of $2q(\pi\cot^2(\pi q\theta)/4)$. For the former case we shall show that the terms do not make a substantial contribution to G_{odd}.

We deal first with the case where $q/(\pi^2 R) < \{q\theta\} < \frac{1}{2} - \epsilon$. If $q < q_0$ then the disc was chosen to contain the periodic islands so we only need to cover the case $q \geq q_0$. Using Lemma 20 as before the potential periodic points with codes $\epsilon^{(a)}_{p/q}$ lie inside $D_{R(1+\epsilon/4)}$ and the potential islands surrounding these points are completely contained in $D_{R(1+\epsilon)}$.

We next show that the periodic orbit is well-behaved. To see this we will verify that $\text{Im } z^{(a)}_{p/q} \geq \text{Im } z^{(1/4)}_{p/q}$ for $a \in Z_q \cap (0, q/4)$, the other quadrants being similar. Lemma 22 establishes this for $1/4 < a < \epsilon q/8$. If $\epsilon q/8 < a < q/4$ then we use Lemma 20 to say that $|z^{(a)}_{p/q} - qe(\frac{a}{q})/(\pi^2\{q\theta\})| < (\epsilon/(4\pi^2))q/\{q\theta\}$. We then have $\text{Im } z^{(a)}_{p/q} > q/(\pi^2\{q\theta\})(\sin(2\pi\epsilon/8) - \epsilon/4) > \epsilon q/(4\pi^2\{q\theta\})$. Since $\epsilon q > 2\pi$ we see $\text{Im } z^{(a)}_{p/q} > 1/(2\pi\{q\theta\}) > \cot(\pi q\theta)/2 = \text{Im } z^{(1/4)}_{p/q}$ as required.

The p/q-coded periodic islands therefore lie completely inside $D_{R(1+\epsilon)}$ and have combined area $2q\pi\cot^2(\pi q\theta)/4$. Since the periodic islands are disjoint we see that

$$\sum_{\substack{\frac{q}{\pi^2 R}<\{q\theta\}<\frac{1}{2} \\ \gcd(q,\lfloor q\theta\rfloor)=1 \\ q \text{ odd}, q\geq q_0}} q\pi\frac{\cot^2(\pi q\theta)}{2} \leq |D_{(1+\epsilon)R} \cap P_{\text{odd}}(\theta)|.$$

This gives us the inequality

$$\frac{1}{2R^2} \sum_{\substack{\frac{q}{\pi^2 R} < \{q\theta\} < \frac{1}{2} - \epsilon \\ \gcd(q, \lfloor q\theta \rfloor)=1 \\ q \text{ odd}, q \geq q_0}} q \cot^2(\pi q \theta) < \frac{(1+\epsilon)^2 |D_{(1+\epsilon)R} \cap P_{\text{odd}}(\theta)|}{D_{(1+\epsilon)R}}. \tag{14}$$

We now consider the terms in $G_{\text{odd}}(\theta, R)$ where $\frac{1}{2} - \epsilon < \{q\theta\} < \frac{1}{2}$. For these terms $\cot(\pi q\theta) < \tan(\pi\epsilon) < 4\epsilon$. In order for the inequality $q/(\pi^2 R) < \{q\theta\} < \frac{1}{2}$ to be satisfied we must have $q < \pi^2 R/2 < 5R$. The sum over those terms in $G_{\text{odd}}(\theta, R)$ for which $\frac{1}{2} - \epsilon < \{q\theta\} < \frac{1}{2}$ may therefore be trivially bounded above by $(1/(2R^2)) \sum_{q<5R} 16\epsilon^2 q < 100\epsilon^2$ giving

$$\frac{1}{2R^2} \sum_{\substack{\frac{q}{\pi^2 R} < \{q\theta\} < \frac{1}{2} \\ \gcd(q, \lfloor q\theta \rfloor)=1 \\ q \text{ odd}, \{q\theta\} \geq \frac{1}{2} - \epsilon}} q \cot^2(\pi q\theta) < 100\epsilon^2. \tag{15}$$

