LIMITS OF FUNCTIONS ON TREES

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CERTIFICATION OF APPROVAL

I certify that I have read *LIMITS OF FUNCTIONS ON TREES* by Cyrous Ali Ahmadieh and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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We present various formulations for the limit of a function from a tree to the reals. The formulations will not only generalize the notion of limit of a sequence of reals, but will also have potential applications in obtaining upper and lower bounds for the Hausdorff dimension of certain classes of Cantor sets. One formulation will involve the branches of the tree, another the maximal antichains, and the third will involve the nonstandard concept of a slice. Our branch formulation for limit will be the simplest to define, while the other two formulations will require us to make both the set of maximal antichains and the set of slices into directed sets. We then study and compare these and notice that they do not always agree (in value), although we do get some nice inequalities. While investigating the interactions amongst the branches, maximal antichains, and slices, we shall see that we can get equality between some or all of our formulations by placing different sorts of mild restrictions on the function or the tree. We end by discussing a particular class of trees, called the omega-trees, which are useful for representating Cantor sets, and then use our observations to reveal a more constructive proof for one of our earlier lemmas restricted to the class of pruned omega-trees.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Chapter 0

Introduction

This chapter is a brief look at what this paper is all about. To aid us in our discussion, we will be including some definitions and results which will be stated without justification.

Important : All definitions, propositions, and theorems presented in this chapter will be reintroduced and justified and/or elaborated upon throughout the course of the paper. Moreover, outside of this chapter, no references will be made to anything specific in this chapter. To further emphasize this fact, we will not number the definitions, propositions, and theorems of this chapter.

0.1 Purpose and Motivation

Let X be a topological space.

Definition. By a tree T, we mean a pair $(T, \prec)$ such that $\prec$ is a strict partial order of T and, for all $y \in T$, $\{x \in T : x \prec y\}$ is well-ordered by $\prec$.

For ease of discussion, we will refer to a function $f : T \to X$, where T is a tree, as a tree in X. We want to think of a tree in X as a generalization of a sequence in X, in much the same way that a *net* is a generalization of a sequence:

Definition. By a directed set A, we mean a pair $(A, \sqsubseteq)$ s.t. $\sqsubseteq$ is transitive and reflexive on A; and furthermore, given $a, b \in A$, there is a $c \in A$ s.t. $a \sqsubseteq c$ and $b \sqsubseteq c$.

Definition. A net (in X) is a function $\mathcal{G} : A \to X$ where A is a directed set.

As with nets, we want our formulation for *limit of a tree in* X to generalize the notion of limit of a sequence in X. But we'd like more than this.

When $X = \mathbb{R}$ (with the standard topology) there is a natural way to define the notions of *lim sup* and *lim inf* of a net s.t. the following three properties hold.

Proposition. *Let $\mathcal{G}$ be a net in $\mathbb{R}$. We then have:*

1. *lim inf* $\mathcal{G} \leq$ *lim sup* $\mathcal{G}$
2. *lim inf* $\mathcal{G} =$ *lim sup* $\mathcal{G}$ *iff lim* $\mathcal{G}$ *exists.*
3. *The limit of any convergent subnet is in the interval* [*lim inf* $\mathcal{G}$, *lim sup* $\mathcal{G}$] *(cf Definitions D.2 and D.4).*

We wish to apply our results in the context of computing Hausdorff dimension and have analogs of *lim sup* and *lim inf* for trees in X s.t. they generalize the usual notions whenever the tree is well-ordered, and s.t. 1, 2, and 3 hold. However, since it is not clear what the analogs would be in an arbitrary topological space, we shift our attention to trees in $\mathbb{R}$. (In fact, we do not expect there to be analogs when $X \neq \mathbb{R}$).

The goal is to develop various formulations for *lim*, *lim sup*, and *lim inf* of a tree in $\mathbb{R}$ (which generalize the usual notions and satisfy 1, 2, and 3); and furthermore, to compare and contrast these formulations.

0.2 Formulations

Let T be a tree and $f : T \to \mathbb{R}$ a function.

There are many different competing formulations for *lim sup* f that might serve to generalize the notion of lim sup of a sequence in $\mathbb{R}$. We discuss and compare three such formulations which (we believe) are also promising in satisfying 1, 2, and 3; and moreover, are potentially useful in the context of computing Hausdorff dimension.

0.2.1 Branch Approach

Definition. $\beta \subseteq T$ is a branch if β is a maximal chain.

Definition. We denote the set of branches by $B(T)$.

Definition. sup lim sup $f_\beta := \sup\{\limsup f \upharpoonright \beta : \beta \in B(T)\}$. In other words, look at the lim sup along each branch, and then take sup lim sup f_β to be the supremum of all such lim sup's.

Our *Branch Formulation* is to define *lim sup* f to be *sup lim sup* f_β.

0.2.2 Maximal Antichain Approach

Definition. $A \subseteq T$ is an antichain if all elements of A are pairwise incompatible.

Definition. We denote the set of maximal antichains by $\mathfrak{M}(T)$.

Definition. Given $A, A' \in \mathfrak{M}(T)$, we write $A \sqsubseteq A'$ if for every $x \in A$ there is some $y \in A'$ s.t. $x \preccurlyeq y$.

Proposition. $\mathfrak{M}(T)$ *is a directed set with respect to* $\sqsubseteq$.

Definition. Given any $C \subseteq T$, we let $F(C) = \{x \in T : \exists y \in C[y \preccurlyeq x]\}$.

Definition. We define the net $\hat{f} : \mathfrak{M}(T) \to \overline{\mathbb{R}}$ s.t:

$$\hat{f}(A) = \sup\{f(x) : x \in F(A)\}$$

Notice that $\hat{f}$ is a monotone decreasing net, so it always converges (to an extended real number). Our *Maximal Antichain Formulation* is to define *lim sup* f to be *lim* $\hat{f}$.

0.2.3 Slice Approach

Definition. $\alpha \subseteq T$ is a slice if α is an antichain and $\alpha \cap \beta \neq \emptyset$ for each $\beta \in B(T)$.

Definition. We denote the set of slices by $A(T)$.

Proposition. *$A(T)$ is a directed set with respect to $\sqsubseteq$.*

Definition. We define the net $\bar{f} : A(T) \to \overline{\mathbb{R}}$ s.t:

$$\bar{f}(\alpha) = \sup\{f(x) : x \in F(\alpha)\}$$

Notice that $\bar{f}$ is a monotone decreasing net, so it always converges (to an extended real number). Our *Slice Formulation* is to define *lim sup* f to be *lim* $\bar{f}$.

0.2.4 A Comparison

Although there is an ω-tree and function $f : T \to \mathbb{R}$ s.t. each of sup lim sup f_β, *lim* $\hat{f}$, and *lim* $\bar{f}$ takes a different value, our main result says:

Theorem. *Given any tree T and function $f : T \to \mathbb{R}$ we have:*

$$lim\ \hat{f} \leq sup\ lim\ sup\ f_\beta \leq lim\ \bar{f}$$

.

Apart from proving the above theorem, we shall also explore placing different kinds of *mild* restrictions on the function or the tree which will give equality between some or all three formulations. The order in which we will (re)introduce the formulations is different than the order given here. We will first set about introducing the branch and slice formulations, without making any mention of maximal antichains. We will then compare these two formulations, provide a counterexample, and discuss some restrictions which give equality. After doing all this, we bring the maximal antichain formulation into the picture and compare this formulation alongside the other two. The last two chapters are concerned with ω-trees.

Chapter 1

Trees

We assume knowledge of basic analysis and set theory. In Chapters 7 and 8, additional knowledge of basic topology is required.

We begin with some standard definitions; most all of which are from Kunen [1]. These are the bare minimum and are the ones most frequently used throughout. The remainder of the notation and terminology which appear are discussed in Appendix A.

1.1 Set-Theoretic Trees

Definition 1.1. A tree is a pair $(T, \prec)$ such that $\prec$ is a strict partial order of T and, for all $y \in T$, $\{x \in T : x \prec y\}$ is well-ordered by $\prec$.

Then for all $x, y \in T$:

- $y\downarrow = \{x \in T : x \prec y\}$

- $y\uparrow = \{x \in T : y \prec x\}$
- $x \preccurlyeq y \iff x \prec y \vee x = y$
- x and y are compatible if $x \prec y \vee y \prec x \vee x = y$.
- incompatible means not compatible.
- $A \subseteq T$ is an antichain if all elements of A are pairwise incompatible.
- $\beta \subseteq T$ is a branch if β is a maximal chain.

Remark 1.2. *Given a tree $(T, \prec)$, it follows that $\prec$ is well-founded on T. (For justification see Appendix A).*

1.2 Slices and Branches

Let T be a tree.

Definition 1.3. We denote the set of branches by $B(T)$.

Definition 1.4. $\alpha \subseteq T$ is a slice if α is an antichain and $\alpha \cap \beta \neq \emptyset$ for each $\beta \in B(T)$.

Remark 1.5. *It follows easily from Definition 1.4 and the definition of a branch, that a slice intersects each branch at precisely one place.*

Definition 1.6. We denote the set of slices by $A(T)$.

Definition 1.7. Given $C \subseteq T$, we let $C_{\prec} = \{x \in T : x \text{ is } \prec\text{-minimal in } C\}$.

1.2.1 Some Easy Consequences

Proposition 1.8. *Let $\beta \in B(T)$ and $x, y \in T$. If $x \preccurlyeq y$ and $y \in \beta$, then $x \in \beta$.*

Proof. The case where $x = y$ is trivial, so suppose we have $x \prec y$ and $y \in \beta$.

Take any $z \in \beta$. If $y \prec z$, then $x \prec z$ by transitivity. If $z = y$, then $x \prec z$ by the assumption that $x \prec y$. And if $z \prec y$, then x and z are compatible since $y\downarrow$ is totally ordered by $\prec$. In any case x and z are compatible. Since $z \in \beta$ was arbitrary, then x is compatible with each member of β. Therefore, since β is a maximal chain, we must have $x \in \beta$.

□

Proposition 1.9. *Let $C \subseteq T$ s.t. $C \cap \beta \neq \emptyset$ for each $\beta \in B(T)$. Then $C_\prec$ is a slice.*

Proof. It is easy to see from Definition 1.7 that $C_\prec$ is an antichain.

Now take any $\beta \in B(T)$. Since $C \cap \beta \neq \emptyset$, then by Remark 1.2, $C \cap \beta$ has a $\prec$-minimal member, call it y. If there were some $x \in C$ s.t. $x \prec y$, then we'd have $x \in \beta$ by Proposition 1.8. But this contradicts the $\prec$-minimality of y. Therefore y is also $\prec$-minimal in C, and so $y \in C_\prec \cap \beta$. This shows that $C_\prec$ intersects every branch and so is a slice.

□

Remark 1.10. *By an easy application of Zorn's Lemma: Given any $x \in T$ (or more generally chain $\mathcal{C} \subseteq T$), there is some $\beta \in B(T)$ s.t. $x \in \beta$ (resp. $\mathcal{C} \subseteq \beta$).*

Chapter 2

Slices

Let T be a tree.

The reader that does not know of directed sets and nets should become familiar with Appendix D before continuing.

2.1 Ordering the Slices

In this section we define an ordering $\preccurlyeq$ on $A(T)$, which makes $(A(T), \preccurlyeq)$ into a directed set.

Definition 2.1. Given any $C \subseteq T$, we let $F(C) = C \cup \bigcup_{y \in C} y \uparrow = \{x \in T : \exists y \in C[y \preccurlyeq x]\}$.

Definition 2.2. We define a function $\Cap : A(T) \times B(T) \to T$ s.t. $\Cap(\alpha, \beta) = \alpha \Cap \beta =$ the unique $z \in T$ s.t. $z \in \alpha \cap \beta$. (This is legitimate by Remark 1.5).

Definition 2.3. Given $\alpha, \alpha' \in A(T)$, we write $\alpha \preccurlyeq \alpha'$ if $\alpha \Cap \beta \preccurlyeq \alpha' \Cap \beta$ for every $\beta \in B(T)$.

This would be a good point for the reader to become familiar with Appendix C, but not necessary.

Proposition 2.4. *$A(T)$ is a directed set with respect to $\preccurlyeq$.*

Proof. Reflexivity is obvious.

For transitivity: Let $\alpha_0, \alpha_1, \alpha_2 \in A(T)$ s.t. $\alpha_0 \preccurlyeq \alpha_1$ and $\alpha_1 \preccurlyeq \alpha_2$, and take $\beta \in B(T)$ arbitrary. We then have $\alpha_0 \Cap \beta \preccurlyeq \alpha_1 \Cap \beta$ and $\alpha_1 \Cap \beta \preccurlyeq \alpha_2 \Cap \beta$. If $\alpha_0 \Cap \beta = \alpha_1 \Cap \beta$ or $\alpha_1 \Cap \beta = \alpha_2 \Cap \beta$, then clearly $\alpha_0 \Cap \beta \preccurlyeq \alpha_2 \Cap \beta$ in either case. Otherwise, $\alpha_0 \Cap \beta \prec \alpha_1 \Cap \beta$ and $\alpha_1 \Cap \beta \prec \alpha_2 \Cap \beta$, in which case we have $\alpha_0 \Cap \beta \prec \alpha_2 \Cap \beta$ by transitivity of $\prec$.

