SYMBOLIC CHARACTERIZATION OF COUNTEREXAMPLES TO
LITTLEWOOD’S CONJECTURE

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Miguel Cardoso
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CERTIFICATION OF APPROVAL

I certify that I have read *SYMBOLIC CHARACTERIZATION OF COUNTEREXAMPLES TO LITTLEWOOD’S CONJECTURE* by Miguel Cardoso and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

Yitwah Cheung  
Professor of Mathematics

Alexander Schuster  
Professor of Mathematics

Chun-kit Lai  
Associate Professor of Mathematics
In the 1930’s, John Edensor Littlewood proposed the Littlewood conjecture. The conjecture states that given any two real numbers $\alpha$ and $\beta$, $\inf_{n>0} n||n\alpha||n\beta|| = 0$, where $||.||$ is the distance to the nearest integer. This problem is currently unsolved in mathematics. This paper will start by establishing a framework for any collection of real numbers. We start by discussing a group of diagonal matrices acting on lattices that lead to a tiling interpretation of the Littlewood conjecture. The Rollback theorem is then developed so we can relate the lattices of the real numbers from the Littlewood conjecture to the lattices of their pivots. This is followed by defining windows and boomerangs, which use the Rollback theorem to get an ordering on these structures and their relationships to tiling. A counterexample to the Littlewood conjecture is then characterized using nearly-nested sequences of windows. This characterization is left for further refining.

I certify that the Abstract is a correct representation of the content of this thesis.
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I thank... (optional).
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Chapter 1

Introduction

In the 1930’s, Littlewood made a conjecture that is yet unproven. It refers to how close the integer multiple of two real numbers can get to being an integer itself. The conjecture is stated as follows:

**Conjecture 1.1.** For all $\alpha, \beta \in \mathbb{R}$

$$\inf_{n>0} n||n\alpha|| ||n\beta|| = 0$$

where $||.||$ is the distance to the nearest integer.

It is already known that if either $\alpha \in \mathbb{Q}$ or $\beta \in \mathbb{Q}$, the conjecture holds. This is true because if $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{Q}$, then $||n\frac{p}{q}|| = 0$ whenever $n$ is a multiple of $q$. And since $n = q < \infty$ and $||q\beta|| < \infty$, we have the infimum of the product is 0. For example, if $\alpha = \frac{2}{3}$, then $||n\frac{2}{3}|| = 0$ whenever $n$ is a multiple of 3. So the product is 0 for any multiples of 3, making the infimum 0.
It has also been determined that if the continued fraction of either number has unbounded partial quotients, the conjecture holds. This means that if $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$ and $\sup a_k = \infty$, then the conjecture is satisfied. [1].

This means that conjecture 1.1 holds for uncountably many irrationals. In fact, it was proven that the set of exceptions to conjecture 1.1 has Hausdorff dimension 0. [2].

We are inspired by the MA thesis of Samantha Lui [3]. In her thesis, Lui defines a tiling of the plane for each $(\alpha, \beta)$ so that having uniformly bounded tiles implies that we have a counterexample. It was already known that having the counterexample implies that these tiles are uniformly bounded, so this statement became if and only it. To follow this, the MA thesis of Lucy Odom showed that these tiles do not overlap.

In this paper, we hope to further explore this topic. However, we will set up framework for the more general conjecture:

**Conjecture 1.2.** For all $\theta = (\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{R}^d$

$$\inf_{n > 0} n \prod_{i=1}^{d} ||n\theta_i|| = 0$$

where $||.||$ is the distance to the nearest integer.

Again, conjecture 1.2 is known to hold if any $\theta_i$ is rational for the same reason that conjecture 1.1 holds. In fact, conjecture 1.1 is just the special case of conjecture
1.2 where \( d=2 \). This paper will focus on an arbitrary \( d \) dimensions, only reverting back to the case where \( d=2 \) for example and when necessary.

In section 2, we define the \( A \) group. We then show how this group acts on lattices. From there, we discuss the \( A_+ \) and \( A_- \) subgroups and their relation to the Littlewood conjecture.

Section 3 start by discussing balance time. This special element of the \( A \) group depends on both the particular \( \theta \) and rational point \( v \in \mathbb{R} \) that we are looking at. We determine its uniqueness in this section as well.

From there section 3 continues on to develop the Roll-back theorem. This theorem relates the linear maps from the \( A \) group to the lattices involved in the Littlewood conjecture. More precisely, the Roll-back theorem gives us relationships between the lattices of \( \theta \) and \( v \), along with the \( A_+ \) and \( A_- \) groups using the balance time.

In section 4, we return to focusing on the case where \( d=2 \). We first define a window, which has rational endpoints and arise naturally from domains of approximation. This, along with the Roll-back theorem, leads us to define boomerangs. We end this section by determining how to take a sequence of windows with special properties and return a counterexample to the Littlewood conjecture.

Section 5 focuses on encoding information about a counterexample to the Littlewood conjecture when \( d=2 \) in windows. We determine how to create the sequence of windows that converge down to the counterexample. We also determine a list of
properties that arise naturally from this process. This section concludes by showing that our process yields the same list of special properties assumed in the final theorem of section 5.

Finally, chapter 6 concludes our work. We discuss the results, as well as future work that can be done after this thesis.
Chapter 2

A-action on the Space of Lattices

2.1 The A group

We start by defining the group A.