Letting A be the sum of the terms of the sum for which $q < q_0$ and combining (14) and (15) we obtain

$$G_{\text{odd}}(\theta, R) \leq (1+\epsilon)^2 H_{\text{odd}}(\theta, R) + \frac{A}{2R^2} + 100\epsilon^2. \tag{16}$$

From (16) (and using the fact that ϵ can be chosen arbitrarily) we obtain that the limits superior and inferior of G_{odd} as $R \to \infty$ are less than those of H_{odd}. Using (13) we obtain the converse inequalities completing the proof of the theorem in the odd case. The even case is essentially identical except that Lemma 21 is used in place of Lemma 20 □

To complete the proof of Theorem 19 we now see that it is sufficient to calculate the limits $\lim_{R\to\infty} G_{\text{odd}}(\theta, R)$ and $\lim_{R\to\infty} G_{\text{even}}(\theta, R)$.

Proof of Theorem 19. The results of Section 3 do not immediately apply since the cotangent function has a singularity at 0 (and the tangent function has a singularity at $\pi/2$). The functions therefore fail to be Riemann integrable. We split the function into two parts: a bounded part and the singularity. Lemma 7 will apply to the bounded part. We argue that the part near the singularity makes an arbitrarily small contribution.

We deal first with $\lim_{R\to\infty} G_{\mathrm{odd}}(\theta, R)$. Let $r > 0$ and write $f(x) = \cot^2(\pi x)\mathbf{1}_{[2^{-r},1/2)}(x)$ and $g(x) = \cot^2(\pi x)\mathbf{1}_{[0,2^{-r}]}(x)$. We then have

(17)
$$G_{\mathrm{odd}}(\theta, R) = \frac{1}{2R^2} \sum_{\substack{\frac{q}{\pi^2 R} < \{q\theta\} \\ \gcd(q,\lfloor q\theta\rfloor)=1 \\ q \text{ odd}}} q \cdot f(\{q\theta\}) + \frac{1}{2R^2} \sum_{\substack{\frac{q}{\pi^2 R} < \{q\theta\} \\ \gcd(q,\lfloor q\theta\rfloor)=1 \\ q \text{ odd}}} q \cdot g(\{q\theta\})$$

Clearly f is Riemann integrable. We apply Remark 16. We are studying the first term of (17):

$$\frac{1}{2R^2} \sum_{\substack{\frac{q}{\pi^2 R} < \{q\theta\} \\ \gcd(q,\lfloor q\theta\rfloor)=1 \\ q \text{ odd}}} q \cdot f(\{q\theta\}).$$

Notice that in order to get a contribution, q must be smaller than $\pi^2 R$ (otherwise the condition $\frac{q}{\pi^2 R} < \{q\theta\}$ cannot be satisfied). This first term can then be rewritten:

$$\frac{\pi^4}{2(\pi^2 R)^2} \sum_{\substack{q < \pi^2 R \\ \gcd(q,\lfloor q\theta\rfloor)=1 \\ q \text{ odd}}} q \cdot f(\{q\theta\})\mathbf{1}_{(q/(\pi^2 R),1)}(\{q\theta\}).$$

We now apply Lemma 7 (with $\pi^2 R$ replacing R) to show that the first term in (17) converges as $R \to \infty$ to

$$\frac{\pi^4}{2} \cdot \frac{4}{\pi^2} \cdot \frac{1}{2} \int_{2^{-r}}^{1/2} x^2 \cot^2(\pi x)\, dx$$

As r approaches ∞, this tends to

(18)
$$\pi^2 \int_0^{1/2} x^2 \cot^2 x\, dx = \log 2 - \frac{\pi^2}{24}.$$

We then need to control the second term of (17). We will use the estimate $g(x) \le 2^{2\ell}$ on the interval $[2^{-(\ell+1)}, 2^{-\ell}]$ for $\ell \ge r$. It suffices to control the second term for R's of the form $2^t/\pi^2$ for large t. Let $N_{s,\ell}$ be the number of q's in the range $(2^{s-1}, 2^s]$ such that $\gcd(q, \lfloor q\theta\rfloor) = 1$ and $\{q\theta\} \in [2^{-(\ell+1)}, 2^{-\ell})$.