For the upper bound property: Take any $\alpha_0, \alpha_1 \in A(T)$. For each $\beta \in B(T)$, let x_β be the $\prec$-maximum of $\alpha_0 \Cap \beta$ and $\alpha_1 \Cap \beta$. Let $C = \{x_\beta : \beta \in B(T)\}$. Notice that C meets every branch and also $C \subseteq F(\alpha_0), F(\alpha_1)$. Take any distinct $y, z \in C$ and suppose to the contrary that $y \prec z$. We know then that z is the $\prec$-maximum of $\alpha_0 \Cap \beta$ and $\alpha_1 \Cap \beta$ while $\alpha_0 \Cap \beta', \alpha_1 \Cap \beta' \preccurlyeq y$ for some $\beta, \beta' \in B(T)$. If $z = \alpha_0 \Cap \beta$, then we have $\alpha_0 \Cap \beta' \prec \alpha_0 \Cap \beta$ by transitivity, contradicting the fact that α_0 is an antichain. Likewise, if $z = \alpha_1 \Cap \beta$, then we have $\alpha_1 \Cap \beta' \prec \alpha_1 \Cap \beta$, contradicting the fact that α_1 is an antichain. In either case we have a contradiction. Therefore C is an antichain which meets every branch (ie $C \in A(T)$). And furthermore, we have $\alpha_0 \preccurlyeq C$ and $\alpha_1 \preccurlyeq C$ by Corollary C.8. □

2.2 Formulations

In the definitions that follow, the reader must take care in noticing the pathologies which may arise and make appropriate adjustments. For instance, we will define nets which may take the value ∞, but in our definitions of net and eventual upper bound (cf Appendix D) we made no reference to the extended reals.

Let T be a tree and $f : T \to \mathbb{R}$ a function. $\overline{\mathbb{R}}$ denotes the extended reals.

2.2.1 Our Nets

The following definitions are legitimate by Proposition 2.4 and the fact that branches are totally ordered:

Definition 2.5. We define the net $f^+ : A(T) \to \overline{\mathbb{R}}$ s.t:

$$f^+(\alpha) = \sup_{x \in \alpha} f(x) = \sup\{f(x) : x \in \alpha\}$$

Definition 2.6. We define the net $\bar{f} : A(T) \to \overline{\mathbb{R}}$ s.t:

$$\bar{f}(\alpha) = \sup_{x \in F(\alpha)} f(x) = \sup\{f(x) : x \in F(\alpha)\}$$

Proposition 2.7. *$\bar{f}$ is monotone decreasing.*

Proof. Let $\alpha, \alpha' \in A(T)$ arbitrary and suppose $\alpha \preccurlyeq \alpha'$. Then $F(\alpha') \subseteq F(\alpha)$ by

Corollary C.8, and so we have:

$$\bar{f}(\alpha') = \sup_{x \in F(\alpha')} f(x) \leq \sup_{x \in F(\alpha)} f(x) = \bar{f}(\alpha)$$

.

□

Proposition 2.8. *If inf* $\{\bar{f}(\alpha) : \alpha \in A(T)\} \in \mathbb{R}$*, then lim* $\bar{f}$ = *inf* $\{\bar{f}(\alpha) : \alpha \in A(T)\}$.

Proof. Let $I = \inf \{\bar{f}(\alpha) : \alpha \in A(T)\}$ and $\epsilon > 0$ arbitrary. Then there is some $\bar{f}(\alpha') \in \{\bar{f}(\alpha) : \alpha \in A(T)\}$ s.t. $\bar{f}(\alpha') < I + \epsilon$. By Proposition 2.7, we have $\alpha' \preccurlyeq \alpha$ and $\alpha \in A(T)$ implies $\bar{f}(\alpha) \leq \bar{f}(\alpha') < I + \epsilon$. And by definition of the infimum, we also have $I \leq \bar{f}(\alpha)$ given any $\alpha \in A(T)$. Thus $\alpha' \preccurlyeq \alpha$ and $\alpha \in A(T)$ implies $I - \epsilon < I \leq \bar{f}(\alpha) \leq \bar{f}(\alpha') < I + \epsilon$ ie $|\bar{f}(\alpha) - I| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\lim \bar{f} = I$.

□

2.2.2 Branch and Slice Formulations

If we define $\lim \bar{f} = -\infty$ whenever $\inf \{\bar{f}(\alpha) : \alpha \in A(T)\} = -\infty$, and $\lim \bar{f} = \infty$ whenever $\inf \{\bar{f}(\alpha) : \alpha \in A(T)\} = \infty$ we then have:

$$\lim \bar{f} = \lim\sup \bar{f} = \lim\inf \bar{f}$$

by Propositions 2.8 and D.8.

Definition 2.9.

$$\sup \limsup f_\beta := \sup_{\beta \in B(T)} \limsup f \restriction \beta = \sup\{\limsup f \restriction \beta : \beta \in B(T)\}$$

We shall take our slice and branch formulations for $\limsup f$ to be:

$$\lim \bar{f}$$

and

$$\sup \limsup f_\beta$$

respectively.

2.2.3 Defining the Limit

The following is a rather informal discussion on how we would define our slice and branch formulations for $\liminf f$ and $\lim f$ in accordance with the discussion in the introduction:

We shall take our slice formulation for $\liminf f$ to be $\lim \underline{f}$, where the function $\underline{f}$ is defined analogously to $\bar{f}$ by just swapping 'sup' with 'inf' in the definition.

We shall take our slice formulation for $\lim f$ to be defined in terms of $\limsup f$ and $\liminf f$, namely we will say that our slice formulation for $\lim f$ is defined just in case lim $\bar{f}$ = lim $\underline{f}$, in which case we define it to be equal to this value.

Likewise, we shall take our branch formulation for lim inf f to be inf lim inf f_β, which is defined analogously to sup lim sup f_β by just swapping 'sup' with 'inf' in the definition.

And likewise, we shall take our branch formulation for lim f to be defined just in case sup lim sup f_β = inf lim inf f_β, in which case we define it to be equal to this value.

Important : We need not worry that these definitions are informal; we will not be referring to the slice or branch formulations for lim f or lim inf f outside of this subsection. In the next two chapters our primary focus will be on comparing lim $\bar{f}$ and sup lim sup f_β. We will introduce the maximal antichain formulation for lim sup f in Chapter 5.

Chapter 3

First Comparisons

We will now begin to compare $\lim \bar{f}$ with $\sup \limsup f_\beta$. In the proofs of the theorems in this chapter and the next, the reader will notice that we are actually comparing $\limsup f^+$ and $\sup \limsup f_\beta$. However, it really makes no difference since we have $\lim \bar{f} = \limsup \bar{f} = \limsup f^+$ by Proposition 3.2 (see below). We choose to work with $\limsup f^+$ rather than $\limsup \bar{f}$ because the definition of f^+ is easier to work with. This is one reason we introduced f^+ in Definition 2.5 in the first place. The other reason will become apparent in the next chapter.

Let T be a tree and $f : T \to \mathbb{R}$ a function.

3.1 Chance for Equality

Lemma 3.1. *Let $\alpha \in A(T)$. Then for every $m \in F(\alpha)$ there is some $\alpha' \in A(T)$ s.t. $m \in \alpha'$ and $\alpha \preccurlyeq \alpha'$.*

Proof. Let $\alpha \in A(T)$ and take any $m \in F(\alpha)$. This means that $n \preccurlyeq m$ for some $n \in \alpha$. If $n = m$, then simply take $\alpha' = \alpha$. So suppose $n \prec m$. Let

$$C = \{y \in T \setminus m\downarrow\, : n \prec y \wedge y\downarrow\, \subseteq m\downarrow\}$$

and then set

$$\alpha' = C \cup \alpha \setminus \{n\}$$

We need to see that $m \in \alpha'$, $\alpha' \in A(T)$, and $\alpha \preccurlyeq \alpha'$. To see that $m \in \alpha'$, notice that $m \not\prec m$ while also $n \prec m$ and $m\downarrow \subseteq m\downarrow$. To see that α' is a slice:

First, notice that $\alpha \setminus \{n\}$ is an antichain since it is a subset of the antichain α. Now, take any two distinct $x, y \in C$. If say, $x \prec y$, then $x \in y\downarrow\, \subseteq m\downarrow$, which contradicts the fact that $x \in C$. Therefore C is an antichain. Let $y \in C$. Since $n \prec y$ and n is incompatible with each member of $\alpha \setminus \{n\}$, then y must also be incompatible with each member of $\alpha \setminus \{n\}$. Therefore α' is an antichain.

Take any $\beta \in B(T)$. If $\alpha \Cap \beta = y \neq n$, then $y \in \alpha \setminus \{n\}$ giving us $y \in \alpha' \cap \beta$; and if $m \in \beta$, then since $m \in \alpha'$, we have $m \in \alpha' \Cap \beta$. So suppose $\alpha \Cap \beta = n$ and $m \notin \beta$. Suppose to the contrary that whenever $y \in \beta$ and $n \prec y$, then $y \in m\downarrow$. Since $n \prec m$, then $y \prec n$ implies $y \prec m$ by transitivity. Therefore m is compatible with each member of β, contradicting maximality of β. Therefore $\{y \in T : y \in \beta \wedge n \prec y \wedge y \notin m\downarrow\} \neq \emptyset$. So let $y \in T$ be $\prec$-minimal s.t. $y \in \beta$, $n \prec y$, and $y \notin m\downarrow$. Now let $x \in T$ s.t. $x \prec y$. Then $x \in \beta$ by Proposition 1.8. If

$n \prec x$, then we must have $x \in m\downarrow$ by $\prec$-minimality of y; while if $x \preccurlyeq n$, then we also have $x \in m\downarrow$ by transitivity. Hence $y\downarrow \subseteq m\downarrow$. Therefore $y \in C \cap \beta$.

This completes the argument that $\alpha' \in A(T)$. Finally, by definition of α', it is easy to see that given any $x \in \alpha$ there is some $y \in \alpha'$ s.t. $x \preccurlyeq y$. Therefore $\alpha \preccurlyeq \alpha'$ by Corollary C.8.

□

Proposition 3.2. *lim sup* $\bar{f}$ = *lim sup* f^+.

Proof. We show that $\bar{f}$ and f^+ have the same eventual upper bounds.

Let $U \in \mathbb{R}$ an arbitrary eventual upper bound for $\bar{f}$ and take any $\epsilon > 0$. Then there is some $\alpha' \in A(T)$ s.t. $\alpha' \preccurlyeq \alpha$ implies $\bar{f}(\alpha) < U + \epsilon$. So take any $\alpha \in A(T)$ where $\alpha' \preccurlyeq \alpha$ and consider any $f(n)$ where $n \in \alpha$. Then clearly $n \in F(\alpha)$ as well, and so we have $f(n) \leq \bar{f}(\alpha) < U + \epsilon$. Therefore $f^+(\alpha) \leq \bar{f}(\alpha) < U + \epsilon$ whenever $\alpha' \preccurlyeq \alpha$. Since $\epsilon > 0$ was arbitrary, then U is an eventual upper bound for f^+.

Now, let $U \in \mathbb{R}$ an eventual upper bound for f^+ and take any $\epsilon > 0$. Then there is some $\alpha' \in A(T)$ s.t. $\alpha' \preccurlyeq \alpha$ implies $f^+(\alpha) < U + \epsilon$. So take any $\alpha \in A(T)$ where $\alpha' \preccurlyeq \alpha$ and consider any $f(n)$ where $n \in F(\alpha)$. Then by Lemma 3.1, there is some $\alpha^* \in A(T)$ s.t. $n \in \alpha^*$ and $\alpha \preccurlyeq \alpha^*$. By transitivity, we have $\alpha' \preccurlyeq \alpha^*$, which implies that $f(n) \leq f^+(\alpha^*) < U + \epsilon$. Therefore $\bar{f}(\alpha) \leq U + \epsilon$ whenever $\alpha' \preccurlyeq \alpha$. Since $\epsilon > 0$ was arbitrary, then U is an eventual upper bound for $\bar{f}$.

□

Theorem 3.3. *sup lim sup* $f_\beta \leq$ *lim* $\bar{f}$

Proof. Take $U \in \mathbb{R}$ an eventual upper bound for f^+ and take any $\beta \in B(T)$. Let $\epsilon > 0$ arbitrary. Then there is $\alpha' \in A(T)$ s.t. $\alpha' \preccurlyeq \alpha$ implies $f^+(\alpha) < U + \epsilon$. By Lemma 3.1, taking any $n \in \beta$ with $\alpha' \Cap \beta \preccurlyeq n$, we can find $\alpha^* \in A(T)$ s.t. $n \in \alpha^*$ and $\alpha' \preccurlyeq \alpha^*$; in which case it follows that $f(n) \leq f^+(\alpha^*) < U + \epsilon$. Since $\epsilon > 0$ was arbitrary, then U is an eventual upper bound for $f \restriction \beta$. As a result, we have lim sup $f \restriction \beta \leq U$. Since $\beta \in B(T)$ was arbitrary, we have sup lim sup $f_\beta \leq U$. Since U was an arbitrary eventual upper bound for f^+, we have sup lim sup $f_\beta \leq$ lim sup f^+.

□

It is worth noting that we can directly show sup lim sup $f_\beta \leq \lim \bar{f}$ without any mention of f^+ or appeal to Lemma 3.1. In fact, it is an easy conseqeunce of the following:

Proposition 3.4. *sup lim sup $f_\beta \leq \bar{f}(\alpha)$ for every $\alpha \in A(T)$.*

Proof. Let $\alpha \in A(T)$ and take any $\beta \in B(T)$. Notice that $\alpha \Cap \beta \preccurlyeq z$ and $z \in \beta$ (which is equivalent to saying $z \in F(\{\alpha \Cap \beta\}) \cap \beta$) implies $f(z) \leq \sup\{f(x) : x \in F(\{\alpha \Cap \beta\}) \cap \beta\} \leq \bar{f}(\alpha) \leq \bar{f}(\alpha) + \epsilon$ whenever $\epsilon > 0$. Therefore $\bar{f}(\alpha)$ is an eventual upper bound for $f \restriction \beta$, which gives us lim sup $f \restriction \beta \leq \bar{f}(\alpha)$. Since $\beta \in B(T)$ was arbitrary, then sup lim sup $f_\beta \leq \bar{f}(\alpha)$.

□

3.2 A Counterexample

There are simpler counterexamples, but the reason we pick this particular one will become apparent in Section 4.3. This would be a good point for the reader to become familiar with Appendix A, in particular Section A.4.

