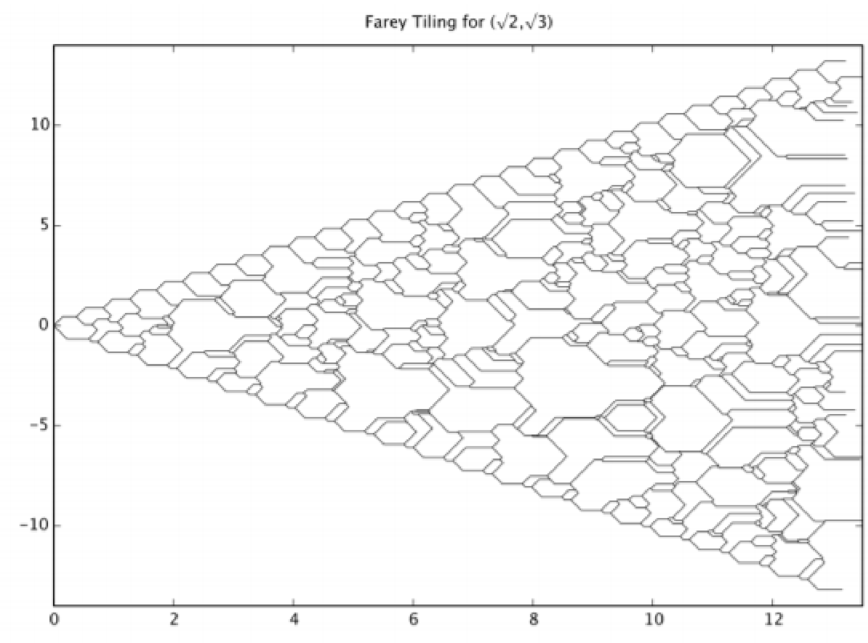
An Attempt To Understand Tilling Approach In proving The Littlewood Conjecture



Final Report For Math 899- Dr. Cheung

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Dr. Cheung I really appreciate your acceptance in adding me to your student's list this semester. I enjoyed your lecture, and I felt how much you believe in the truth and validity of The Littlewood Conjecture. First I register for Math 899 my main goal was to study the construction of other student's thesis project in our Math department as a preparation to my Master's Degree final project. What I felt during our meeting about your belief in the conjecture directed my learning effort to deeply looking at the concept of these papers.

I had a chance to read three students' thesis -Samantha Lui, Lucy Odom, and Deborah Damon- who worked directly with you. It was so challenging to understand every single detail since I believe that I'm not mathematically prepared yet also the time span was not enough, and lack of resources that cover the topic.

In this class I achieved the following objectives:

- Fully and deeply understanding of "Continued Fraction" as an important topic in Number Theory.
- A good understanding of the implication in the Littlewood conjecture and why it is true.
- A good understanding to the definitions of lattice, Pivots, shears, boxes in three diminutions, tilling and all Linear Algebra that is involved.

I totally understand that I should start my summary with Damon's thesis, but I believe that the same material that is required to write my report was covered during lecture.

An Attempt To Understand The Littlewood Conjecture "Tilling Approach"

Littlewood conjecture is an open problem (as of 2017) in Diophantine approximation, proposed by John Edensor Littlewood around 1930. It states that for any two real numbers α and β, $\liminf_{n\to\infty} n\,\|n\alpha\|\,\|n\beta\| = 0$. Where $||.||$ measures the distance to the nearest integer. This conjecture concerns the approximation of any two real numbers by rationals with the same denominator [5].

- ***Continued Fraction***

An Expression

$$a_0 + \cfrac{1}{a_{1+\frac{1}{a_{2+\frac{1}{a_{3+\dots}}}}}}\quad \frac{1}{a_n}$$

is called a regular finite continued fraction of a real number, where a_i such that $i \in \{1, 2, ..., n\}$ are positive integer except for a_0 might be a negative integer. Any real number can be expressed as a continued fraction using the Euclidean algorithm. If a real number is rational then it's continued fraction is finite; however, irrational numbers have infinite continued fraction expansion. Every real number has it's unique continued fraction representation. Extra to the advantage of knowing the value of a real number by determining its continued fraction representation, of course to some degree of accuracy, they reproduce properties of the real number they represent. For example, operating in irrationals, continued fractions will assure having a real number output by best approximating the irrational in the inputs [1].