The quantity that we need to control is then overestimated by

(19)
$$\frac{1}{2^{2t}} \sum_{s=1}^{t} \sum_{\ell=r}^{t-s} N_{s,\ell} 2^{s+2\ell} = \frac{1}{2^{2t}} \sum_{s=1}^{t-r} \sum_{\ell=r}^{t-s} N_{s,\ell} 2^{s+2\ell}.$$

We separate this sum into three ranges: $s \le \ell \le t - r$, $\ell > t - r$ and $\ell < s$. These ranges are illustrated in Figure 5.

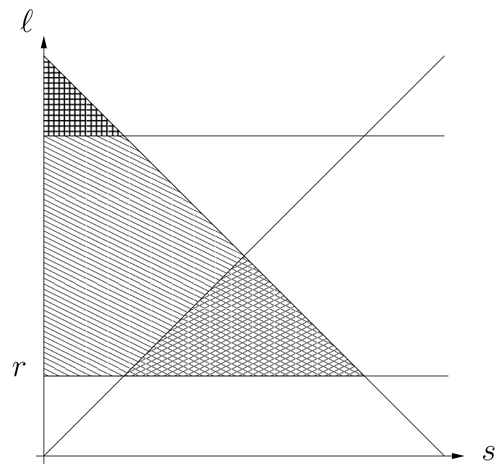

FIGURE 5. Regions of Summation

We start by controlling the range $\ell < s$. By Lemma 13 provided that $\ell < s$, we have $N_{s,\ell} \leq 2^{s-\ell+1}$. This part of the sum is then estimated by

$$\begin{aligned}\frac{1}{2^{2t}} \sum_{s=r}^{t-r} \sum_{\ell=r}^{\min(s,t-s)} 2^{s-\ell+1} \cdot 2^{s+2\ell} &= \frac{2}{2^{2t}} \sum_{s=r}^{t-r} \sum_{\ell=r}^{\min(s,t-s)} 2^{2s+\ell} \\ &\leq \frac{2}{2^{2t}} \sum_{s=r}^{t-r} \sum_{\ell=r}^{t-s} 2^{2s+\ell} \\ &\leq \frac{4}{2^{2t}} \sum_{s=r}^{t-r} 2^{t+s} \\ &\leq \frac{8}{2^{2t}} 2^{2t-r} = 8 \cdot 2^{-r}.\end{aligned}$$

Outside this range we have $\ell \geq s$ so that Lemma 13 ensures that $N_{s,\ell} \leq 1$. The part of (19) corresponding to the range $s \leq \ell \leq t-r$ is bounded above by

$$\begin{aligned}\frac{1}{2^{2t}} \sum_{s=1}^{\lfloor t/2 \rfloor} \sum_{\ell=\max(r,s)}^{\min(t-s,t-r)} 2^{s+2\ell} &\leq \frac{2}{2^{2t}} \left(\sum_{s=1}^{r} 2^{s+2t-2r} + \sum_{s=r+1}^{t/2} 2^{2t-s} \right) \\ &\leq \frac{4}{2^{2t}} 2^{2t-r} = 4 \cdot 2^{-r}.\end{aligned}$$

The final part of (19) remaining to be bounded is the range $\ell > t-r$. We assume that $t > 2r$. The contribution is then given by

$$\frac{1}{2^{2t}}\sum_{s=1}^{r}\sum_{\ell=t-r}^{t-s} N_{s,\ell}2^{2\ell}.$$

Let $\delta = \min_{q\le 2^r}\{q\theta\}$. If $2^{-(t-r+1)} < \delta$ then all of the $N_{s,\ell}$ appearing in this sum are 0, so that for sufficiently large t, this range makes no contribution.