Proposition 3.5. *There is a tree T and function $f : T \to \mathbb{R}$ s.t. sup lim sup $f_\beta <$ lim $\bar{f}$.*

Proof. Let $T = 2^{<\omega} = \{0,1\}^{<\omega}$ and $\prec\, = \,\subset$. Let $f(x) = \frac{1}{\mathrm{dom}(x)+1}$ whenever $x \in \{\langle\rangle\ , \langle 0\rangle\ , \langle 0,0\rangle\ , \langle 0,0,0\rangle\ , \ldots\} = \beta_L$. And when $x \notin \beta_L$, let $f(x) = \frac{1}{\mathrm{dom}(z)+1}$ where $z \in 2^{<\omega}$ and $y \in \beta_L$ are unique s.t. $x = y^\frown\langle 1\rangle^\frown z$. Notice that $\limsup f \restriction \beta = 0$ for each $\beta \in B(T)$, and therefore $\sup \limsup f_\beta = 0$.

Suppose to the contrary that $0 \leq \limsup f^+ < 1$, and let $\eta = 1 - \limsup f^+$. We know there is an eventual upper bound U for f^+ s.t. $U < \limsup f^+ + \frac{\eta}{2}$. And so there is an $\alpha' \in A(T)$ s.t. $\alpha' \preccurlyeq \alpha$ implies that $f^+(\alpha) < U + \frac{\eta}{2} < \limsup f^+ + \eta = 1$. Let $x' = \alpha' \Cap \beta_L$ and $S_{x'} = \{y \in T : x' \prec y \wedge \mathrm{height}(y) = \mathrm{height}(x') + 1\}$, and then let

$$\alpha^* = \alpha' \setminus \{x'\} \cup S_{x'}$$

It is clear then that $\alpha^* \in A(T)$ and $\alpha' \preccurlyeq \alpha^*$. Therefore $f^+(\alpha^*) < 1$. However, since $x'^\frown\langle 1\rangle \in S_{x'}$ and $f(x'^\frown\langle 1\rangle) = 1$, then $f^+(\alpha^*) \geq 1$. A contradiction. Therefore $\sup \limsup f_\beta = 0 < 1 \leq \limsup f^+$ by Theorem 3.3.

$\square$

An alternative way of seeing $0 < \lim \bar{f}$ is to notice that $\bar{f}$ is constant on $A(T)$ and equal to 1; and this implies $\lim \bar{f} = 1$.

Chapter 4

Restrictions on our Function

Let T be a tree and $f : T \to \mathbb{R}$ a function.

In Section 3.2 we saw that we do not in general have $\sup \limsup f_\beta = \lim \bar{f}$. In this chapter we will place certain restrictions on our function f and show that equality will hold.

4.1 Monotonicity

Definition 4.1. Given a net $\mathcal{G} : A \to \mathbb{R}$, we say that $\mathcal{G}$ is monotone if $\mathcal{G}$ is monotone increasing or monotone decreasing.

4.1.1 Comparing Branch-wise and Slice-wise Monotonicity

Proposition 4.2. *If $f \upharpoonright \beta$ is monotone decreasing for each $\beta \in B(T)$ then f^+ is monotone decreasing, and likewise, if $f \upharpoonright \beta$ is monotone increasing for each*

$\beta \in B(T)$ then f^+ is monotone increasing.

Proof. First suppose $f \upharpoonright \beta$ is monotone decreasing for each $\beta \in B(T)$. Take any $\alpha, \alpha' \in A(T)$ s.t. $\alpha \preccurlyeq \alpha'$. Take $x \in \alpha'$ arbitrary. Then $x = \alpha' \Cap \beta$ for some $\beta \in B(T)$ by Remark 1.10. Since $\alpha \preccurlyeq \alpha'$, then $y = \alpha \Cap \beta \preccurlyeq \alpha' \Cap \beta = x$. And since $f \upharpoonright \beta$ is monotone decreasing, then $f(x) \leq f(y) \leq f^+(\alpha)$. Since $x \in \alpha'$ was arbitrary, then $f^+(\alpha') \leq f^+(\alpha)$. Therefore f^+ is monotone decreasing.

Now suppose $f \upharpoonright \beta$ is monotone increasing for each $\beta \in B(T)$. Again, take any $\alpha, \alpha' \in A(T)$ s.t. $\alpha \preccurlyeq \alpha'$. Take $x \in \alpha$ arbitrary. Then $x = \alpha \Cap \beta$ for some $\beta \in B(T)$ by Remark 1.10. Since $\alpha \preccurlyeq \alpha'$, then $x = \alpha \Cap \beta \preccurlyeq \alpha' \Cap \beta = y$. And since $f \upharpoonright \beta$ is monotone increasing, then $f(x) \leq f(y) \leq f^+(\alpha')$. Since $x \in \alpha$ was arbitrary, then $f^+(\alpha) \leq f^+(\alpha')$. Therefore f^+ is monotone increasing.

□

Does the converse of Proposition 4.2 hold?

Proposition 4.3. *There is a tree T and a function $f : T \to \mathbb{Q}_{\geq 0}$ s.t. f^+ is monotone decreasing but not monotone increasing, while $f \upharpoonright \beta$ is strictly monotone increasing for some $\beta \in B(T)$. Likewise, the same claim holds with 'decreasing' and 'increasing' interchanged.*

Proof. Let $T = (\{0\}^{<\omega} \cup \{1\}^{<\omega}) \setminus \{\langle\rangle\}$ and $\prec = \subset$. Let

$$f_0(x) = \begin{cases} 1 + \frac{1}{\mathrm{dom}(x)} & \text{if } x \in \{0\}^{<\omega} \setminus \{\langle\rangle\} \\ 1 - \frac{1}{\mathrm{dom}(x)} & \text{if } x \in \{1\}^{<\omega} \setminus \{\langle\rangle\} \end{cases}$$

$$f_1(x) = \begin{cases} \mathrm{dom}(x) & \text{if } x \in \{0\}^{<\omega} \setminus \{\langle\rangle\} \\ \frac{1}{\mathrm{dom}(x)} & \text{if } x \in \{1\}^{<\omega} \setminus \{\langle\rangle\} \end{cases}$$

Then it can easily be checked that f_0^+ is monotone decreasing but not monotone increasing while $f_0 \restriction \{1\}^{<\omega} \setminus \{\langle\rangle\}$ is strictly monotone increasing. Likewise it can easily be checked that f_1^+ is monotone increasing but not monotone decreasing while $f_1 \restriction \{1\}^{<\omega} \setminus \{\langle\rangle\}$ is strictly monotone decreasing.

□

So our answer to the above question is “no”.

4.2 Equality

We are ready to show that we have sup lim sup $f_\beta = \lim \bar{f}$ whenever f^+ is monotone. Once we know this, then, by Proposition 4.2, we will also know that the equality will hold whenever either of the following hold:

- $f \restriction \beta$ is monotone increasing for each $\beta \in B(T)$

- $f \restriction \beta$ is monotone decreasing for each $\beta \in B(T)$

We assume sup lim sup $f_\beta \neq -\infty$ for the remainder of this section. The cases where sup lim sup $f_\beta = -\infty$ are treated in Appendix B.

4.2.1 Monotone Decreasing

Lemma 4.4. *Given $\epsilon > 0$ and $\beta \in B(T)$, there is a $y \in \beta$ s.t. $y \preccurlyeq x$ and $x \in \beta$ implies $f(x) <$ sup lim sup $f_\beta + \epsilon$. In particular, there is a $y \in \beta$ s.t. $f(y) <$ sup lim sup $f_\beta + \epsilon$.*

Proof. Take any $\epsilon > 0$ and $\beta \in B(T)$. If lim sup $f \restriction \beta = \infty$, then the conclusion holds trivially.

Suppose $-\infty <$ lim sup $f \restriction \beta < \infty$. Notice then that there must be an eventual upper bound $U \in \mathbb{R}$ for $f \restriction \beta$ s.t. $U <$ lim sup $f \restriction \beta + \frac{\epsilon}{2}$. So, there is a $y \in \beta$ s.t. $y \preccurlyeq x$ and $x \in \beta$ implies $f(x) < U + \frac{\epsilon}{2}$ by definition of an eventual upper bound. Therefore $y \preccurlyeq x$ and $x \in \beta$ implies $f(x) < U + \frac{\epsilon}{2} <$ lim sup $f \restriction \beta + \epsilon \leq$ sup lim sup $f_\beta + \epsilon$. And in particular we have $f(y) <$ sup lim sup $f_\beta + \epsilon$.

Now suppose lim sup $f \restriction \beta = -\infty$. Then the set of eventual upper bounds for $f \restriction \beta$ is not bounded below. Since sup lim sup $f_\beta \neq -\infty$, then there must be U an eventual upper bound for $f \restriction \beta$ s.t. $U <$ sup lim sup f_β. And the result follows as in the previous case.

$\square$

Theorem 4.5. *If f^+ is monotone decreasing, then lim $\bar{f} \leq$ sup lim sup f_β.*

Proof. Let $\epsilon > 0$. Then for each $\beta \in B(T)$, we can find a $y_\beta \in \beta$ s.t. $f(y_\beta) <$ sup lim sup $f_\beta + \epsilon$ by Lemma 4.4. Let $C = \{y_\beta : \beta \in B(T)\}$ and then let $\alpha' = C_\prec$ which is a slice by Proposition 1.9. Now, taking any $y \in \alpha'$, since $y = y_\beta$ for some $\beta \in B(T)$, then $f(y) <$ sup lim sup $f_\beta + \epsilon$. Hence $f^+(\alpha') \leq$ sup lim sup $f_\beta + \epsilon$. Since f^+ is monotone decreasing, then whenever $\alpha' \preccurlyeq \alpha$, we have $f^+(\alpha) \leq f^+(\alpha') \leq$ sup lim sup $f_\beta + \epsilon$. Since $\epsilon > 0$ was arbitrary, then sup lim sup f_β is an eventual upper bound for f^+. Since lim sup f^+ is the infimum of the eventual upper bounds for f^+, then lim sup $f^+ \leq$ sup lim sup f_β.

□

4.2.2 Monotone Increasing

Lemma 4.6. *If f^+ is monotone increasing, then given any $\epsilon > 0$ and any $\alpha \in A(T)$ s.t. $f^+(\alpha) \in \mathbb{R}$, there is some $\beta \in B(T)$ s.t. $\alpha \Cap \beta \preccurlyeq x$ and $x \in \beta$ implies $f^+(\alpha) \leq f(x) + \epsilon$.*

Proof. Take any $\alpha \in A(T)$ where $f^+(\alpha) \in \mathbb{R}$ and $\epsilon > 0$. Suppose to the contrary that for each $\beta \in B(T)$, there is some $x_\beta \in \beta$ s.t. $\alpha \Cap \beta \preccurlyeq x_\beta$ and $f(x_\beta) < f^+(\alpha) - \epsilon$. Let $C = \{x_\beta : \beta \in B(T)\}$. Now let $\alpha' = C_\prec$ and notice that we have $\alpha' \in A(T)$ and $\alpha \preccurlyeq \alpha'$ by Proposition 1.9 and Corollary C.8.

Given any $x \in \alpha'$, we know that $x = x_\beta$ for some $\beta \in B(T)$ s.t $f(x) = f(x_\beta) < f^+(\alpha) - \epsilon < f^+(\alpha)$. Therefore $f^+(\alpha') \leq f^+(\alpha) - \epsilon < f^+(\alpha)$. But since f^+ is monotone increasing, we must have $f^+(\alpha) \leq f^+(\alpha')$. A contradiction.

Therefore we can conclude that there is some $\beta \in B(T)$ s.t. $\alpha \Cap \beta \preccurlyeq x$ and $x \in \beta$ implies $f^+(\alpha) \leq f(x) + \epsilon$.

□

Lemma 4.7. *If f^+ is monotone increasing then $f^+(\alpha) \leq$ sup lim sup f_β whenever $f^+(\alpha) \in \mathbb{R}$.*

Proof. Take any $\alpha \in A(T)$, where $f^+(\alpha) \in \mathbb{R}$, and $\epsilon > 0$. We know that there is some $\beta \in B(T)$ s.t. $\alpha \Cap \beta \preccurlyeq x$ and $x \in \beta$ implies $f^+(\alpha) \leq f(x) + \frac{\epsilon}{2}$ by Lemma 4.6. By Lemma 4.4, there is a $y \in \beta$ s.t. $y \preccurlyeq x$ and $x \in \beta$ implies $f(x) <$ sup lim sup $f_\beta + \frac{\epsilon}{2}$. Let z be the $\prec$-maximum of y and $\alpha \Cap \beta$. Then $f^+(\alpha) \leq f(z) + \frac{\epsilon}{2} <$ sup lim sup $f_\beta + \epsilon$. Since $\epsilon > 0$ was arbitrary, then $f^+(\alpha) \leq$ sup lim sup f_β.

□

Theorem 4.8. *If f^+ is monotone increasing and $f^+(\alpha) \in \mathbb{R}$ for each $\alpha \in A(T)$, then lim $\bar{f} \leq$ sup lim sup f_β.*

Proof. Since $f^+(\alpha) \in \mathbb{R}$ for each $\alpha \in A(T)$, then sup lim sup f_β is an eventual upper bound for f^+ by Lemma 4.7. Since lim sup f^+ is the infimum of the eventual upper bounds for f^+, we then have lim sup $f^+ \leq$ sup lim sup f_β.

□

Theorem 4.9. *Suppose f^+ is monotone increasing. If there is some $\alpha \in A(T)$ s.t. $f^+(\alpha) = \infty$, then sup lim sup $f_\beta = \infty$.*

Proof. Suppose to the contrary that there is some $\alpha \in A(T)$ s.t. $f^+(\alpha) = \infty$ and that sup lim sup $f_\beta = N < \infty$.