Canonical representation:

A regular finite continued fraction of a real number α can be written in the form

$$[a_0; a_1, a_2, ..., a_n].$$

Stopping at any value of n, the different continued fractions $a_0, a_0+\frac{1}{a_1}, a_0+\frac{1}{a_{1+\frac{1}{a_{2+\dots}}}}$ are called the convergent of α. Therefore, $\frac{p}{q}$ is the n-ht order of convergent if

$$\frac{p}{q} = [a_0; a_1, a_2, ..., a_n] = [a_0; r_1] = a_0 + \frac{1}{r_1}$$

Where r_1 itself is the continued fraction $\frac{p_{n-1}}{q_{n-1}} = [a_1; a_2, a_3, ..., a_n]$. Then we can write α as

$$\frac{p}{q} = a_0 + \frac{p_{n-1}}{q_{n-1}} = \frac{a_0 + p_{n-1}}{q_{n-1}}$$

.

The Canonical Representation is useful in finding the $k-$ht order of convergent using the following recurrence relation

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2}$$

This recurrence relation can be easily proved by induction.

For the infinite continued fraction, it is important to test the convergence which in fact can be determined by the convergent of the series $\sum_{n=1}^{\infty} a_n$. The divergent of that series implies the convergent of the corresponding continued fraction. [1] Rational numbers have finite number of convergent while the irrationals have infinite numbers of convergent thus unique.

Convergent of a real number can be used as an approximation to that real number with some degree of accuracy. The following definitions will have the essential conditions to determine the best approximation

The rational number $\frac{a}{b}$ for $b > 0$ and $\gcd(a, b) = 1$ is a *Best Approximation of the First Kind* of a real number α if every other rational fraction with the same or smaller denominator differs from α by a greater amount, in other words, if the inequalities $0 < b \leq d$ and $a/b \neq c/d$ imply

$$\left|\alpha - \frac{c}{d}\right| > \left|\alpha - \frac{a}{b}\right|.$$

This definition follows naturally, but it is more convenient in Number Theory to consider the following definition:

The rational fraction a/b for $b > 0$ and $\gcd(a, b) = 1$is a *Best approximation of the second kind* of a real number α if the inequalities $\frac{c}{d} \neq \frac{a}{b}$ and $0 < d \leq b$ imply

$$\Big|d\alpha - c\Big| > \Big|b\alpha - a\Big|.$$

It is clear that best approximation 2^{nd} is derived by multiplying the difference in each absolute value in the best approximation 1^{st} kind with the denominator of the associated fraction. This implies that every best approximation of the second kind is a best approximation of the first kind, but the converse doesn't hold.

Lagrange Law of Best Approximates states that best approximates 2^{nd} kind to a real number α are precisely the convergences of α.

- **Best Approximation Geometric Interpretation.**

Davenport in his book *The Higher Arithmetic* includes a great interpretation to the geometric representation of the continued fraction best approximation. Let $\frac{p}{q}$ be the best approximation to the real number α. Consider the line $y = \alpha x$ that pass through the origin. A peg is inserted in every point in the plane with positive integers. Then a string is fixed at a certain point (remote point) from one side and the other side is fixed at the origin such that it coincides with $x = ay$. If we bull this string away from the line in both direction it will touch one of these pegs. The coordinate of this point is the numerator and the denominator of the best approximation to α.

The figure below shows p/q as best approximation to $a = \alpha$(hard to write α while drawing the graph). According to Davenport explanation (p, q) is the first peg the string touches when it is stretched a way from the line $x = ay$.

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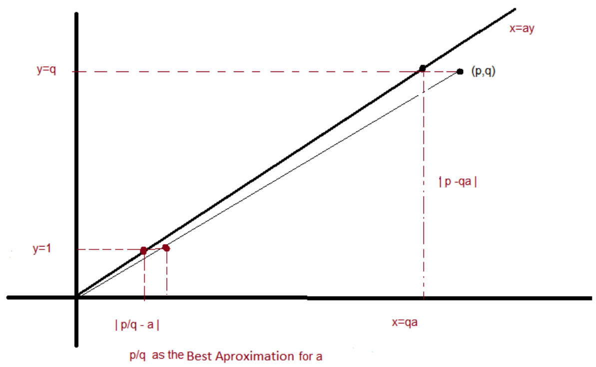


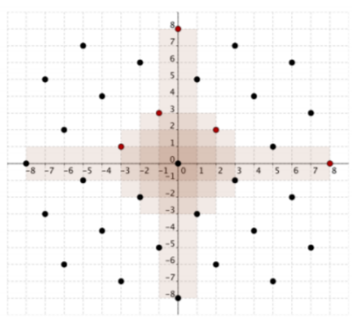
- **How Continued Fractions are related to the Littlewood Conjecture " Tilling Approach".**

The Littlewood conjecture in fact concerns the simultaneous approximation of two real numbers by rationals with the same denominator. The conjecture states that for any two real numbers α and β, $\liminf_{n\to\infty} n\,\|n\alpha\|\,\|n\beta\| = 0$ where n is the denominator of the pair (α, β). Dr. Cheung in his research creates a tilling approach to prove the Littlewood Conjecture. In this approach continued fraction best approximation for any real number is used to create boxes and rectangular with special measurements and properties related to the conjecture.