**Definition 2.1 (The Group A).** In a $d+1$ dimensional space, define

$$A = \left\{ \begin{pmatrix} a_1 & 0 & \ldots & 0 & 0 \\ 0 & a_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & a_d & 0 \\ 0 & 0 & \ldots & 0 & a_{d+1} \end{pmatrix} : \prod_{i=1}^{d+1} a_i = 1 \text{ and } a_i > 0 \text{ for } i = 1, 2, \ldots, d, d+1 \right\}$$

Note that the condition of $\prod_{i=1}^{d+1} a_i = 1$ means that one term can be written as a product of the others. This means we are in a $d$-dimensional space. This condition and the fact that all of the $a_i$ are positive leads to another definition.
Definition 2.2. We put coordinates on $A$ by defining

$$A = \left\{ \begin{pmatrix} e^{t_1} & 0 & \ldots & 0 & 0 \\ 0 & e^{t_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & e^{t_d} & 0 \\ 0 & 0 & \ldots & 0 & e^{t_{d+1}} \end{pmatrix} : \sum_{i=1}^{d+1} t_i = 0 \right\}$$

where $t_i = \log(a_i)$.

We call $(t_1, t_2, \ldots, t_{d+1})$ the homogeneous coordinates. In the case $d=2$, we use $(t,s)$-coordinates given by

$$A = \left\{ \begin{pmatrix} e^{t+s} & 0 & 0 \\ 0 & e^{t-s} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} : t, s \in \mathbb{R} \right\}$$

Then the sum of the exponents is 0 as desired, and we can look at the 2-dimensional plane to learn about these structures.

We now look at two important subgroups of our group $A$.

Definition 2.3 ($A_+$).

$$A_+ = \{ a \in A : \min(a_1, a_2, \ldots, a_d) \geq 1 \geq a_{d+1} \}$$
We call this subgroup the strongly unstable subgroup.

This group has a very strict condition that the last coordinate in our diagonal matrix must be less than or equal to 1, while all other entries of these matrices must be greater than or equal to 1. When \( d=2 \), this is equivalent to \( 0 \leq |s| \leq t \). Now we look at a subgroup with a less strict condition.

**Definition 2.4** \((A_-)\).

\[
A_- = \{ a \in A : a_{d+1} \geq \max(a_1, a_2, \ldots, a_d) \}
\]

We call this subgroup the stable subgroup.

This is the subgroup where the last term dominates the others. However, it is not as strict because we only condition that the last term is biggest with no other restrictions on the other terms. When \( d=2 \), this is equivalent to \( 0 \leq |s| \leq -3t \).

These two subgroups lead us to the following theorem:

**Theorem 2.1.** \( A_+ \cap (\text{any coset of } A_-) \) is bounded as a subset of \( A \).

**Proof.** Let \( a \in A_+ \cap e^{t} A_- \). Since \( a \in e^{t} A_- \), we have \( e^{-t} a \in A_- \). So by definition 2.4, this means

\[
a_i e^{-t_i} \leq a_i e^{-t_{i+1}} \forall i = 1, 2, \ldots, d
\]

This implies that
\[ a_i e^{t_n^* - t_i^*} \leq a_n \]

But by Definition 2.3, we have \( a_i \geq 1 \geq a_n \forall i = 1, 2, \ldots, d \). So

\[ e^{t_n^* - t_i^*} \leq a_i e^{t_n^* - t_i^*} \leq a_n \leq 1 \]

So we have \( 1 \leq a_i \leq e^{t_n^* - t_i^*} \) and \( e^{t_n^* - t_i^*} \leq a_n \leq 1 \). So we have that the intersection is bounded in \( A \).

\[ \square \]

### 2.2 Lattices

We now work towards defining the set on which the \( A \) group acts on. First, we define a lattice.

**Definition 2.5** (Lattice). A lattice is a discrete subgroup of \( \mathbb{R}^{d+1} \) that is of full rank. We denote the collection of lattices by

\[ \mathcal{L}_{d+1} = \{ \Lambda \subset \mathbb{R}^{d+1} : \Lambda \text{ is a full-rank lattice in } \mathbb{R}^{d+1} \} \]

Now we can define a shearing matrix.
Definition 2.6. Given $\theta \in \mathbb{R}^d$, define the $(d+1)$-by-$(d+1)$ matrix

\[
    h_\theta = \begin{pmatrix}
        1 & 0 & 0 & \ldots & -\theta_1 \\
        0 & 1 & 0 & \ldots & -\theta_2 \\
        0 & 0 & 1 & \ldots & -\theta_3 \\
        \vdots & \vdots & \vdots & \ddots & \vdots \\
        0 & 0 & 0 & \ldots & 1
    \end{pmatrix}
\]

Now we give notation to our sheared lattice, which we refer to as the sheared integer lattice.

Definition 2.7. Given a $(d+1)$-dimensional lattice and a $\theta \in \mathbb{R}^d$, we define

\[
    \Lambda_\theta = h_\theta \mathbb{Z}^{d+1} = \{ h_\theta x : \text{where } x \in \mathbb{Z}^{d+1} \}
\]

With the sheared lattice, we act upon it with the A group defined above. We will start the next section by looking at a special property from these actions.
Chapter 3

Roll-Back Theorem

3.1 Balance Time

Having the A group acting on the space of lattices, we first need to determine a norm to use for magnitudes of these vectors. We use:

**Definition 3.1** (*l*∞ Norm). Let \( x = (x_1, x_2, \ldots, x_d, x_{d+1} \in \mathbb{R}^{d+1}) \). Then the *l*∞ norm is

\[
||x||_{\infty} = \max_{i=1,2,\ldots,d,d+1} (|x_i|)
\]

With this norm, we can determine magnitudes of vectors after they have been transformed by the A-action. We wish to follow the norm of a vector as it evolves under the A-action. It is clear that we can make these vectors have large magnitude since our norm looks at the absolute value of the maximum coordinate. Thinking about how small we can make the magnitude leads to the following definition.
**Theorem 3.1.** Given $v \in \mathbb{R}^{d+1} \setminus \{\text{coordinate planes}\}$, there exists a unique $a \in A$ such that $||av||_\infty$ is minimized.