By Lemma 4.4, given any $\beta \in B(T)$, there is a $y_\beta \in \beta$ s.t. $y_\beta \preccurlyeq x$ and $x \in \beta$ implies $f(x) < N + 1$. Now for each $\beta \in B(T)$, let z_β be the $\prec$-maximum of y_β and $\alpha \Cap \beta$. Let $C = \{z_\beta : \beta \in B(T)\}$ and then let $\alpha' = C_\prec$. We then have $\alpha' \in A(T)$ and $\alpha \preccurlyeq \alpha'$ by Proposition 1.9 and Corollary C.8. And since f^+ is monotone increasing, then $f^+(\alpha) \leq f^+(\alpha')$.

Now, take any $z \in \alpha'$. We know that $z = z_\beta$ for some $\beta \in B(T)$ where $y_\beta \preccurlyeq z_\beta$ and $z_\beta \in \beta$. This means that $f(z) < N + 1$. Since $z \in \alpha'$ was arbitrary, then $f^+(\alpha') \leq N + 1$. A contradiction.

□

4.3 Other Restrictions?

We've only discussed monotonicity so far, and we won't be considering any other sorts of restrictions on our function. So at this point, we'll just briefly mention what other function restrictions won't work.

Definition 4.10. Let $\mathcal{G} : A \to \mathbb{R}$ be a net where A is a directed set ordered by some binary relation $\sqsubseteq$.

- We say that $\mathcal{G}$ is constant if $\forall x, y \in A[\mathcal{G}(x) = \mathcal{G}(y)]$.

- We say that $\mathcal{G}$ is eventually constant if $\exists z \in A \forall x \in A[z \sqsubseteq x \implies \mathcal{G}(z) = \mathcal{G}(x)]$.
- We say that $\mathcal{G}$ is eventually monotone decreasing if $\exists z \in A \forall x, y \in A[z \sqsubseteq x, y \wedge x \sqsubseteq y \implies \mathcal{G}(y) \leq \mathcal{G}(x)]$ and we say that $\mathcal{G}$ is eventually monotone increasing if $\exists z \in A \forall x, y \in A[z \sqsubseteq x, y \wedge x \sqsubseteq y \implies \mathcal{G}(x) \leq \mathcal{G}(y)]$.

Proposition 4.11. *There is an ω-tree T, and function $f : T \to (0,1] \cap \mathbb{Q}$ s.t:*

- *$f \restriction \beta$ is eventually monotone decreasing for each $\beta \in B(T)$.*
- *$f \restriction \beta$ is not eventually constant for each $\beta \in B(T)$.*
- *$\bar{f}$ is constant.*
- *sup lim sup $f_\beta = 0$ and lim $\bar{f} = 1$.*

Proof. Take the same tree and function as in the proof of Proposition 3.5.

□

This is why we picked a more complicated counterexample than apparently necessary for Proposition 3.5.

Lastly, notice that the focus of this chapter was to see what would happen when f^+ is monotone. If we had tried to do the same with $\bar{f}$, then our efforts would've been in vain:

We know that $\bar{f}$ is already always monotone decreasing by Proposition 2.7. The other possibility would be to consider the case where $\bar{f}$ were also monotone increasing. But even this property wouldn't guarantee equality by Proposition 4.11.

Chapter 5

Maximal Antichains

In this chapter we will introduce our maximal antichain formulation for lim sup.

Let T be a tree and $f : T \to \mathbb{R}$ a function.

5.1 Ordering the Maximal Antichains

Definition 5.1. We denote the set of maximal antichains by $\mathfrak{M}(T)$.

Remark 5.2. *By another easy application of Zorn's Lemma: Given any antichain $A \subseteq T$, there is some $A' \in \mathfrak{M}(T)$ s.t. $A \subseteq A'$.*

Definition 5.3. Given $A, A' \in \mathfrak{M}(T)$, we write $A \sqsubseteq A'$ if for every $x \in A$ there is some $y \in A'$ s.t. $x \preccurlyeq y$.

Proposition 5.4. *$\mathfrak{M}(T)$ is a directed set with respect to $\sqsubseteq$.*

Proof. Reflexivity is obvious.

For transitivity: Suppose we have $A \sqsubseteq A'$ and $A' \sqsubseteq A^*$ for $A, A', A^* \in \mathfrak{M}(T)$. Then given any $x \in A$, there is some $y \in A'$ s.t. $x \preccurlyeq y$. And furthermore, there is some $z \in A^*$ s.t. $y \preccurlyeq z$. Therefore $x \preccurlyeq z$ by transitivity. Hence $A \sqsubseteq A^*$.

For the upper bound property: Take any $A, A' \in \mathfrak{M}(T)$. Let

$$C_A = F(A') \cap A = \{x \in A : \exists y \in A'[y \preccurlyeq x]\}$$

$$C_{A'} = F(A) \cap A' = \{x \in A' : \exists y \in A[y \preccurlyeq x]\}$$

Then set

$$C = C_A \cup C_{A'}$$

First, let's see that C is an antichain: We already know that C_A and $C_{A'}$ are antichains since they are both subsets of antichains. Now, take any distinct $x_0 \in C_A$ and $x_1 \in C_{A'}$, and suppose to the contrary that they are compatible; say we have $x_0 \prec x_1$. Since $x_0 \in C_A$, then there is some $y \in A'$ s.t. $y \preccurlyeq x_0$, and therefore $y \prec x_1$ by transitivity. This contradicts the fact that A' is an antichain. This shows that C is an antichain.

Now we will see that given any $x \in A$, there is some $z \in C$ s.t. $x \preccurlyeq z$: So take any $x \in A$. If $x \in A'$, then $x \in C_{A'} \subseteq C$. Otherwise, $x \notin A'$. Since A' is a maximal antichain, then there is some $z \in A'$ s.t. x is compatible with z. If $x \preccurlyeq z$, then $z \in C_{A'} \subseteq C$, and if $z \prec x$, then $x \in C_A \subseteq C$. In any case, there is some $z \in C$ s.t. $x \preccurlyeq z$. An analogous argument also shows that given any $x \in A'$, there is some

$z \in C$ s.t. $x \preccurlyeq z$. Now taking any maximal antichain $A^* \in \mathfrak{M}(T)$ s.t. $C \subseteq A^*$, we have $A \sqsubseteq A^*$ and $A' \sqsubseteq A^*$.

□

5.2 The Maximal Antichain Formulation

The following definitions are legitimate by Proposition 5.4:

Definition 5.5. We define the net $f^* : \mathfrak{M}(T) \to \overline{\mathbb{R}}$ s.t:

$$f^*(A) = \sup_{x \in A} f(x) = \sup\{f(x) : x \in A\}$$

Definition 5.6. We define the net $\hat{f} : \mathfrak{M}(T) \to \overline{\mathbb{R}}$ s.t:

$$\hat{f}(A) = \sup_{x \in F(A)} f(x) = \sup\{f(x) : x \in F(A)\}$$

Lemma 5.7. *Let $A \in \mathfrak{M}(T)$. Then for every $m \in F(A)$ there is some $A' \in \mathfrak{M}(T)$ s.t. $m \in A'$ and $A \sqsubseteq A'$.*

Proof. Take any $m \in F(A)$. This means that $n \preccurlyeq m$ for some $n \in A$. If $n = m$, then simply take $A' = A$. So suppose $n \prec m$. Set

$$D = C \cup A \setminus \{n\}$$

where C is the same C appearing in the proof of Lemma 3.1. Notice then that the same argument as in Lemma 3.1 shows that $m \in D$ and D is an antichain.

It is also clear that given any $x \in A$, there is some $y \in D$ s.t. $x \preccurlyeq y$. Therefore taking any $A' \in \mathfrak{M}(T)$ with $D \subseteq A'$, we have $A \sqsubseteq A'$ and $m \in A'$.

□

Proposition 5.8. *lim sup* $\hat{f} =$ *lim sup* f^*.

Proof. The proof is identical to the proof of Proposition 3.2, except that we use Lemma 5.7 instead of Lemma 3.1.

□

Proposition 5.9. $\hat{f}$ *is monotone decreasing.*

Proof. Let $A, A' \in \mathfrak{M}(T)$ arbitrary and suppose $A \sqsubseteq A'$. Then $F(A') \subseteq F(A)$ by Corollary C.5, and so we have:

$$\hat{f}(A') = \sup_{x \in F(A')} f(x) \leq \sup_{x \in F(A)} f(x) = \hat{f}(A)$$

.

□

Proposition 5.10. *If inf* $\{\hat{f}(A) : A \in \mathfrak{M}(T)\} \in \mathbb{R}$, *then lim* $\hat{f} =$ *inf* $\{\hat{f}(A) : A \in \mathfrak{M}(T)\}$.

Proof. Identical to the proof of Proposition 2.8, except that we use Proposition 5.9 instead of Proposition 2.7.

□

So as with the slices, we define lim $\hat{f} = -\infty$ whenever inf $\{\hat{f}(A) : A \in \mathfrak{M}(T)\} = -\infty$, and lim $\hat{f} = \infty$ whenever inf $\{\hat{f}(A) : A \in \mathfrak{M}(T)\} = \infty$ and then obtain:

$$\lim \hat{f} = \limsup \hat{f} = \liminf \hat{f}$$

by Propositions 5.10 and D.8.

We shall take our maximal antichain formulation for lim sup f to be

$$\lim \hat{f}$$

Similar to what we did in Subsection 2.2.3, we shall take our maximal antichain formulation for lim inf f to be lim $\underset{\smile}{f}$, where the function $\underset{\smile}{f}$ is defined analogously to $\hat{f}$ by just swapping 'sup' with 'inf' in the definition.

We shall take our maximal antichain formulation for lim f to be defined in terms of lim sup f and lim inf f, namely we will say that our maximal antichain formulation for lim f is defined just in case lim $\hat{f} =$ lim $\underset{\smile}{f}$, in which case we define it to be equal to this value.

5.3 More Comparisons

This section is analogous to what we did in Chapter 3. We will actually be comparing sup lim sup f_β and lim sup f^*, since the definition of f^* is easier to work with. Again, as in the slice case, it makes no difference since we know that lim $\hat{f}$ = lim sup $\hat{f}$ = lim sup f^* by Proposition 5.8.

5.3.1 Comparing with the Branch Formulation

In the following theorem, we assume sup lim sup $f_\beta \neq -\infty$. The case where sup lim sup $f_\beta = -\infty$ is treated in Appendix B.

Theorem 5.11. *lim $\hat{f} \leq$ sup lim sup f_β*

Proof. Let $\epsilon > 0$ and let $E = \{x \in T : f(x) < \text{sup lim sup } f_\beta + \epsilon\}$. By Lemma 4.4, for each $\beta \in B(T)$, there is some $y \in \beta$ s.t. $y \preccurlyeq x$ and $x \in \beta$ implies $x \in E$. So for each $\beta \in B(T)$, let $y_\beta \in \beta$ be $\prec$-least with the property that $y_\beta \preccurlyeq x$ and $x \in \beta$ implies $x \in E$. Let

$$C = \{y_\beta : \beta \in B(T)\}$$

$$M = \{x \in T : x \text{ is } \prec \text{-maximal in } C\}$$

Now we'd like to see that for each $y \in C$ there is some $x \in M$ s.t. $y \preccurlyeq x$. By Zorn's Lemma, it suffices to show that given any chain $\mathcal{C} \subseteq C$ (w.r.t the ordering $\prec$ of T restricted to C), there is some $y \in C$ s.t. $z \preccurlyeq y$ for each $z \in \mathcal{C}$.

Take any chain $\mathcal{C} \subseteq C$. Then $\mathcal{C}$ is a chain in T as well. Therefore there is some $\beta \in B(T)$ s.t. $\mathcal{C} \subseteq \beta$ by Remark 1.10. Consider $y_\beta \in C$, and suppose to the contrary that $y_\beta \prec z$ for some $z \in \mathcal{C}$. Since $z \in \mathcal{C} \subseteq C$, then $z = y_{\beta'}$ for some $\beta' \in B(T)$. Since we now have $y_\beta \prec y_{\beta'}$, then there must be some $x \in \beta'$ s.t. $y_\beta \preccurlyeq x$ and $x \notin E$ by $\prec$-minimality of $y_{\beta'}$. Notice x and $y_{\beta'}$ are compatible as they both belong to β'. We cannot have $y_{\beta'} \preccurlyeq x$, since otherwise we would have $x \in E$. Therefore we must have $y_\beta \preccurlyeq x \prec y_{\beta'}$. Since $y_{\beta'} \in \mathcal{C} \subseteq \beta$, then $x \in \beta$ by Proposition 1.8; but this is a contradiction since in this case we must also have $x \in E$. Therefore we must have $z \preccurlyeq y_\beta$ for each $z \in \mathcal{C}$.

It is clear that M is an antichain, so let $A' \in \mathfrak{M}(T)$ be any maximal antichain s.t. $M \subseteq A'$. Take any $A \in \mathfrak{M}(T)$ s.t. $A' \sqsubseteq A$, and let $z \in A$ arbitrary. Then there is some $\beta \in B(T)$ s.t. $z \in \beta$ by Remark 1.10. Take $y_\beta \in C$ where $y_\beta \preccurlyeq v$ and $v \in \beta$ implies $v \in E$, and suppose to the contrary that $z \prec y_\beta$. Since $y_\beta \in C$, then there is some $x \in M$ s.t. $y_\beta \preccurlyeq x$. Therefore $z \prec x$ by transitivity. Since $x \in M \subseteq A'$ and $A' \sqsubseteq A$, then there is some $w \in A$ s.t. $x \preccurlyeq w$. Therefore $z \prec w$ by transitivity, contradicting the fact that A is an antichain. Therefore we must have $y_\beta \preccurlyeq z$, which means that $z \in E$. Since $z \in A$ was arbitrary, then $f^*(A) \leq \sup \limsup f_\beta + \epsilon$. Since $\epsilon > 0$ was arbitrary, then $\sup \limsup f_\beta$ is an eventual upper bound for f^*. Since $\limsup f^*$ is the infimum of the eventual upper bounds for f^*, then $\limsup f^* \leq \sup \limsup f_\beta$.