- ***Tilling in the Littlewood Conjecture.***

Dr. Cheung lecture helped me to understand how the continued fraction geometric interpretation for a single real number is used to construct a tile. To help me understand his approach he first introduce the idea of tilling in two dimensional space - a single continued fraction for a real number is used. Back to the figure above, if we create a parallel line to $x = ay$ starts at the point (p, q) -best approximation 2^{nd} kind to α down to the $x-$axis, and a horizontal line from that point to the same line then consider the reflection to these two lines around the line $x = \alpha y$ we will have a parallelogram. This parallelogram is divided by the line $x = \alpha y$ in to two congruent parts where the point (p, q) is a top corner point. The Dr Cheung introduced the flowing definitions:

- The shearer $\begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}$ is used as a linear transformation to the lines of the parallelogram which creates a rectangle with sides parallel to both axis.
- The Lattice Λ in $\mathbb{R}^2$is a basis with integer points. In other words, it is a $\mathbb{Z}-$module of rank two. The coordinate of (p,q) is shifted to the point $(p-q\alpha,q)$ which represent the best approximation to α.
- The shear $\begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}$ is used as a linear transformation to the lines of the parallelogram which creates a rectangle with sides parallel x and y axis.
- The point $h_\alpha = (p-q\alpha,q)$ is called a pivot of Λ_α if it is the only nonzero lattice point in the rectangle.
- The figure below is the lattice and pivots points associated to $(1,5,8)$ [6].



$\mathcal{L}(1,5,8)$

- ***Lok Shum Lui thesis Project "Tiling Problem For Littlewood Conjecture"***

Lui in her project used the continued fraction expansion of two irrational numbers (α,β) to formulate a tiling problem of the plain that is equivalent to the Littlewood conjecture. She showed that $\mathbb{R}^2$is covered by the non- overlapping tiles where the convergent of the continued fraction is used as a motivation to define a pivot of the tilling. Pivots are used to determinate the size of the tilling and more precisely she showed that the diameter of the tile is comparable to $-\log n \|n\alpha\| \|n\beta\| = 0$ where n is the denominator of the associated convergent of the pair (α,β).

Lui include the following definition and observations:

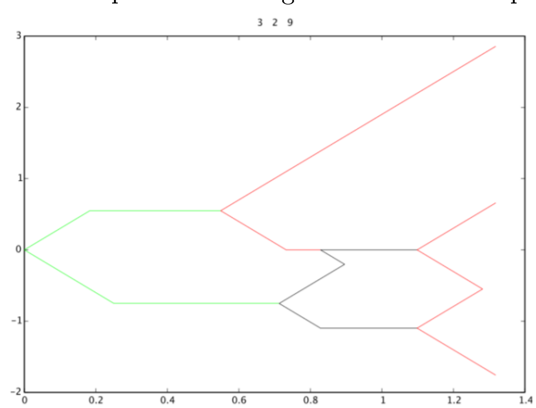
- A lattice $\Lambda_{\alpha,\beta}$, a shear transformation,and a piecewise functions $f(u)$ that is associated $u \in \mathbb{Z}^3 \setminus \{0\}$ where $u = (m_1, m_2, n)$ is the vector that is used to define the tile of the tilling.
- The vector u is (α, β) good if m_1 is the nearest integer to $n\alpha$ and m_2 is the nearest integer to n_β.
- A rectangle $R(u)$ maybe think about it as a three-dimensional object. She defined the boundaries and the corner of $R(u)$.
- The vector $u \in \Lambda$ is a pivot if it is the non-zero point in the corner of i.e is the only non-zero integer point the lattice.
- The tile associated to nonequivalent pivots of $\Lambda_{\alpha,\beta}$ cover $\mathbb{R}^2$and they don't overlap.
- If $h_{\alpha,\beta}u$ is a pivot of the lattice $\Lambda_{\alpha,\beta}$ then u is (α, β) good and it is the convergent of (α, β) .
- There are two types of pivoted: degenerate and non-degenerate. The degenerate pivots has one or more *zero* coordinates the only degenerate pivots of the lattice $\Lambda_{\alpha,\beta}$ are the vectors $\pm e_1$ and $\pm e_2$.
- If $h_{\alpha,\beta}u$ is a pivot of the lattice $\Lambda_{\alpha,\beta}$ then u is a non-degenerate pivot of $\Lambda_{\alpha,\beta}$ with the diameter $|u|$.
- The diameter $|u|$ of the non-degenerate pivot $h_{\alpha,\beta}u$ is equivalent to $-\log n \|n\alpha\| \|n\beta\|$ which is comparable to the Littlewood Conjecture.
- The irrational pairs (α, β) is a counter example of the Littlewood Conjecture iff the diameter of the tile associated with the degenerate pivots are uniformly unbounded.
- Dr. Cheung states the following theorem when I asked him about the relation of pivots of a box and best approximation