**Proof.** Let $v = (x_1, x_2, \ldots, x_d, x_{d+1}) \in \mathbb{R}^{d+1} \setminus \{\text{coordinate planes}\}$. For any $a \in A$, Definition 3.1 tells us that

$$||av||_\infty = \max(|a_1 x_1|, |a_2 x_2|, |a_3 x_3|, \ldots, |a_d x_d|, |a_{d+1} x_{d+1}|)$$

$$= \max(a_1 |x_1|, a_2 |x_2|, a_3 |x_3|, \ldots, a_d |x_d|, a_{d+1} |x_{d+1}|)$$

by Definition 2.1. If there exists a number, $r \in \mathbb{R}$, such that $||av||_\infty$ is minimized, we would have

$$a_1 |x_1| = a_2 |x_2| = a_3 |x_3| = \cdots = a_d |x_d| = a_{d+1} |x_{d+1}| = r$$

This is because $x_1, x_2, \ldots, x_d, x_{d+1}$ is fixed, so if one of these terms was lower than the other would be higher since $\prod_{i=1}^{d+1} a_i = 1$, and $||av||_\infty$ would be higher since it is the max. With this in place, we consider multiplying these terms.

$$\prod_{i=1}^{d+1} a_i |x_i| = r^{d+1}$$

$$\implies \prod_{i=1}^{d+1} |x_i| = r^{d+1}$$

since $\prod_{i=1}^{d+1} a_i = 1$. So we have
\[ r = \frac{d+1}{\prod_{i=1}^{d+1} |x_i|} \]

We want \( a_i|x_i| = r \) for all \( i \). So we have

\[ a_i = \frac{r}{|x_i|} \]

for all \( i \). Each of these terms is positive since \( r \) and \( |x_i| \) are positive and nonzero since no \( x_i \) is zero. The product of the \( a_i \) is 1. So this \( a \) exists in \( A \). It is also unique since this is the only way to get \( a_i|x_i| = r \) for \( i=1,2,\ldots,d+1 \).

We now name this unique element of \( A \).

**Definition 3.2 (Balance Time).** Let \( v \in \mathbb{R}^{d+1} \setminus \{ \text{coordinate planes} \} \). Then the balance time of \( v \) is a unique \( a \in A \) such that \( ||av||_{\infty} \) is minimized.

By theorem 3.1, the balance time exists and is unique. If we have a \( \theta \in \mathbb{R}^d \) and \( v \in \mathbb{Z}^{d+1} \), we denote the balance time of \( h_{\theta}(v) \) as \( t_*(v,\theta) \). We call the coset \( e^{t_*(v,\theta)}A_- = A_-(v;\theta) \) the stable sector. We will use this to get more relationships about \( v \) and \( \theta \), but we must discuss properties of linear maps first.
3.2 Norms of Linear Maps

We first note that every $a \in A$ and all $h_\theta$ are linear maps. So we can apply the following definition.

**Definition 3.3.** Let $f : (X, ||.||_X) \to (Y, ||.||_Y)$ be a linear map. Then we define the norm to be

$$||f|| = \sup_{x \in X, ||x|| \neq 0} \frac{||f(x)||_Y}{||x||_X}$$

It is also known that

$$||f|| = \sup_{x \in X, ||x|| = 1} ||f(x)||_Y$$

In our case, every $a \in A$ and all $h_\theta$ are linear maps from $(\mathbb{R}^{d+1}, ||.||_\infty)$ to $(\mathbb{R}^{d+1}, ||.||_\infty)$. For $x = (x_1, x_2, \ldots, x_{d+1}) \in \mathbb{R}^{d+1}$ to have $||x||_\infty = 1$, at least one $x_j = 1$ or $-1$ and we have $-1 \leq x_i \leq 1$ for $i = 1, 2, \ldots, d + 1$.

**Definition 3.4 (K-Lipschitz).** Let $f : (X, d_X) \to (Y, d_Y)$, where $f$ is linear and $d_X, d_Y$ are the metrics on $X$ and $Y$ respectively. Then $f$ is K-Lipschitz if for all $x, y \in X$ we have

$$d_Y(f(x), f(y)) \leq Kd_X(x, y)$$

We call $K$ the Lipschitz constant.
For an easier way to calculate the Lipschitz constant, we have the following well-known theorem.

**Theorem 3.2.** Let \( f : (\mathbb{R}^{d+1}, \ell_\infty) \to (\mathbb{R}^{d+1}, \ell_\infty) \) be a linear map. Then the Lipschitz constant \( K \) is equal to the norm of the linear map, or

\[
K = ||f|| = \sup_{x \in \mathbb{R}^{d+1}, ||x|| = 1} ||f(x)||_\infty
\]

This means we can find the Lipschitz constant by calculating the norm of our linear maps. Now we find a way to calculate the norm of \( h_\theta \).

**Lemma 3.3.** For \( \theta \in \mathbb{R}^d \), we have

\[
||h_\theta|| = 1 + \max_{i=1,2,...,d} |\theta_i|
\]

**Proof.** Let \( \theta \in \mathbb{R}^d \) and \( x = (x_1, x_2, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \) where \( ||x||_\infty = 1 \). By the definition of \( h_\theta \), we have

\[
h_\theta x = \begin{pmatrix}
x_1 - \theta_1 x_{d+1} \\
x_2 - \theta_2 x_{d+1} \\
\vdots \\
x_d - \theta_d x_{d+1}
\end{pmatrix}
\]

Taking the \( \ell_\infty \) norm of this gets us
\[ ||h_{\theta} x||_\infty = \max_{i=1,2,...,d} |x_i - \theta_i x_{d+1}| \]

But since \( ||x||_\infty = 1 \), we know that \( |x_i| \leq 1 \) for \( i = 1, 2, \ldots, d + 1 \). So for each \( i \), we have a max when \( x_i = 1 \) and \( -\theta_i x_{d+1} = |\theta_i| \). So taking the supremum over all \( ||x|| = 1 \) yields

\[ ||h_{\theta}|| = 1 + \max_{i=1,2,...,d} |\theta_i| \]

If \( f \) is invertible, we also get the following definition.