□

Notice the essential property of E which made the proof work was that E contained a tail of each $\beta \in B(T)$. Though we could've said something more general, it would've been a bit of a digression.

5.3.2 More Counterexamples

Proposition 5.12. *There is an ω-tree T and function $f : T \to \{0, 1, 2\}$ s.t. lim $\hat{f}$ < sup lim sup f_β < lim $\bar{f}$.*

Proof. Let $T = 2^{<\omega}$ and $\prec = \subset$, and let

$f(x) = 1$ whenever $x \in \{\langle\rangle, \langle 0\rangle, \langle 0, 0\rangle, \langle 0, 0, 0\rangle, \ldots\} = \beta_L$.

$f(x) = 2$ whenever $x \in \{\langle 1\rangle, \langle 0, 1\rangle, \langle 0, 0, 1\rangle, \langle 0, 0, 0, 1\rangle, \ldots\} = C \in \mathfrak{M}(T)$.

$f(x) = 0$ whenever $x \notin \beta_L \cup C$.

A similar argument to the one appearing in the proof of Proposition 3.5 gives $2 \leq \limsup f^+$ (alternatively, just notice that $\limsup f^+ = \lim \bar{f} = 2$ since $\bar{f}$ is constant on $A(T)$ and equal to 2). Also notice that $\limsup f \restriction \beta_L = 1$, while $\limsup f \restriction \beta = 0$ whenever $\beta \neq \beta_L$. Therefore $\sup \limsup f_\beta = 1$.

Consider the maximal antichain

$$A' = \{\langle 1, 0\rangle, \langle 1, 1\rangle, \langle 0, 1, 0\rangle, \langle 0, 1, 1\rangle, \langle 0, 0, 1, 0\rangle, \langle 0, 0, 1, 1\rangle, \ldots\}$$

and take any $\epsilon > 0$ and any $A \in \mathfrak{M}(T)$ s.t. $A' \sqsubseteq A$. Let $x \in A$. Since $A' \cap \beta_L = \emptyset$, then $A \cap \beta_L = \emptyset$ by Proposition C.6. Therefore $x \notin \beta_L$. Suppose to the contrary that $x \in C$. Then since there is some $y \in A'$ s.t. $x \prec y$, and $A' \sqsubseteq A$, we get $x \prec z$ for some $z \in A$. This contradicts the fact that A is an antichain. Therefore $x \notin \beta_L \cup C$, and so $f(x) = 0$. Since $x \in A$ was arbitrary, then $f^*(A) = 0 < \epsilon$. And since $\epsilon > 0$ was arbitrary, then 0 is an eventual upper bound for f^*. Therefore lim sup $f^* \leq 0 < 1 =$ sup lim sup $f_\beta < 2 \leq$ lim sup f^+.

□

Notice that our choice of 0, 1, and 2 was somewhat arbitrary. We could've just as well picked any $r_0, r_1, r_2 \in \mathbb{R}$ where $r_0 < r_1 < r_2$ to play the corresponding roles.

Proposition 5.13. *There is an ω-tree T and function $f : T \to [0, 1) \cap \mathbb{Q}$ s.t. f^+ is monotone increasing, not constant, and with lim $\hat{f} <$ sup lim sup f_β.*

Proof. Let $T = 2^{<\omega}$ and $\prec = \subset$, and let

$f(x) = 1 - \frac{1}{\text{dom}(x)+2}$ whenever $x \in \{\langle\rangle, \langle 0 \rangle, \langle 0, 0 \rangle, \langle 0, 0, 0 \rangle, \ldots\} = \beta_L$.

$f(x) = 0$ whenever $x \notin \beta_L$.

Since every slice must intersect β_L, it is easy to see that $f^+(\alpha) = f(\alpha \Cap \beta_L) > 0$ for each $\alpha \in A(T)$. Using this, we see that f^+ is monotone increasing and not constant on $A(T)$. Furthermore, sup lim sup $f_\beta = 1$ since lim sup $f \upharpoonright \beta_L = 1$ while lim sup $f \upharpoonright \beta = 0$ whenever $\beta \neq \beta_L$. Moreover, taking the maximal antichain $A' = \{\langle 1 \rangle, \langle 0, 1 \rangle, \langle 0, 0, 1 \rangle, \langle 0, 0, 0, 1 \rangle, \ldots\}$, a similar argument as in the proof of Proposition 5.12 gives us that $A' \sqsubseteq A$ and $A \in \mathfrak{M}(T)$ implies $f^*(A) = 0 < \epsilon$ whenever $\epsilon > 0$. Hence we have lim sup $f^* \leq 0 < 1 =$ sup lim sup f_β.

□

Proposition 5.14. *There is an ω-tree T and function $f : T \to (0,1] \cap \mathbb{Q}$ s.t:*

- *$f \restriction \beta$ is monotone decreasing for each $\beta \in B(T)$.*
- *$f \restriction \beta$ is not eventually constant for each $\beta \in B(T)$.*
- *lim $\hat{f}$ < sup lim sup f_β*

Proof. Let $T = 2^{<\omega}$ and $\prec\, = \,\subset$, and let

$f(x) = \frac{1}{2} + \frac{1}{\text{dom}(x)+2}$ whenever $x \in \{\langle\rangle, \langle 0\rangle, \langle 0,0\rangle, \langle 0,0,0\rangle, \ldots\} = \beta_L$.

$f(x) = \frac{1}{\text{dom}(x)+1}$ whenever $x \notin \beta_L$.

It is not difficult to see that $f \restriction \beta$ is monotone decreasing and not eventually constant for each $\beta \in B(T)$. Notice sup lim sup $f_\beta = \frac{1}{2}$ since lim sup $f \restriction \beta_L = \frac{1}{2}$ while lim sup $f \restriction \beta = 0$ whenever $\beta \neq \beta_L$.

Now consider the maximal antichain $A' = \{\langle 1\rangle, \langle 0,1\rangle, \langle 0,0,1\rangle, \langle 0,0,0,1\rangle, \ldots\}$ and take any $\epsilon > 0$. Take any $N < \omega$ s.t. $\frac{1}{\epsilon} < N$, and consider the maximal antichain $\mathfrak{L}_N(T)$. Let $A^* \in \mathfrak{M}(T)$ s.t. $A', \mathfrak{L}_N(T) \sqsubseteq A^*$. Take any $A \in \mathfrak{M}(T)$ s.t. $A^* \sqsubseteq A$ and any $x \in A$. Since $A' \sqsubseteq A$ by transitivity, and $A' \cap \beta_L = \emptyset$, then $A \cap \beta_L = \emptyset$ by Corollary C.5 and Proposition C.6. Therefore $x \notin \beta_L$ and so $f(x) = \frac{1}{\text{dom}(x)+1}$. Since $\mathfrak{L}_N(T) \sqsubseteq A$ by transitivity, we have that there is some $y \in \mathfrak{L}_N(T)$ s.t. $y \preccurlyeq x$ by Corollary C.5. Since $\text{dom}(y) = N$ and $y \preccurlyeq x$, we then have $\frac{1}{\text{dom}(x)+1} \leq \frac{1}{\text{dom}(y)+1} = \frac{1}{N+1} < \frac{1}{N} < \epsilon$. Since $x \in A$ was arbitrary, then $f^*(A) < \epsilon$. And since $\epsilon > 0$ was arbitrary, then 0 is an eventual upper bound for f^*. Hence we have lim sup $f^* \leq 0 < \frac{1}{2} =$ sup lim sup f_β. ∎

Chapter 6

Slices and Maximal Antichains

Let T be a tree and $f : T \to \mathbb{R}$ a function.

In this chapter we shall see when the set of slices and set of antichains are equal, and then using this idea, we will place a restriction on our tree and show that $\lim \hat{f} = \sup \limsup f_\beta = \lim \bar{f}$. We shall also show that all three are equal when f^* is monotone increasing, and then conclude with some additional results regarding f^*.

6.1 When are they Equal?

We know that there are trees T s.t. $\mathfrak{M}(T) \nsubseteq A(T)$. For instance, the maximal antichain C in the proof of Proposition 5.12 is not a slice since $C \cap \beta_L = \emptyset$.

Proposition 6.1. $A(T) \subseteq \mathfrak{M}(T)$

Proof. Every slice is already an antichain by definition, so all we need to see is that every slice is maximal.

Take any $\alpha \in A(T)$ and any $x \in T \setminus \alpha$. Take any $\beta \in B(T)$ s.t. $x \in \beta$. Since β is a chain, then x and $\alpha \Cap \beta$ must be compatible. This shows that α is maximal.

□

Proposition 6.2. *$A(T) = \mathfrak{M}(T)$ iff for every $\beta \in B(T)$ there is some $x \in \beta$ s.t. $x \uparrow \subseteq \beta$.*

Proof. First suppose that for each $\beta \in B(T)$, there is some $x_\beta \in \beta$ s.t. $x_\beta \uparrow \subseteq \beta$. Take any $A \in \mathfrak{M}(T)$ and $\beta \in B(T)$. If $x_\beta \in A$, then $A \cap \beta \neq \emptyset$. Otherwise, $x_\beta \notin A$, and since A is maximal, then x_β is compatible with some $y \in A$. If $y \prec x_\beta$, then $y \in \beta$ by Proposition 1.8. And if $x_\beta \prec y$, then $y \in x_\beta \uparrow \subseteq \beta$. In either case, $A \cap \beta \neq \emptyset$, and since $\beta \in B(T)$ was arbitrary, then $A \in A(T)$.

Now suppose there is some $\beta \in B(T)$ s.t. $x \uparrow \not\subseteq \beta$ whenever $x \in \beta$. Let $C = \{y \in F(\beta) : y \notin \beta\}$, and consider the antichain $C_\prec$. Take any $A \in \mathfrak{M}(T)$ s.t. $C_\prec \subseteq A$. Suppose to the contrary that $A \cap \beta \neq \emptyset$ and take any $z \in A \cap \beta$. We know that $z \uparrow \not\subseteq \beta$, so let $v \in z \uparrow \setminus \beta$. Since $z \prec v$ and $v \notin \beta$, then $v \in C$. If $v \in C_\prec$, then $v \in A$, but this contradicts the fact that A is an antichain. Therefore $v \notin C_\prec$, so take $u \in C_\prec$ where $u \prec v$. Then u and z must be compatible, but not equal, since $z \in \beta$ while $u \notin \beta$. But in this case we also have a contradiction, since now u and z are distinct compatible members of the antichain A. This shows that $A \cap \beta = \emptyset$, and so $A \notin A(T)$.

□

6.2 Equality of all Formulations

6.2.1 A Restriction on our Tree

Theorem 6.3. *If for every $\beta \in B(T)$ there is some $x \in \beta$ s.t. $x \uparrow \subseteq \beta$, then $lim\ \hat{f} = lim\ \bar{f}$.*

Proof. Since $A(T) = \mathfrak{M}(T)$ by Proposition 6.2, and $\alpha \preccurlyeq \alpha' \iff \alpha \sqsubseteq \alpha'$ by Corollary C.8, then $f^* = f^+$ as nets. Therefore lim sup $f^* =$ lim sup f^+.

□

6.2.2 A Restriction on our Function

Theorem 6.4. *If f^* is monotone increasing, then $lim\ \bar{f} \leq lim\ \hat{f}$.*

Proof. Let $U \in \mathbb{R}$ an eventual upper bound for f^* and $\epsilon > 0$ arbitrary. This means there is some $A' \in \mathfrak{M}(T)$ s.t. $A' \sqsubseteq A$ and $A \in \mathfrak{M}(T)$ implies $f^*(A) < U + \epsilon$. Take any $\alpha \in A(T)$, and let $A'' \in \mathfrak{M}(T)$ s.t. $A', \alpha \sqsubseteq A''$. Then, since $A' \sqsubseteq A''$, we have $f^*(A'') < U + \epsilon$. And since f^* is monotone increasing and $\alpha \sqsubseteq A''$, then $f^+(\alpha) = f^*(\alpha) \leq f^*(A'') < U + \epsilon$. Since $\epsilon > 0$ was arbitrary, then U is an eventual upper bound for f^+. And since U was an arbitrary eventual upper bound for f^*, we have lim sup $f^+ \leq$ lim sup f^*.

□

Notice that the proof of Proposition 6.4 implicitly shows that given any monotone increasing net $\mathcal{G} : A \to \mathbb{R}$, any eventual upper bound $U \in \mathbb{R}$ for $\mathcal{G}$, and any $\epsilon > 0$, we have $\mathcal{G}(x) < U + \epsilon$ for every $x \in A$.

6.2.3 Some Additional Results

Proposition 6.5. *If f^* is monotone increasing (respectively decreasing), then f^+ is monotone increasing (respectively decreasing).*

Proof. Easy consequence of Corollary C.8.

□

Proposition 6.6. *If $f \restriction \beta$ is monotone decreasing for each $\beta \in B(T)$ then f^* is monotone decreasing, and likewise, if $f \restriction \beta$ is monotone increasing for each $\beta \in B(T)$ then f^* is monotone increasing.*

Proof. First suppose $f \restriction \beta$ is monotone decreasing for each $\beta \in B(T)$. Take any $A, A' \in \mathfrak{M}(T)$ s.t. $A \sqsubseteq A'$. Take $x \in A'$ arbitrary. Then we have $y \preccurlyeq x$ for some $y \in A$ by Corollary C.5. And then taking any $\beta \in B(T)$ s.t. $x, y \in \beta$, since $f \restriction \beta$ is monotone decreasing, we have $f(x) \leq f(y) \leq f^*(A)$. Since $x \in A'$ was arbitrary, then $f^*(A') \leq f^*(A)$. Therefore f^* is monotone decreasing.