We have therefore shown that as $t\to\infty$, the contribution to (17) coming from the second term is bounded above by $12\cdot 2^{-r}$. By taking r large, this term can be made arbitrarily small.

It follows that $\lim_{R\to\infty} G_{\mathrm{odd}}(\theta,R) = \log 2 - \pi^2/24$.

A similar argument works to compute $\lim_{R\to\infty} G_{\mathrm{even}}(\theta,R)$. We recall the definition:

$$G_{\mathrm{even}}(\theta,R) = \frac{1}{4R^2}\sum_{\substack{\{q\theta\}>q/(\pi^2R)\\ \gcd(q,\lfloor q\theta\rfloor)=1\\ q\text{ even}}} q\tan^2\left(\frac{\pi q\theta}{2}\right).$$

Notice that the summand has period 2 in the variable $q\theta$ rather than 1 so that a priori the earlier results do not apply. The ambiguity is resolved by observing that in order for $\gcd(q,\lfloor q\theta\rfloor)$ to be 1 when q is even, one must have that $\lfloor q\theta\rfloor$ is odd so that $\{q\theta/2\}$ must lie in $[\frac{1}{2},1)$. Notice that in this case $\tan(\pi q\theta/2) = -\cot(\pi\{q\theta\}/2)$. We therefore re-express the sum as

$$G_{\mathrm{even}}(\theta,R) = \frac{1}{4R^2}\sum_{\substack{\{q\theta\}>q/(\pi^2R)\\ \gcd(q,\lfloor q\theta\rfloor)=1\\ q\text{ even}}} q\cot^2\left(\frac{\pi\{q\theta\}}{2}\right),$$

thereby restoring the periodicity of the summand.

From this point the calculations are exactly similar to the odd case (using Remark 16, Lemma 7 and the splitting into singular and non-singular parts) yielding:

$$\begin{aligned}\lim_{R\to\infty} G_{\mathrm{even}}(\theta,R) &= \frac{\pi^4}{4}\cdot\frac{2}{\pi^2}\cdot\frac{1}{2}\int_0^1 x^2\cot^2\left(\frac{\pi x}{2}\right)\,dx\\ &= 2\log 2 - \frac{\pi^2}{12} = 2\lim_{R\to\infty} G_{\mathrm{odd}}(\theta,R).\end{aligned}$$

The combined asymptotic density is therefore given by $3\log 2-\pi^2/8$. □

6. Open Questions

Question 1. *Does the set of aperiodically-coded points have positive (upper) density?*

Question 2. *It is known that there periodic points whose codings do not come from any rational rotation. Is there a general description of these periodic irrotational codings? What is the density of the corresponding islands?*

Note that an answer to Question 2 together with establishing Conjecture 18 would answer Question 1.

Question 3. *For a fixed θ, let $\mathcal{L}_\theta$ denote the itinerary language: the set of all finite words that can appear in the itinerary of a point. What can be said about the growth rate of the number of words of length n? Buzzi* [5] *proves that piecewise isometries have sub-exponential numbers of 'names' but his proof relies on compactness of the ambient space. For now it is not clear whether his proof extends to the general non-compact case.*

Question 4. *Prove or disprove Conjecture 18.*

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(Cheung and Goetz) Department of Mathematics, San Francisco State University, 1600 Holloway Avenue, San Francisco, CA 94132, U.S.A.

(Quas) Department of Mathematics and Statistics, University of Victoria, Victoria BC, V8W 3R4, Canada

E-mail address: cheung(a)math.sfsu.edu

E-mail address: goetz(a)sfsu.edu

E-mail address: aquas(a)uvic.ca