**Definition 3.5** (BiLipschitz Constant). Let \( f \) be an invertible linear map with Lipschitz constant \( K_1 \) and let the inverse of \( f \) have a Lipschitz constant of \( K_2 \). Then \( f \) has a biLipschitz constant of \( K = \max(K_1, K_2) \).

Note that for all \( a \in A \), \( ah_{\theta-\bar{a}^{-1}} = h_{\bar{\theta}} \)

where \( \bar{\theta}_i = \frac{\alpha_i}{a_n} \left( \theta - \frac{p_i}{q} \right) \).

This leads us to:

**Lemma 3.4.** The biLipschitz constant of \( ah_{\theta-\bar{a}^{-1}} = h_{\bar{\theta}} \) is at most 2 if and only if for all \( i=1,2,\ldots \ d \) we have \( |\theta_i - \frac{p_i}{q}| \leq \frac{\alpha_{i+1}}{\alpha_i} \).
**Proof.** ($\Rightarrow$) Let the biLipschitz constant of $h_{\tilde{\theta}}$ be at most 2. Then by theorem 3.2, we know that $||h_{\tilde{\theta}}|| \leq 2$. By theorem 3.3 this means

$$||h_{\tilde{\theta}}|| = 1 + \max_{i=1,2,\ldots,d} |\tilde{\theta}_i| \leq 2$$

We may subtract 1 from both sides and use the definition of $\tilde{\theta}_i$ to get

$$\left| \frac{a_i}{a_n} \left( \theta - \frac{p_i}{q} \right) \right| \leq 1$$

For $i=1,2,\ldots,d$. But since all $a_i$ and $a_{d+1}$ are positive, we may pull them out and divide to yield

$$\left| \left( \theta - \frac{p_i}{q} \right) \right| \leq \frac{a_{d+1}}{a_i}$$

For $i=1,2,\ldots,d$. Thus, the forward direction is complete.

($\Leftarrow$) Assume that $|\theta_i - \frac{p_i}{q}| \leq \frac{a_{d+1}}{a_i}$ for all $i=1,2,\ldots,d$. This means $\left| \frac{a_i}{a_{d+1}} \theta_i - \frac{p_i}{q} \right| \leq 1$ for all $i=1,2,\ldots,d$. Adding 1 to both sides yields

$$1 + \left| \frac{a_i}{a_{d+1}} \theta_i - \frac{p_i}{q} \right| \leq 2$$

So by theorem 3.3, we have that the Lipschitz constant for $h_{\tilde{\theta}}$ is at most 2. To get the Lipschitz constant for the inverse, we will observe that
\[ h^{-1}_\tilde{\theta} = \begin{pmatrix} 1 & 0 & \ldots & \frac{a_1}{a_{d+1}} \theta_1 \\ 0 & 1 & \ldots & \frac{a_2}{a_{d+1}} \theta_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \]

But by 3.3, we have

\[ ||h^{-1}_\tilde{\theta}|| = 1 + \frac{a_i}{a_d + 1} \theta_i - \frac{p_i}{q} \]

But by the assumption, we know this value is at most 2. So the biLipschitz constant of \( h_{\tilde{\theta}} \) is at most 2. \qed

With this in place, we may proceed to the Roll-back Theorem.

**Theorem 3.5** (Roll-back Theorem). Let \( \theta \in \mathbb{R}^d \) and \( v \in \mathbb{Z}^{d+1} \). Then \( a \in A_-(v, \theta) \iff \) the bilipschitz constant of \( ah_{\theta - \epsilon}a^{-1} \) is at most 2.

**Proof.** Let \( \theta \in \mathbb{R}^d \) and \( v \in \mathbb{Z}^{d+1} \). Then \( a \in A_-(v, \theta) \iff a \in e^{t_(v; \theta)}A_- \iff ae^{-ts(v; \theta)} \in A_- \)

By definition 2.4 and the proof of theorem 3.1, this is true if and only if

\[ \frac{a_i}{r} \left| \theta_i - \frac{p_i}{q} \right| \leq \frac{a_n}{r} \]
For all $i=1,2,\ldots,d$. But by theorem 3.4, this is true if and only if the biLipschitz constant of $ah_{\theta-v}a^{-1}$ is at most 2.

This now leads us to a useful corollary.

**Corollary 3.6.** Let $\theta \in \mathbb{R}^d$ and $v \in \mathbb{Z}^{d+1}$. Then for any $a \in A_-(v, \theta)$, the lattices $a\Lambda_{\theta}$ and $a\Lambda_v$ are isomorphic by a linear map of biLipschitz constant at most 2.

This is true by the Roll-back theorem. The isomorphism from $a\Lambda_v$ to $a\Lambda_{\theta}$ is defined in the Roll-back theorem as $ah_{\theta-v}a^{-1}$. Since this is a composition of isomorphisms, this map is also an isomorphism, and the Roll-back theorem gives us that it has a biLipschitz constant that is at most 2.
Chapter 4

Windows

To build to our next new objects, we will introduce some objects that have been previously developed to aid in resolving the Littlewood conjecture.

**Definition 4.1 (Box).** Given a lattice \( \Lambda \) and \( v \in \Lambda \), we define the box of \( v \) to be

\[
B(v) = \{ x \in \mathbb{R}^d : |x_i| \leq |v_i| \}
\]

With boxes in place, we may now define the stable pivots of a lattice.

**Definition 4.2 (stable pivot).** \( v \) is a stable pivot if (i) \( \Lambda \cap \text{Int}B(v) = \{0\} \) and (ii) \( B(u)=B(v) \) for any nonzero \( u \in \Lambda \cap B(v) \).