Now suppose $f \restriction \beta$ is monotone increasing for each $\beta \in B(T)$. Again, take any $A, A' \in \mathfrak{M}(T)$ s.t. $A \sqsubseteq A'$. Take $x \in A$ arbitrary. Then, since $A \sqsubseteq A'$, there is some $y \in A'$ s.t. $x \preccurlyeq y$. And then taking any $\beta \in B(T)$ s.t. $x, y \in \beta$, since $f \restriction \beta$

is monotone increasing, we have $f(x) \leq f(y) \leq f^*(A')$. Since $x \in A$ was arbitrary, then $f^*(A) \leq f^*(A')$. Therefore f^* is monotone increasing.

$\square$

Chapter 7

Omega-trees and Topology

In this chapter we will study some topology and the structure of ω-trees and their slices. This would be a good point to become familiar with Section A.3 of Appendix A, if not already done so.

7.1 Why Omega-trees?

The main reasons we care to distinguish and study ω-trees is because various Cantor sets, such as the Cantor ternary set, can be naturally represented by ω-trees. The properties of ω-trees we will be discussing are geared in this direction and do not generalize to arbitrary trees. Those familiar with descriptive set theory will notice that the ideas presented in this chapter are influenced and motivated by the study of Baire space.

In the next chapter, we will use the results of this chapter to give a more constructive proof of Lemma 4.6 (without any mention of ϵ) when T is a pruned ω-tree.

7.2 A Useful Topology

Let T be a non-empty tree.

Definition 7.1. For each $x \in T$, we let $\mathcal{B}_x = \{\beta \in B(T) : x \in \beta\}$.

Proposition 7.2. *Let $x, y \in T$. If $x \preccurlyeq y$ then $\mathcal{B}_y \subseteq \mathcal{B}_x$; and if x and y are incompatible then $\mathcal{B}_x \cap \mathcal{B}_y = \emptyset$.*

Proof. If $x \preccurlyeq y$: Given any $\beta \in \mathcal{B}_y$, we have $y \in \beta$. This implies $x \in \beta$ by Proposition 1.8, and so $\beta \in \mathcal{B}_x$.

If $\mathcal{B}_x \cap \mathcal{B}_y \neq \emptyset$: Taking any $\beta \in \mathcal{B}_x \cap \mathcal{B}_y$, we have $x, y \in \beta$. Therefore x and y are compatible since β is a chain.

□

Proposition 7.3. $\{\mathcal{B}_x : x \in T\}$ *is a basis for a topology on* $B(T)$*.*

Proof. It is enough to show that the intersection of any two members of $\{\mathcal{B}_x : x \in T\}$ can be written as a (possibly empty) union of members of $\{\mathcal{B}_x : x \in T\}$ and also that $\bigcup\{\mathcal{B}_x : x \in T\} = B(T)$; both of which are clear by appealing to Proposition 7.2.

□

Definition 7.4. $\mathcal{B}_T$ denotes the topology on $B(T)$ generated by $\{\mathcal{B}_x : x \in T\}$.

Proposition 7.5. *Each $\mathcal{B}_x$ is clopen in $\mathcal{B}_T$.*

Proof. Let $I_x = \{y \in T : y \text{ is incompatible with } x\}$. Take any $\beta \in B(T) \setminus \mathcal{B}_x$. If it was the case that every member of β is compatible with x, then we would have $x \in \beta$ by maximality of β, which is a contradiction. Therefore there is some $y \in \beta$ incompatible with x. Hence $\beta \in \bigcup_{y \in I_x} \mathcal{B}_y$.

Now take any $\beta \in \bigcup_{y \in I_x} \mathcal{B}_y$. Then $\beta \in \mathcal{B}_y$ for some y where x and y are incompatible. Therefore $x \notin \beta$ since β is a chain. Hence $\beta \in B(T) \setminus \mathcal{B}_x$.

This shows that $B(T) \setminus \mathcal{B}_x = \bigcup_{y \in I_x} \mathcal{B}_y$, which means that $\mathcal{B}_x$ is closed.

□

7.3 Compact Spaces

Proposition 7.6. *$B(T)$ is a compact space iff every $\alpha \in A(T)$ is finite.*

Proof. Suppose $B(T)$ is a compact space and take any $\alpha \in A(T)$. Since α intersects each $\beta \in B(T)$, then $\{\mathcal{B}_x : x \in \alpha\}$ is an open cover for $B(T)$. Therefore there is a finite $F \subseteq \alpha$ s.t. $B(T) \subseteq \bigcup_{x \in F} \mathcal{B}_x$. Notice F is an antichain, since it is a subset of an antichain, and moreover, intersects every branch, as $B(T) \subseteq \bigcup_{x \in F} \mathcal{B}_x$. Therefore F is a slice. Since all slices are maximal antichains by Proposition 6.1, we then have $F = \alpha$. Therefore α is finite.

Now suppose every slice is finite. Let $\{\mathcal{B}_x : x \in C\}$ be an arbitrary open cover for $B(T)$ where $C \subseteq T$. Let $\alpha = C_{\prec}$ and notice that $\alpha \in A(T)$ by Proposition 1.9. Furthermore, $\{\mathcal{B}_x : x \in \alpha\}$ is a finite subcover for $B(T)$ since every branch must intersect α.

□

The proof of the following proposition (also known as Brouwer's Fan Theorem) is due to Moschovakis [2].

Proposition 7.7. *If T is an ω-tree, then $B(T)$ is a compact space.*

Proof. We first show that every $\alpha \in A(T)$ is finite.

Take any $\alpha \in A(T)$ and consider the tree $T^* = \{x \in T : \exists y \in \alpha[x \preccurlyeq y]\}$ with the induced order. Notice that $\mathfrak{L}_n(T^*)$ is finite for each $n < \omega$ since $(T^*, \prec)$ is a reduct of $(T, \prec)$.

Suppose to the contrary that height$(T^*) = \omega$. Then by König's Lemma, there is a chain $C \subseteq T^*$ s.t. $C \cap \mathfrak{L}_n(T^*) \neq \emptyset$ for each $n < \omega$. Since C is an infinite chain and the leaves of T^* are precisely the members of the antichain α, then C cannot intersect α. Since $\mathfrak{L}_n(T^*) \subseteq \mathfrak{L}_n(T)$, and since C intersects each $\mathfrak{L}_n(T^*)$, then C is a maximal chain (ie branch) in T. However, this is a contradiction since α is a slice and must intersect each branch in T. Therefore T^* has finite height which means that T^* is finite. Since $\alpha \subseteq T^*$, then α is finite as well.

This shows that every $\alpha \in A(T)$ is finite and so $B(T)$ is compact by Proposition 7.6.

□

7.3.1 And Their Slices

Corollary 7.8. *Let T be an ω-tree and $f : T \to \mathbb{R}$. Then for every $\alpha \in A(T)$ there is some $x \in \alpha$ s.t. $f(x) = f^+(\alpha)$.*

Proof. All slices are non-empty. Furthemore, by Propositions 7.6 and 7.7, we know that each $\alpha \in A(T)$ is finite. Therefore, given any $\alpha \in A(T)$, there is some $x \in \alpha$ s.t. $f(x) = \max\{f(y) : y \in \alpha\} = \sup\{f(y) : y \in \alpha\} = f^+(\alpha)$.

□

Notice that there are ω-trees with infinite maximal antichains; for instance consider the tree $(2^{<\omega}, \subset)$ and the maximal antichain C defined in the proof of Proposition 5.12.

But given a tree T with each level finite, is it the case that $B(T)$ is compact regardless of the height of T? And since we want pruned trees for the next chapter, then what if we were to also assume that T is pruned?

Proposition 7.9. *There is a pruned tree T with each level finite, height $\omega + \omega$, and with an infinite slice.*

Proof. Let f_k be the constant function 1 with domain $\omega + k$, and consider the tree $T = 2^{<\omega} \cup \{f_k : k < \omega\}$ with $\prec\, = \,\subset$. Notice $\mathfrak{L}_n(T)$ is occupied by some member of $2^{<\omega}$ whenever $n < \omega$. Furthermore, $f_k \in \mathfrak{L}_{\omega+k}(T)$ since $\text{height}(f_k) = \omega + k$. Moreover, $\mathfrak{L}_{\omega+\omega}(T) = \emptyset$, since each member of $2^{<\omega}$ has finite height and since $\text{height}(f_k) = \omega + k < \omega + \omega$. Notice that the levels $\mathfrak{L}_n(T)$ are finite. Furthermore, the levels $\mathfrak{L}_{\omega+k}(T)$ have only one member, namely, given $k < \omega$, $\mathfrak{L}_{\omega+k}(T) = \{f_k\}$. Therefore T is pruned, with each level of T finite, and height$(T) = \omega + \omega$.

Now let $\alpha = \{\langle 0\rangle, \langle 1,0\rangle, \langle 1,1,0\rangle, \langle 1,1,1,0\rangle, \ldots\} \cup \{f_0\}$. (In fact, we could have picked f_k for any $k < \omega$, the choice of f_0 is arbitrary.) Notice that any two distinct elements of $\{\langle 0\rangle, \langle 1,0\rangle, \langle 1,1,0\rangle, \langle 1,1,1,0\rangle, \ldots\}$ are incompatible. Moreover, since $f_0(i) = 1$ for each $i < \omega$, then f_0 is incompatible with each member of $\{\langle 0\rangle, \langle 1,0\rangle, \langle 1,1,0\rangle, \langle 1,1,1,0\rangle, \ldots\}$. Therefore α is an antichain.

Now take any $\beta \in B(T)$. Clearly there is only one branch of length larger than ω, and in fact it has length $\omega+\omega$ and is equal to $\{\langle\rangle, \langle 1\rangle, \langle 1,1\rangle, \langle 1,1,1\rangle, \ldots,\} \cup \{f_k : k < \omega\}$; in which case it contains f_0. (Here when we speak of the length of a branch β, we are actually referring to type(β)). If β has length ω, then it must contain some $x \in 2^{<\omega}$ s.t. $x(n) = 0$ for some $n < \omega$. Let $n < \omega$ be least s.t. $x(n) = 0$. Then if we take $y : n+1 \to \{0,1\}$ s.t. $y(i) = 1$ for $i < n$ and $y(n) = 0$, we then have $y \preccurlyeq x$, and therefore $y \in \beta$ by Proposition 1.8. Since $y \in \{\langle 0\rangle, \langle 1,0\rangle, \langle 1,1,0\rangle, \langle 1,1,1,0\rangle, \ldots\}$, then α intersects β. Hence $\alpha \in A(T)$ and is infinite.

$\square$

Chapter 8

A Constructive Approach

Lemma 4.6 is interesting since it basically says that if f^+ is monotone increasing, then given a slice α with $f^+(\alpha) \in \mathbb{R}$ and $\epsilon > 0$, there is some branch β with a tail starting at $\alpha \Cap \beta$ which will 'dominate' α to within ϵ. However, the proof of Lemma 4.6 was non-constructive. It gave us no feel or idea as to what such a branch would look like, only that there must be one with the property.

It turns out, that in the case of ω-trees, we can do better. We will not only be able to drop the ϵ of Lemma 4.6, but we will be able to get a non-empty closed set $\mathbb{B}$ of branches which witness this 'domination', and moreover if a particular branch does not belong to $\mathbb{B}$, we will know after finitely many stages in our construction.

Lastly, our theorem below would also hold even without the assumption that T is pruned, but we will impose this condition since the Cantor sets are typically represented by pruned ω-trees; in addition, the pruned condition will also make for a more straightforward construction.

8.1 Redoing Lemma 4.6

8.1.1 The Setup

Theorem 8.1. *Let T be a pruned ω-tree s.t. f^+ is monotone increasing. Then given any $\alpha \in A(T)$ there is a non-empty closed set $\mathbb{B} \subseteq B(T)$ s.t. $f^+(\alpha) \leq f(x)$ whenever $\beta \in \mathbb{B}$, $x \in \beta$, and $\alpha \Cap \beta \preccurlyeq x$.*

Proof. First, we define a system $\langle \alpha_i : i < \omega \rangle$ of subsets of T by recursion on ω:

$$\alpha_0 = \alpha$$

$$\alpha_{i+1} = K_i \cup \alpha_i \setminus M_i$$

where

$$M_i = \{x \in \alpha_i : f(x) = f^+(\alpha_i)\}$$

$$K_i = \bigcup_{x \in M_i} \{y \in T : x \prec y \wedge \text{height}(y) = \text{height}(x) + 1\}$$

.

8.1.2 Breaking for some Propositions

Proposition 8.2. *$\alpha_i \in A(T)$ for each $i < \omega$.*

Proof. We have $\alpha_0 = \alpha \in A(T)$ by assumption.

Now suppose $\alpha_i \in A(T)$. To see that $\alpha_{i+1} = K_i \cup \alpha_i \setminus M_i$ is an antichain: Take any distinct $u, v \in K_i$. If $u, v \in \{y \in T : x \prec y \wedge \text{height}(y) = \text{height}(x) + 1\}$ for some $x \in M_i$, then u and v have the same height, which must mean they are incompatible. Otherwise, $u \in \{y \in T : x \prec y \wedge \text{height}(y) = \text{height}(x) + 1\}$ and $v \in \{y \in T : z \prec y \wedge \text{height}(y) = \text{height}(z) + 1\}$ for distinct $x, z \in M_i$. If, say $u \prec v$ were the case, then we would have $x \prec v$ and $z \prec v$, which means that x and z are compatible. But $M_i \subseteq \alpha_i$ is an antichain. A contradiction. Therefore K_i is an antichain. Since α_i is an antichain and $\alpha_i \setminus M_i \subseteq \alpha_i$, then $\alpha_i \setminus M_i$ is an antichain. Now take any distinct $u \in K_i$ and $v \in \alpha_i \setminus M_i$. We then have $x \prec u$ for some $x \in M_i$. If $v \prec u$, then v and x would be comptabile. But $x \in M_i \subseteq \alpha_i$, and so $x, v \in \alpha_i$ are distinct, which is a contradiction. And if $u \prec v$, then $x \prec v$, and we have a contradiction just as in the previous case. This concludes the argument that α_{i+1} is an antichain.