Theorem. *(Analogue to Lagrange) The pair $\left(\frac{p_1}{q}, \frac{p_2}{q}\right)$ is a pivot of (α, β) iff it is the best approximation with respect to some rectangular norm.*

- ***Lucy Hanh Odom thesis project "An Overlap Criterion For The Tiling Program Of The Littlewood Conjecture"***

Odom studies the tile of a pair of rational numbers (α, β) as an attempt to understand the tilling of rationals since rationals has finite convergent hence finite tilling which is going to be more convenient to study and discover. I will summaries his work and discoveries as follows:

- For the rational pair $\left(\frac{p_1}{q}, \frac{p_2}{q}\right)$ Lui's result is preserved if the $\gcd(p_1, q) = 1$ and $\gcd(p_2, q) = 1$ i.e there exist a non-overlapping in the tiling associated to that rational pair.
- If the rational pair $\left(\frac{p_1}{q}, \frac{p_2}{q}\right)$ is not totally reduced i.e the $\text{gad}(p_1, q) \neq 1$ and $\text{gad}(p_2, q) \neq 1$ the associated tile will overlap.
- She used the pair $\left(\frac{1}{3}, \frac{2}{9}\right)$ as an example of overlapping of two degenerate tile. The figure below shows the tiling in red where the overlap takes place in a portion of the portion of the green with another portion of the black.



- The irrational pair $(\sqrt{2}, \sqrt{3})$ is tiled by using different order of convergents. The tile doesn't terminate and the upper and lower boundary lines has slopes $\sqrt{2}$, $\sqrt{3}$ respectively. This is due to their infinite Canonical Representations $\sqrt{2} = [1; 2, 2, ...]$ and $\sqrt{3} = [1; 1, 2, 1, ...]$.

- Odom investigates the cause of the tiles overlapping and define the following criterion that reformulate and extend the original tiling to a higher dimension.

Theorem. *(Overlap Criterion). For any $\theta \in \mathbb{R}$, let $\Lambda_\emptyset$ denote the sheared integer lattice* $\Lambda_\theta := \begin{pmatrix} 1 & & & -\theta \\ & \ddots & & \vdots \\ & & 1 & -\theta_d \\ & & & 1 \end{pmatrix} \mathbb{Z}^{\dagger|}$ *. Then no two tiles in n $\tau(u) : u \in \prod(\Lambda_\emptyset)$ overlap provided each coordinate of $\emptyset$ is irrational, i.e.$\emptyset \in (\mathbb{R} \setminus \mathbb{Q})^d$, or if $\emptyset \in \mathbb{Q}$and has uniform height, i.e, gcd $(pi, q) = 1$, and where $\theta i = pi/q$.*

- The first part of the overlapping criterion shows that there is no overlap for a pair of irrationals $\theta = (\alpha, \beta)$. The second part states that the overlap of a pair of rationals exist when rational pair $(\alpha, \beta) = \left(\frac{p_1}{q}, \frac{p_2}{q}\right)$ has uniform height or not reduced totally.

Deborah Damon thesis project "Lattices Of Approximation In Tow Dimensions"

I have to state in this report that Damon thesis was a motivation to Lui and Odom thesis project. In her project 2011 she first introduced pivots and lattices. Her discoveries to what she called Strand was a so important to build other students thesis and projects of the same topic.

Damon defined a Strand as a sequence of positive integers that is inherent to each lattice. Strands - as she stated- are so important strand to describe the lattice, determine the ratios of coordinates of a lattice's pivots in terms of continued fractions and also to provide the coefficients for the recurrence relation among the pivots [6].