This allows us to define stable pivot classes.

**Definition 4.3 (stable pivot classes).** Given a lattice, \( \Lambda \), we define an equivalence relation by \( u,v \in \Lambda, \ u \sim v \) if \( B(u) = B(v) \). We denote the collection of all stable pivot classes by \( \Pi(\Lambda) \).
For the next definition, we need a bit of notation.

**Definition 4.4** ($\hat{v}$). Given $v=(x_1, x_2, \ldots, x_{d+1}) \in \mathbb{R}^{d+1}$, we define

$$\hat{v} = \left( \frac{x_1}{x_{d+1}}, \frac{x_2}{x_{d+1}}, \ldots, \frac{x_d}{x_{d+1}} \right) \in \mathbb{R}^d$$

Now we define a convergent.

**Definition 4.5** (convergent). Let $\theta \in \mathbb{R}^d$ and $v \in \mathbb{Z}^{d+1}$. We say that $\hat{v}$ is a convergent of $\theta$ if $h_\theta v \in \Pi(\Lambda_\theta)$. The collection of all convergents is denoted by $\chi(\theta)$, and we say $v \in \chi(\theta)$.

Stable pivots from definition 4.2 lead us the to following collection:

**Definition 4.6** (Domain of Approximation). Given $v \in \mathbb{R}^{d+1}$, we define the domain of approximation as

$$\Delta(v) = \{ \theta \in \mathbb{R}^d : h_\theta(v) \text{ is a stable pivot of } \Lambda_\theta \}$$

In a thesis by Nuguid, we get a grasp on the shape of the domain of approximation [4].

**Theorem 4.1** ([4]). *The domain of approximation is a rectilinear region with sides parallel to the coordinate axes that is star convex with respect to $v$.*

With this shape defined, we may now start creating new objects related to this shape.
Definition 4.7 (Windows). A window, $W$, is the maximal rectangle contained in the domain of approximation, $\Delta(v)$, with one vertex at $\hat{v}$. The opposite vertex is another rational, $\hat{w}$. The window is open on $\partial \Delta(v)$, but closed on $\text{Int}(\Delta(v))$.

We now look at how this relates to the stable sectors.

Definition 4.8. Let $W$ be a window in $\Delta(v)$. We define

$$A_-(W) = \bigcap_{\theta \in W} A_-(v; \theta)$$

This allows us to get a new definition for a sequence of windows.

Definition 4.9 (stably monotone). Let $(W_k)_{k=1}^{\infty}$ be a sequence of windows. We say the sequence is stably monotone if $A_-(W_k) \subset A_-(W_{k+1})$.

We may also define one more property for a sequence of windows.

Definition 4.10 (Exhaustion of $A_+$). Let $(W_k)_{k=1}^{\infty}$ be a sequence of windows. We say that this sequence exhausts $A_+$ if $\bigcup_{k=1}^{\infty} A_-(W_k) \supset A_+$.

These two properties guarantee that the windows are not piling up at a limit in their stable sectors and that their stable sectors have an order to them. Next, we acknowledge that being nested is not always possible, so we make another property that is close enough to work.
**Definition 4.11** (uniform shrinking). Let \((W_k)_{k=1}^{\infty}\) be a sequence of windows. We say the sequence is \(\rho\) nearly nested if \(\exists \rho > 1\) such that \(\forall k \in \mathbb{Z}_+\) we have

\[ \rho W_{k+1} \subset \rho W_k \]

where \(\rho W_k = \{\rho(w - \hat{v}) + \hat{v} : w \in W_k \text{ and } W_k \text{ is maximal in } \Delta(v)\}\)

In this condition, we scale up the size of the windows with respect the \(\hat{v}\) from the \(\Delta(v)\) that the window is maximal in. Then the scaled up windows will be nested instead of the original windows. We know transition to a bounded structure that represents the difference between the stable sectors of two windows.

**Definition 4.12.** Given two windows, \(W' \subset W\), we define the boomerang to be

\[ B(W, W') := (A_-(W') \setminus \text{Int}(A_-(W))) \cap A_+ \]

Before moving on to our next property, we need the systole function. Given a lattice \(\Lambda\) under a norm \(||.||_{\Lambda}\), the systole function denoted \(\text{sys}(\Lambda)\) is the magnitude of the shortest vector in \(\Lambda\) under the given norm. In our case, we use the \(\ell_\infty\) norm from definition 3.1.

We may now use boomerangs and the systole function to define a property that relates lattices to boomerangs.

**Definition 4.13** (The Boomerang Property). Let \(\theta \in \mathbb{R}^d\). Let \((W_k)_{k=1}^{\infty}\) be windows containing \(\theta\), with \(W_{k-1} \subset W_k\). Then the sequence has the boomerang property if
\( \exists \delta > 0 \) such that \( \forall a \in B(W_k, W_{k+1}) \) we have \( \text{sys}(a\Lambda) \geq \delta \)

Using this, we may now proceed to our next theorem. The conditions in this theorem are motivated by the theorem in the following section.

**Theorem 4.2** (Retrieving information from encodings). Let \( d=2 \). If \( (W_k)_{k=1}^{\infty} \) be a sequence of windows having the following properties,

(i) (stably monotone) \( A_-(W_k) \subset A_-(W_{k+1}) \) for all \( k \).