To see that every $\beta \in B(T)$ intersects α_{i+1}: Take any $\beta \in B(T)$. If $\alpha_i \Cap \beta \in \alpha_i \setminus M_i$, then clearly β intersects α_{i+1}. Otherwise, $\alpha_i \Cap \beta \in M_i$, and so β must intersect $\{y \in T : \alpha_i \Cap \beta \prec y \wedge \text{height}(y) = \text{height}(\alpha_i \Cap \beta) + 1\} \subseteq K_i$ by maximality of β and the fact that T is pruned. Therefore $\alpha_{i+1} \in A(T)$.

This shows that $\alpha_i \in A(T)$ for each $i < \omega$.

$\square$

Proposition 8.3. *$\alpha_i \preccurlyeq \alpha_{i+1}$ for each $i < \omega$.*

Proof. Take any $x \in \alpha_{i+1} = K_i \cup \alpha_i \setminus M_i$. If $x \in K_i$, then $y \prec x$ for some $y \in M_i \subseteq \alpha_i$. In any case, given any $x \in \alpha_{i+1}$, there is some $y \in \alpha_i$ s.t. $y \preccurlyeq x$, and therefore $\alpha_i \preccurlyeq \alpha_{i+1}$ by Corollary C.8.

□

Proposition 8.4. *Given any $x \in M_{i+1}$, there is a $y \in M_i$ s.t. $y \prec x$.*

Proof. Take any $x \in M_{i+1}$ and notice $M_{i+1} \subseteq \alpha_{i+1} = K_i \cup \alpha_i \setminus M_i$. Suppose to the contrary that $x \in \alpha_i \setminus M_i$. Then it must be the case that $f(x) < f^+(\alpha_i)$. We know that $\alpha_i \preccurlyeq \alpha_{i+1}$ by Proposition 8.3. Since f^+ is monotone increasing, we must have $f^+(\alpha_i) \leq f^+(\alpha_{i+1})$. Therefore we have $f(x) < f^+(\alpha_i) \leq f^+(\alpha_{i+1}) = f(x)$ which is a contradiction. Hence $x \in K_i$, which means that $y \prec x$ for some $y \in M_i$.

□

Proposition 8.5. $\bigcup_{x \in M_i} \mathcal{B}_x \supseteq \bigcup_{x \in M_{i+1}} \mathcal{B}_x$ *for each $i < \omega$.*

Proof. Take any $\mathcal{B}_x$ where $x \in M_{i+1}$. We know there is some $y \in M_i$ s.t. $y \prec x$ by Proposition 8.4. We then have $\mathcal{B}_x \subseteq \mathcal{B}_y$ by Proposition 7.2. This shows $\bigcup_{x \in M_i} \mathcal{B}_x \supseteq \bigcup_{x \in M_{i+1}} \mathcal{B}_x$.

□

8.1.3 Back to the Proof

Since T is an ω-tree, then $B(T)$ is compact (w.r.t. $\mathcal{B}_T$) by Proposition 7.7. Therefore each slice is finite by Proposition 7.6. By Proposition 7.5, we have that each $\mathcal{B}_x$ is clopen. Furthermore, since each α_i is finite, then each $\bigcup_{x\in M_i}\mathcal{B}_x$ is clopen in the compact space $B(T)$. By Corollary 7.8, we get that each $M_i \neq \emptyset$. Therefore each $\bigcup_{x\in M_i}\mathcal{B}_x \neq \emptyset$ by Proposition 1.10. Notice the $\bigcup_{x\in M_i}\mathcal{B}_x$'s are nested by Proposition 8.5. Therefore Cantor's Intersection Theorem gives us that $\bigcap_{i<\omega}\bigcup_{x\in M_i}\mathcal{B}_x \neq \emptyset$.

So, let $\mathbb{B} = \bigcap_{i<\omega}\bigcup_{x\in M_i}\mathcal{B}_x$ which is non-empty and closed. Take any $\beta^* \in \mathbb{B}$. We need to see that β^* has the desired property, but first we need a couple more propositions.

8.1.4 Another Break

Proposition 8.6. *$f(\alpha_i \Cap \beta^*) = f^+(\alpha_i)$ for each $i < \omega$.*

Proof. Take any $i < \omega$. Since $\beta^* \in \bigcup_{x\in M_i}\mathcal{B}_x$, then $\beta^* \in \mathcal{B}_x$ for some $x \in \alpha_i$ s.t. $f(x) = f^+(\alpha_i)$. Hence $x \in \alpha_i \cap \beta^*$ which must mean that $\alpha_i \Cap \beta^* = x$. Therefore $f(\alpha_i \Cap \beta^*) = f^+(\alpha_i)$.

□

Proposition 8.7. *Suppose $x \in \beta^*$ s.t. $\alpha \Cap \beta^* \preccurlyeq x$, and let $k = height(x) - height(\alpha \Cap \beta^*)$. Then $x \in \alpha_k$.*

Proof. We induct on the difference in height between x and $\alpha \Cap \beta^*$.

The result clearly holds when height(x) = height$(\alpha \Cap \beta^*)$. So now suppose the result holds for a height difference of k. Let $x \in \beta^*$ s.t. $\alpha \Cap \beta^* \prec x$ and height(x) − height$(\alpha \Cap \beta^*) = k + 1$. Take $y \in \beta^*$ to be the immediate predecessor of

x (ie s.t. $y \prec x$ and $\text{height}(x) = \text{height}(y) + 1$). Since we now have $\alpha \Cap \beta^* \preccurlyeq y \prec x$ where $\text{height}(y) - \text{height}(\alpha \Cap \beta^*) = k$, then $y \in \alpha_k$. By Proposition 8.6, we know that $f(y) = f(\alpha_k \Cap \beta^*) = f^+(\alpha_k)$. So since $\text{height}(x) = \text{height}(y) + 1$ where $y \prec x$ and $y \in M_k$, then $x \in K_k \subseteq \alpha_{k+1}$.

$\square$

8.1.5 Finishing the Proof

So now take any $x \in \beta^*$ s.t. $\alpha \Cap \beta^* \preccurlyeq x$. By Proposition 8.7, we have $x \in \alpha_k$ where $k = \text{height}(x) - \text{height}(\alpha \Cap \beta^*)$. We then have $f(x) = f(\alpha_k \Cap \beta^*) = f^+(\alpha_k)$ by Proposition 8.6. Notice $\alpha = \alpha_0 \preccurlyeq \alpha_k$ by Proposition 8.3; and since f^+ is monotone increasing, then $f^+(\alpha) \leq f^+(\alpha_k) = f(x)$.

$\square$

To conclude, notice that given any $\beta \in B(T)$, we have $\beta \notin \mathbb{B} = \bigcap_{i<\omega} \bigcup_{x \in M_i} \mathcal{B}_x$ iff there is some $i < \omega$ s.t. $\beta \notin \bigcup_{x \in M_i} \mathcal{B}_x$ iff there is some $i < \omega$ s.t. $x \notin \beta$ for each $x \in M_i$.

This is what we meant when we said, "if a particular branch does not belong to $\mathbb{B}$, we will know after finitely many stages in our construction".

Appendix A: Supplementary Definitions

A.1 Logical and Set-Theoretic Notation

Remark A.1. *The symbols '* $\vee$ *' and '* $\wedge$ *' are logical connectives read 'or' and 'and' respectively.*

Remark A.2. *The set minus symbol, '* $\setminus$ *', always takes precedence over the set union symbol, '* $\cup$ *', and the set intersection symbol, '* $\cap$ *'. For instance,* $A \cup B \setminus C$ *is actually read* $A \cup (B \setminus C)$*.*

Remark A.3. *Given* $n < \omega$*, we often identify* n *with* $\{0, 1, \ldots, n-1\}$ *if* $n > 0$ *and with* $\emptyset$ *if* $n = 0$ *(as is common practice in set theory).*

A.2 Well-Founded Relations and Chains

Definition A.4. Let R be a binary relation on a set X. Then $y \in X$ is R-minimal (or R-least) in X if there is no $z \in X$ s.t. zRy, and R-maximal in X if there is no

$z \in X$ s.t. yRz.

Notice that if R is a binary relation on a set A and $X \subseteq A$, then R is also a binary relation on X by simply restricting R to X.

Definition A.5. Let R be a binary relation on a set A. Then R is well-founded on A if for all non-empty $X \subseteq A$, there is a $y \in X$ s.t. y is R-minimal in X.

Proposition A.6. *Let $(T, \prec)$ be a tree. Then $\prec$ is well-founded on T.*

Proof. Let $C \subseteq T$ s.t. $C \neq \emptyset$. Since C is non-empty, there is a $y \in C$. If y is not $\prec$-minimal in C, then a $\prec$-minimal element in $y \downarrow \cap C \neq \emptyset$ would also be $\prec$-minimal in C.

□

Remark A.7. *We will often have a tree T and be working in the context of some $X \subseteq T$, and we'll just say '$\prec$-minimal' (or '$\prec$-least'); in which case it is understood that we really mean $\prec$-minimal in X.*

Definition A.8. Let $<$ be a strict partial order of a set A. Then $C \subseteq A$ is a chain if C is totally ordered by $<$; C is a maximal chain if in addition there are no chains $X \supsetneq C$.

A.3 More Definitions Concerning Trees

Definition A.9. Let $(T, \prec)$ be a tree.

Then for all $y \in T$, ordinals γ, and cardinals κ:

- y is a leaf if $y\uparrow = \emptyset$
- T is pruned if T has no leaves
- $\text{height}(y) = \text{type}(y\downarrow)$
- $\mathfrak{L}_\gamma(T) = \{y \in T : \text{ height}(y) = \gamma\} = \text{level } \gamma \text{ of T }.$
- $\text{height}(T)$ is the least γ s.t. $\mathfrak{L}_\gamma(T) = \emptyset$.
- T is a κ-tree if each level of T has cardinality less than κ and $\text{height}(T) = \kappa$.

A.4 Notation Appearing in Counterexamples

Definition A.10. For all sets A, B and ordinals γ, ξ:

- $B^A =$ set of all functions f s.t. $f : A \to B$.
- $B^{<\gamma} = \bigcup\{B^\xi : \xi < \gamma\}$
- Given a function of the form $f : A \to B$, we use $\text{dom}(f)$ to denote the domain of f; in this case we have $\text{dom}(f) = A$.

Definition A.11. For all sets B and ordinals γ, we define a strict partial order $\subset$ on $B^{<\gamma}$ s.t. $h \subset g \iff \text{dom}(h) < \text{dom}(g) \wedge g \restriction \text{dom}(h) = h$.

Remark A.12. *Definition A.11 is really just saying that* $h \subset g \iff h \subsetneq g$ *(as sets). So to avoid confusion, we will only use the symbol '* $\subset$ *' in this particular context and we will never use it to mean 'is a subset of'.*

Definition A.13. Given $s : n \to B$ for some $n < \omega$, if $n > 0$ then we use the sequence notation $\langle s(0), s(1), \ldots, s(n-1)\rangle$ to represent s, and if $n = 0$ we use $\langle\rangle$ to represent s.

Definition A.14. Given $s : n \to B$ and $t : m \to B$ for $n, m < \omega$, if s and t have corresponding sequence notations $\langle s(0), s(1), \ldots, s(n-1)\rangle$ and $\langle t(0), s(1), \ldots, t(m-1)\rangle$ respectively, we define the concatenation of s and t, denoted $s^\frown t$, to be the function with corresponding sequence notation $\langle s(0), s(1), \ldots, s(n-1), t(0), t(1), \ldots, t(m-1)\rangle$; otherwise at least one of s and t has empty domain, in which case $s^\frown t = s$ if $\mathrm{dom}(t) = \emptyset$ and $s^\frown t = t$ if $\mathrm{dom}(s) = \emptyset$.

Appendix B: Supplementary Proofs

Let T be a tree and $f : T \to \mathbb{R}$ a function. Furthermore, also assume that $\sup \limsup f_\beta = -\infty$.

B.1 Section 4.2

Theorem B.1. *If f^+ is monotone decreasing, then lim $\bar{f} = -\infty$.*

Proof. We need to see that the set of eventual upper bounds for f^+ is unbounded below.

Take $U \in \mathbb{R}$ and $\epsilon > 0$ arbitrary. Since $\sup \limsup f_\beta = -\infty$, then $\limsup f \restriction \beta = -\infty$ for each $\beta \in B(T)$. So for each $\beta \in B(T)$, there is some $y_\beta \in \beta$ s.t. $f(y_\beta) < U + \epsilon$. Then proceed as in the proof of Theorem 4.5, to get $\alpha' \in A(T)$ s.t. $f^+(\alpha') \leq U + \epsilon$. And since f^+ is monotone decreasing, then $\alpha' \preccurlyeq \alpha$ implies $f^+(\alpha) \leq f^+(\alpha') \leq U + \epsilon$. Since $\epsilon > 0$ was arbitrary, then U is an eventual upper bound for f^+. Since $U \in \mathbb{R}$ was arbitrary, then the set of eventual upper bounds for f^+ is unbounded below.

□

We will use Lemma 4.6 in the proof of the following lemma; which is legitimate since the proof of Lemma 4.6 is independent of sup lim sup f_β.