(ii) (boomerang property) \( \exists \delta > 0 \) such that \( \forall a \in B(W_k, W_{k+1}) \) we have \( \text{sys}(a\Lambda) \geq \delta \)

(iii) (covering) \( \bigcup_{k=1}^{\infty} A_-(W_k) \supset A_+ \)

(iv) (uniform shrinking) \( \exists \rho > 1 \) such that \( \forall k \in \mathbb{Z}_+ \) we have

\[
\rho W_{k+1} \subset \rho W_k
\]

where \( \rho W_k = \{ \rho(w - \hat{v}) + \hat{v} : w \in W_k \text{ and } W_k \text{ is maximal in } \Delta(v) \} \)

Then \( \bigcap_{k=1}^{\infty} \overline{\rho W_k} = \theta = (\alpha, \beta) \) has the property that \( n||n\alpha||||n\beta|| \geq \gamma > 0 \).

**Proof.** Let \( (W_k)_{k=1}^{\infty} \) be a sequence of windows having the given four properties. Since the closure of the \( \rho W_k \) is closed for all \( k \), and the sequence is nested and nonempty, then \( \bigcap_{k=1}^{\infty} \overline{\rho W_k} \) is nonempty, call it \( \theta = (\alpha, \beta) \).
Given any $k$, we know that there exists $\delta > 0$ such that for all $a \in B(W_{k-1}, W_k)$ we have $\sys(a\Lambda_{\omega_{k-1}}) \geq \delta$ from (ii).

But by the Roll-back theorem, we know $a\Lambda_{\omega_{k-1}} \sim_2 a\Lambda_{\theta}$, so $\sys(a\Lambda_{\theta}) \leq 2\sys(a\Lambda_{\omega_{k-1}}) \leq \delta$ for any $k$. So we get that $\sys(a\Lambda_{\theta}) \leq \frac{\delta}{2}$. Therefore,

$$\exists \delta > 0 : \forall a \in A_+ \text{ we have } \sys(a\Lambda_{\theta}) \geq \frac{\delta}{2}.$$  

since $A_+ = \bigcup_{k=1}^{\infty} A_-(W_k)$ and $A_-(W_k) \subset A_-(W_{k+1})$ for all $k$ by properties (i) and (iii).

We now show that $n||n\alpha||||n\beta|| \geq \gamma > 0$. For $n \in \mathbb{Z}_+$, let $u=(m_1, m_2, n) \in \mathbb{Z}^{d+1}$ where $m_1$ is the nearest integer to $n\alpha$ and $m_2$ is the nearest integer to $n\beta$. The volume of the box $B(h_\theta u)$ is $8n||n\alpha||||n\beta||$. Since the volume of the box is preserved under the $A$-action, we have

$$8n||n\alpha||||n\beta|| = \text{vol}(B(h_\theta u)) = \text{vol}(B(ah_\theta u)) \geq 8\sys(a\Lambda_{\theta})^3 \geq 8\left(\frac{\delta}{2}\right)^3$$

Dividing both sides by 8 and noting that $\gamma = \left(\frac{\delta}{2}\right)^3 > 0$, we have that $n||n\alpha||||n\beta|| \geq \gamma > 0$.

Now that we know how to retrieve information about a counterexample to the Littlewood conjecture from an encoding of windows, we will move on to see how to
take the counterexample and encode the information in a sequence of windows in the next section.
Chapter 5

Building a Sequence of Windows from a Counterexample

In this section, we will start with a counterexample to the Littlewood conjecture and show that we can build a sequence of windows with specific properties. But we first need more information about another interpretation of the Littlewood conjecture.

**Definition 5.1 (tiles).** Let $h_\theta v \in \Pi(\lambda_\theta)$. Then the tile of $u$ is

$$\tau(u) = a \in A : \text{sys}(a\Lambda) = ||au||_\infty$$

It was also known that $A = \bigcup_{u \in \Pi(\Lambda_\theta)} \tau(u)$. If $\theta$ is a counterexample to the Littlewood conjecture, then there is no overlap of these tiles [5].

We also know that each tile $\tau_\theta(v)$ is contained in a right triangle, denoted $\Delta_\theta(v)$. The diameter of this triangle is bounded by $diam(\Delta_\theta(v)) \leq -\log \left( n \prod_{i=1}^{d} ||n\theta_i|| \right)$. 
These will be useful as we work toward proving our next theorem.

**Theorem 5.1.** Let $d=2$ and $\theta = (\alpha, \beta)$. If $n||n\alpha||||n\beta|| \geq \delta > 0$ for all $n$, then there exists a sequence of windows $(W_k)_{k=1}^{\infty}$ with the following properties

(i) (stably monotone) $A_-(W_k) \subset A_-(W_{k+1})$ for all $k$.

(ii) (boomerang property) $\exists \delta > 0$ such that $\forall a \in B(W_k, W_{k+1})$ we have $\text{sys}(a\Lambda) \geq \delta$

(iii) (covering) $\bigcup_{k=1}^{\infty} A_-(W_k) \supset A_+$

(iv) (uniform shrinking) $\exists \rho > 1$ such that $\forall k \in \mathbb{Z}_+$ we have

$$\rho W_{k+1} \subset \rho W_k$$

where $\rho W_k = \{ \rho (w - \hat{v}) + \hat{v} : w \in W_k \text{ and } W_k \text{ is maximal in } \Delta(v) \}$

(v) $\bigcup_{k=1}^{\infty} \overline{\rho W_k} = \theta$

First, we can show that given such a $\theta$, there exists a sequence of windows containing $\theta$. First, we need the following definition:

**Definition 5.2.** Generic We define $a \in A$ to be generic if $a$ is not on the boundary of any tile in the tiling of $\Lambda_\theta$.

We now prove the following theorem.
**Theorem 5.2.** Let $d=2$ and $\theta = (\alpha, \beta)$ and $n||n\alpha||||n\beta|| \geq \delta > 0$ for all $n$. Then given a generic $a \in A_+$, there exists a $v \in \chi(\theta)$ such that there is a window $W$ with $\theta \in W \subset \Delta(v)$.

*Proof.* Let $\theta$ be defined as in theorem 5.1. Since $n||n\alpha||||n\beta|| \geq \delta > 0$ for all $n$, there is no overlap in any of the tiles in the tiling of $\Lambda_{\theta}$. So for any generic $a \in A_+$ there exists a unique convergent $\hat{v}$ of $\theta$ such that $a \in \tau_{\theta}(v)$.

This means $||ah_{\theta}v||_\infty = \text{sys}(a\Lambda_{\theta})$. So $ah_{\theta}v$ is a stable pivot of $a\Lambda_{\theta}$. But the $A$-action preserves stable pivots, so this implies that $h_{\theta}v$ is a stable pivot of $\Lambda_{\theta}$. This means that $\theta \in \text{Int}(\Delta(v))$. Therefore, $\theta$ is in the interior of some window $W$ that is a subset of $\Delta(v)$.

The above argument shows us that there are windows for us to pick. We next look for estimates on the area and slope of a window based on the generic $a$ we chose.

Let $n||n\alpha||||n\beta|| \geq \delta > 0$ for all $n$ and pick a generic $a \in A_+$ where $(t,s)$ are the coordinates of $a=e^{(t+s,t-s,-2t)}$. By the theorem 5.2, we can pick a window that contains $\theta$ based off of $a$. Let this window have dimensions $b$ and $c$. We start by determining the area of the window.

By Quillin, we know $\Delta(v)$ is contained in a hyperbola defined by $xy = \frac{1}{q^2}$, so our window is contained in this hyperbola as well, making area($W$) = $bc \leq \frac{1}{q^2}$ [6]. On the other hand, since $n||n\alpha||||n\beta|| \geq \delta > 0$, $\theta$ must lie outside of a hyperbola defined by $xy = \frac{\delta}{q^2}$. Since $W$ contains $\theta$, we get area($W$) $\geq \frac{\delta}{q^2}$. Putting these together yields
the following bounds on the area

\[ \frac{\delta}{q^3} \leq \text{area}(W) \leq \frac{1}{q^3} \]

We now move to creating bounds on the slope of the window. Using the same dimensions for the window as above, we know that the hyperbola created by \( n ||n\alpha|| n\beta \) cuts through the window at \( (\frac{\delta}{bq}, b) \) and \( (c, \frac{\delta}{cq}) \) with respect to \( \hat{v} \).

The slope of \( \theta - \hat{v} \) is \( \frac{||n\beta||}{||n\alpha||} \) with respect to \( \hat{v} \). So we have

\[ \frac{||n\beta||}{||n\alpha||} \leq \frac{b}{\delta} = \frac{b^2q^3}{\delta} \]

This gets us

\[ \frac{||n\beta||}{||n\alpha||} \leq \frac{bcq^3}{\delta} = \frac{1}{\delta} \]

because \( \text{area}(W) = bc \leq \frac{1}{q^3} \). The other side follows similarly, yielding

\[ 0 < \delta \leq \frac{\text{slope}(\theta - \hat{v})}{\text{slope}(W)} = \frac{||n\beta||}{||n\alpha||} \leq \frac{1}{\delta} \]

Taking the log of both sides and dividing by 2 results in

\[ \frac{1}{2} \log(\delta) \leq \frac{1}{2} \log(\text{slope}(W)) - s_* \leq \frac{1}{2} \log \left( \frac{1}{\delta} \right) \]

since \( s_* = \frac{1}{2} \log(\frac{||n\beta||}{||n\alpha||}) \). But \( |s_* - s| \leq \frac{1}{2} \text{diam} \Delta (v) = \frac{1}{2} \log(n ||n\alpha|| ||n\beta||) \leq \)
$\frac{1}{2} \log \left( \frac{1}{\delta} \right)$. So we can replace $s_*$ in the above slope bounds by $s + \frac{1}{2} \log \left( \frac{1}{\delta} \right)$ to get

$$\left| \frac{1}{2} \log (\text{slope}(W)) - s \right| \leq \log \left( \frac{1}{\delta} \right)$$

We will now choose our $a \in A_+$ such that $|s| < \log \left( \frac{1}{\delta} \right)$ so that $a$ is close to the $t$-axis. This gives us new bounds to our slope, namely

$$\left| \frac{1}{2} \log (\text{slope}(W)) \right| \leq 2 \log \left( \frac{1}{\delta} \right)$$

Multiplying by 2 and exponentiating yields

$$\delta^4 \leq \text{slope}(W) \leq \frac{1}{\delta^4}$$

Now we will get estimates on $q$ given any generic $a \in \tau_\theta(v) \subset \Delta_\theta(v)$. We know that $\frac{1}{2} \log(q) \leq t \leq \frac{1}{2} \log(q) - \frac{1}{2} \log(q ||q\alpha|| ||q\beta||) = \frac{1}{2} \log(q) - \frac{1}{2} \log \left( \frac{1}{\delta} \right)$ (see [3]).

Solving for $q$ gets us

$$\delta e^{2t} \leq q \leq e^{2t}$$

This allows us to update our bounds on $\text{area}(W)$ to

$$\delta e^{-6t} \leq \text{area}(W) \leq \frac{1}{\delta^3} e^{-6t}$$

So we now have bounds on both the slope and area that depend solely on the
t-value of our generic a given $|s| < \log\left(\frac{1}{\delta}\right)$. We can now use the bounds on the slope and area to get bounds on the dimensions of the windows. By noting that the area$(W) = bc$ and the slope$(W) = \frac{c}{b}$, we can multiply the area and the slope to get

$$\frac{\delta^5}{q^3} \leq c^2 \leq \frac{1}{\delta^4 q^3}$$

$$\implies \frac{\delta^2}{q^2} \leq c \leq \frac{1}{\delta^2 q^2}$$

Similarly, if we divide the area by the slope, we get

$$\frac{\delta^2}{q^2} \leq b \leq \frac{1}{\delta^2 q^2}$$

Now that we have bounds on the dimensions of the window, we would like to make sure that $W_{k+1}$ is bounded away from the sides of $W_k$ that are on the interior of $\Delta(v)$. We will prove this with $\rho = 2$. Note that since we have a counterexample, $||q_k\alpha|| \geq \frac{\delta}{cq_k} \geq \frac{\delta}{q_k^2}$. Similarly, $||q_k\beta|| \geq \frac{\delta}{q_k^2}$.

So this means we want $||q_k\beta|| \geq \frac{c^3}{q_k^2}$ greater than $\rho = 2$ times the x-dimension of $W_{k+1}$. We have a lower bound for the x-dimension of $W_{k+1}$, so we want

$$\frac{c^3}{q_k^2} > 2\left( \frac{1}{\delta^2 q_k^2} \right)$$

This is only true if
\[ \frac{q_k}{q_{k+1}} \geq \left( \frac{2}{\delta^2} \right)^{3\delta^2} \]

We may do this using the bounds for \( q \) we determined earlier and by choosing \( t_{k+1} \) to make this hold given \( t_k \).

To finish off being verifying that this is \( \rho \) nearly nested, we again let \( \rho = 2 \). We start by focusing on the x-dimension of \( W_k \) and \( W_{k+1} \). Let the x coordinate of \( v_k \) and \( v_{k+1} \) be \( x_k \) and \( x_{k+1} \) respectively. Also let the x-dimension of \( W_k \) and \( W_{k+1} \) be \( b_k \) and \( b_{k+1} \) respectively. Since \( v_{k+1} \in A_k \), we know that \( x_{k+1} < x_k + b_k \). So we want

\[ x_k + 2b_k > x_k + b_k + 2b_{k+1} > x_{k+1} + 2b_{k+1} \]

Using subtracting \( x_k + b_k \) from the first inequality tells us we want

\[ b_k > 2b_{k+1} \]

Using our estimates for the dimension of a window tells us we need

\[ \frac{\delta^2}{q_k^2} > 2 \frac{1}{\delta^2 q_{k+1}^2} \]

So given \( q_k \), we have estimates for \( q_{k+1} \) so we can choose \( a t \) that makes this true as well. The case for the y-dimension follows similarly. So we use this condition and
\[
\frac{q_k}{q_{k+1}} \geq \left( \frac{2}{\delta^5} \right)^\frac{2}{3}
\]

to choose the value of \(t_{k+1}\) given \(t_k\). This construction gives satisfies property (iv) in theorem 5.1. Since we are picking all the \(a \in A_+\) to be near the t-axis and wide apart, and we are continuing this process infinitely, we also get properties (i) and (iii). Since \(\theta \in W_k\) for all \(k\), we also have (v).

To check property (ii), we will use the Roll-back theorem. Let \(a \in B(W_{k-1}, W_k)\). From the Roll-back theorem and its corollary, we know that \(2 \text{sys}(a \Lambda_{\omega_{k-1}}) \geq \text{sys}(a \Lambda_\theta)\). But since \(n||n\alpha||n\beta|| \geq \delta > 0\), \(\text{sys}(a \Lambda_\theta) > \delta\). So we get

\[
\text{sys}(a \Lambda_{\omega_{k-1}}) \geq \frac{\delta}{2} > 0
\]

for all \(a \in B(W_{k-1}, W_k)\). But \(k\) was arbitrary, so this holds for all \(k\). Thus, property (ii) holds. Therefore, theorem 5.1 holds.
Chapter 6

Conclusion

The Littlewood conjecture has been an open problem since the 1930’s and has had many interpretations. We used the ideas of lattices, tilings, and diagonal matrix groups to examine this problem a little closer.

We first developed the Roll-back theorem and its corollary to find a connection between the sheared lattice of a point in $\mathbb{R}^d$ compared to the sheared lattice of one of its convergents. It turns out, they are very similar, only being blurred from one another by a factor of at most 2.

After this, we developed the notions of windows and boomerangs. These structures occur naturally in domains of approximation and the tilings of the $A_+$ groups respectively. These tools can be used to build a counterexample to the Littlewood conjecture, or can encode a given counterexample. Considering theorems 4.2 and 5.1, we may combine these two theorems to get

**Theorem 6.1.** Let $d=2$ and $\theta = (\alpha, \beta)$. $n||n\alpha|| ||n\beta|| \geq \delta > 0$ for all $n$ if and only
if there exists a sequence of windows \((W_k)_{k=1}^{\infty}\) with the following properties

(i) (stably monotone) \(A_-(W_k) \subset A_-(W_{k+1})\) for all \(k\).

(ii) (boomerang property) \(\exists \delta > 0\) such that \(\forall a \in B(W_k, W_{k+1})\) we have \(\text{sys}(a\Lambda) \geq \delta\)

(iii) (exhaustion) \(\bigcup_{k=1}^{\infty} A_-(W_k) \supset A_+\)

(iv) (\(\rho\) nearly nested) \(\exists \rho > 1\) such that \(\forall k \in \mathbb{Z}_+\) we have

\[
\rho W_{k+1} \subset \rho W_k
\]

where \(\rho W_k = \{\rho(w - \hat{v}) + \hat{v} : w \in W_k \text{ and } W_k \text{ is maximal in } \Delta(v)\}\)

(v) \(\bigcup_{k=1}^{\infty} \overline{\rho W_k} = \theta\)

This final theorem shows that a sequence of windows with this property when \(d=2\) is sufficient to build a counterexample, and every counterexample of the Littlewood conjecture comes from a sequence of windows with these properties.

But this leaves some questions. Is there a way to relax or refine any of these conditions? Do we need all of the windows, or can we get this from a more stringent or finite subset? Also, will this work for \(d \geq 2\)? These are some possible questions to investigate further in future research.
Bibliography