Lemma B.2. *If f^+ is monotone increasing then $f^+(\alpha) \neq \mathbb{R}$ for each $\alpha \in A(T)$.*

Proof. Let $\alpha \in A(T)$ and suppose to the contrary that $f^+(\alpha) \in \mathbb{R}$. We know that there is some $\beta^* \in B(T)$ s.t. $\alpha \Cap \beta^* \preccurlyeq x$ and $x \in \beta^*$ implies $f^+(\alpha) \leq f(x) + 1$ by Lemma 4.6. Since sup lim sup $f_\beta = -\infty$, then lim sup $f \restriction \beta^* = -\infty$. Therefore there is some $y \in \beta^*$ s.t. $y \preccurlyeq x$ and $x \in \beta^*$ implies $f(x) < f^+(\alpha) - 1$. Let z be the $\prec$-maximum of y and $\alpha \Cap \beta^*$. Then $f^+(\alpha) \leq f(z) + 1 < f^+(\alpha)$. This is a contradiction.

□

Lemma B.3. *If f^+ is monotone increasing then $f^+(\alpha) \neq \infty$ for each $\alpha \in A(T)$.*

Proof. Let $\alpha \in A(T)$ and suppose to the contrary that $f^+(\alpha) = \infty$. Since we have sup lim sup $f_\beta = -\infty$, then lim sup $f \restriction \beta = -\infty$ for each $\beta \in B(T)$. So given any $\beta \in B(T)$, there is a $y_\beta \in \beta$ s.t. $y_\beta \preccurlyeq x$ and $x \in \beta$ implies $f(x) < 1$. Now for each $\beta \in B(T)$, let z_β be the $\prec$-maximum of y_β and $\alpha \Cap \beta$.

Let $C = \{z_\beta : \beta \in B(T)\}$ and then let $\alpha' = C_\prec$. We then have $\alpha' \in A(T)$ and $\alpha \preccurlyeq \alpha'$ by Proposition 1.9 and Corollary C.8. And since f^+ is monotone increasing, then $f^+(\alpha) \leq f^+(\alpha')$. Now, take any $z \in \alpha'$. We know that $z = z_\beta$ for some

$\beta \in B(T)$ where $y_\beta \preccurlyeq z_\beta$ and $z_\beta \in \beta$. This must mean that $f(z) < 1$. Since $z \in \alpha'$ was arbitrary, then $f^+(\alpha') \leq 1$. This is a contradiction.

□

Theorem B.4. *If $T \neq \emptyset$, then f^+ is not monotone increasing.*

Proof. Suppose to the contrary that f^+ is monotone increasing. Since $T \neq \emptyset$, then $A(T) \neq \emptyset$. So take any $\alpha \in A(T)$. Then $f^+(\alpha) \notin \mathbb{R}$ and $f^+(\alpha) \neq \infty$ by Lemmas B.2 and B.3. Therefore $f^+(\alpha) = -\infty$ which must mean that $\alpha = \emptyset$. But, since $T \neq \emptyset$, then every slice must be non-empty. This is a contradiction.

□

B.2 Section 5.3

Theorem B.5. *lim* $\hat{f} = -\infty$

Proof. We need to see that the set of eventual upper bounds for f^* is unbounded below.

Take $U \in \mathbb{R}$ and $\epsilon > 0$ arbitrary. Since sup lim sup $f_\beta = -\infty$, then lim sup $f \restriction \beta = -\infty$ for each $\beta \in B(T)$. So for each $\beta \in B(T)$, there is some $y \in \beta$ s.t. $y \preccurlyeq x$ and $x \in \beta$ implies $f(x) < U + \epsilon$.

Let $E = \{x \in T : f(x) < U + \epsilon\}$, and proceed as in the proof of Theorem 5.11 to get an $A' \in \mathfrak{M}(T)$ s.t. $A' \sqsubseteq A$ and $A \in \mathfrak{M}(T)$ implies $f^*(A) \leq U + \epsilon$. Since

$\epsilon > 0$ was arbitrary, then U is an eventual upper bound for f^*. Since $U \in \mathbb{R}$ was arbitrary, then the set of eventual upper bounds for f^* is unbounded below.

□

Appendix C: The Various Orderings

Let T be a tree.

Definition C.1. Given $C, D \subseteq T$, we write $C \sqsubseteq_0 D$ if for every $x \in C$ there is some $y \in D$ s.t. $x \preccurlyeq y$, and we write $C \sqsubseteq_1 D$ if for every $x \in D$ there is some $y \in C$ s.t. $y \preccurlyeq x$.

Remark C.2. *Notice that given any $C, D \subseteq T$, we have $C \sqsubseteq_1 D \iff D \subseteq F(C) \iff F(D) \subseteq F(C)$ (cf. Definition 2.1).*

Definition C.3. We denote the set of antichains by $\mathfrak{A}(T)$.

Proposition C.4. *Let $U, U' \in \mathfrak{A}(T)$:*

1. *If $U \sqsubseteq_0 U'$ and U is maximal, then $U \sqsubseteq_1 U'$*

2. *If $U \sqsubseteq_1 U'$ and U' is maximal, then $U \sqsubseteq_0 U'$*

Proof. For 1: Let $x \in U'$ and suppose $x \notin U$. Since U is maximal, then there is some $y \in U$ s.t. x and y are compatible. Suppose to the contrary that $x \prec y$. Since $U \sqsubseteq_0 U'$, then there is some $z \in U'$ s.t. $y \preccurlyeq z$. And then we have $x \prec z$ by transitivity, contradicting the fact that U' is an antichain. Therefore we must have $y \prec x$.

For 2: Let $x \in U$ and suppose $x \notin U'$. Since U' is maximal, then there is some $y \in U'$ s.t. x and y are compatible. Suppose to the contrary that $y \prec x$. Since $U \sqsubseteq_1 U'$, then there is some $z \in U$ s.t. $z \preccurlyeq y$. And then we have $z \prec x$ by transitivity, contradicting the fact that U is an antichain. Therefore we must have $x \prec y$.

□

Notice that $\sqsubseteq_0$ is the same as $\sqsubseteq$ when restricted to $\mathfrak{M}(T)$ (cf. Definitions 5.1 and 5.3). More generally:

Corollary C.5. *Given $A, A' \in \mathfrak{M}(T)$, we have $A \sqsubseteq_0 A' \iff A \sqsubseteq_1 A' \iff A' \subseteq F(A) \iff F(A') \subseteq F(A)$.*

Proposition C.6. *Let $U, U' \in \mathfrak{A}(T)$ and suppose $U \sqsubseteq_1 U'$. Then given any $\beta \in B(T)$, if $U' \cap \beta \neq \emptyset$, then $U \cap \beta \neq \emptyset$.*

Proof. Take any $\beta \in B(T)$ s.t. $U' \cap \beta \neq \emptyset$. Let $x \in U' \cap \beta$. Since $U \sqsubseteq_1 U'$, then there is some $y \in U$ s.t. $y \preccurlyeq x$. And therefore $y \in \beta$ by Proposition 1.8. Hence $U \cap \beta \neq \emptyset$.

□

Proposition C.7. *Let $\alpha, \alpha' \in A(T)$. Then $\alpha \preccurlyeq \alpha' \iff \alpha \sqsubseteq_0 \alpha'$.*

Proof. First suppose $\alpha \preccurlyeq \alpha'$. Take any $x \in \alpha$. Then $x = \alpha \Cap \beta$ for some $\beta \in B(T)$ s.t. $x \in \beta$. Since $\alpha \preccurlyeq \alpha'$, then $x = \alpha \Cap \beta \preccurlyeq \alpha' \Cap \beta$. Since $x \in \alpha$ was arbitrary, then $\alpha \sqsubseteq_0 \alpha'$.

Now suppose $\alpha \sqsubseteq_0 \alpha'$, and take any $\beta \in B(T)$. Suppose to the contrary that $\alpha' \Cap \beta \prec \alpha \Cap \beta$. Since $\alpha \sqsubseteq_0 \alpha'$, then there is some $y \in \alpha'$ s.t. $\alpha \Cap \beta \preccurlyeq y$. Therefore $\alpha' \Cap \beta \prec y$ by transitivity, which contradicts the fact that α' is an antichain. Hence $\alpha \Cap \beta \preccurlyeq \alpha' \Cap \beta$. Since $\beta \in B(T)$ was arbitrary, then $\alpha \preccurlyeq \alpha'$.

□

Since all slices are maximal antichains (cf. Proposition 6.1):

Corollary C.8. *Given $\alpha, \alpha' \in A(T)$, we have $\alpha \preccurlyeq \alpha' \iff \alpha \sqsubseteq_0 \alpha' \iff \alpha \sqsubseteq_1 \alpha' \iff \alpha' \subseteq F(\alpha) \iff F(\alpha') \subseteq F(\alpha)$.*

Appendix D: Directed Sets and Nets

D.1 Directed Sets

Definition D.1. A directed set is a pair $(A, \sqsubseteq)$ s.t. $\sqsubseteq$ is transitive and reflexive on A; and furthermore, given $a, b \in A$, there is a $c \in A$ s.t. $a \sqsubseteq c$ and $b \sqsubseteq c$, which is called an upper bound for a and b.

Definition D.2. Let $(A, \sqsubseteq)$ be a directed set. We say $B \subseteq A$ is cofinal in A if for each $a \in A$ there exists $b \in B$ s.t. $a \sqsubseteq b$.

D.2 Nets

Definition D.3. A net is a function $\mathcal{G} : A \to X$ where A is a directed set and X is a topological space.

Definition D.4. Let $(A, \sqsubseteq_A)$ and $(B, \sqsubseteq_B)$ be directed sets with $\mathcal{G} : A \to X$ a net and $g : B \to A$ a function s.t.:

1. $g(B)$ is cofinal in A

2. Given $i, j \in B$, if $i \sqsubseteq_B j$ then $g(i) \sqsubseteq_A g(j)$

then we say that the composite function $\mathcal{G} \circ g : B \to X$ is a subnet of $\mathcal{G}$.

Definition D.5. Let $\mathcal{G} : A \to \mathbb{R}$ be a net.

1. We say that $\mathcal{G}$ converges to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is some $a \in A$ s.t. $a \sqsubseteq b$ and $b \in A$ implies $|\mathcal{G}(b) - x| < \epsilon$.

2. We say $U \in \mathbb{R}$ is an eventual upper bound for $\mathcal{G}$ if for every $\epsilon > 0$ there is some $a \in A$ s.t. $a \sqsubseteq b$ and $b \in A$ implies $\mathcal{G}(b) < U + \epsilon$; and we say $L \in \mathbb{R}$ is an eventual lower bound for $\mathcal{G}$ if for every $\epsilon > 0$ there is some $a \in A$ s.t. $a \sqsubseteq b$ and $b \in A$ implies $\mathcal{G}(b) > L - \epsilon$.

3. lim sup $\mathcal{G} :=$ inf$\{U \in \mathbb{R} : U$ is an eventual upper bound for $\mathcal{G}\}$, and lim inf $\mathcal{G}$ $:=$ sup$\{L \in \mathbb{R} : L$ is an eventual lower bound for $\mathcal{G}\}$.

Since $\mathbb{R}$ is Hausdorff, then $\mathcal{G}$ can converge to at most one $x \in \mathbb{R}$. So, we write lim $\mathcal{G} = x$ to denote that $\mathcal{G}$ converges to x. Also, given a net $\mathcal{G} : A \to \mathbb{R}$ with $a \in A$, when we write $a \sqsubseteq b$, it is understood that $b \in A$.

Definition D.6. Given a net $\mathcal{G} : A \to \mathbb{R}$, we say that $\mathcal{G}$ is monotone decreasing if $\forall x, y \in A[x \sqsubseteq y \implies \mathcal{G}(y) \leq \mathcal{G}(x)]$ and we say that $\mathcal{G}$ is monotone increasing if $\forall x, y \in A[x \sqsubseteq y \implies \mathcal{G}(x) \leq \mathcal{G}(y)]$.

D.2.1 Some Properties of Nets

Let $\mathcal{G} : A \to \mathbb{R}$ be a net (where A is ordered by $\sqsubseteq_A$ for the sake of Proposition D.9).

We're about to see that nets of this form have some of the same desirable properties as sequences in analysis.

Proposition D.7. *lim inf* $\mathcal{G} \leq$ *lim sup* $\mathcal{G}$

Proof. Let L an eventual lower bound for $\mathcal{G}$, U an eventual upper bound for $\mathcal{G}$ and take $\epsilon > 0$ arbitrary. Then there are $a, a' \in A$ s.t. $a \sqsubseteq b$ implies $\mathcal{G}(b) < U + \frac{\epsilon}{2}$ and $a' \sqsubseteq b$ implies $\mathcal{G}(b) > L - \frac{\epsilon}{2}$. Since A is a directed set, we can take $a, a' \sqsubseteq c$ for some $c \in A$, in which case we have $L - \frac{\epsilon}{2} < \mathcal{G}(c) < U + \frac{\epsilon}{2}$, which implies that $L < U + \epsilon$. Since $\epsilon > 0$ was arbitrary, then $L \leq U$. And since U and L were arbitrary eventual upper and lower bounds for $\mathcal{G}$ respectively, then lim inf $\mathcal{G} \leq$ lim sup $\mathcal{G}$.

□

Proposition D.8. *Given $x \in \mathbb{R}$, we have lim $\mathcal{G} = x$ iff lim inf $\mathcal{G} =$ lim sup $\mathcal{G} = x$.*

Proof. First suppose that lim $\mathcal{G} = x$. By Proposition D.7, it suffices to show that lim sup $\mathcal{G} \leq x \leq$ lim inf $\mathcal{G}$. Since $\mathcal{G}$ converges to x, then given $\epsilon > 0$, there is some $a \in A$ s.t. $a \sqsubseteq b$ implies that $|\mathcal{G}(b) - x| < \epsilon$. And this means that $-\epsilon < \mathcal{G}(b) - x < \epsilon$ ie $-\epsilon + x < \mathcal{G}(b) < \epsilon + x$. Therefore x is both an eventual lower bound and an eventual upper bound for $\mathcal{G}$, which means that lim sup $\mathcal{G} \leq x \leq$ lim inf $\mathcal{G}$.

Now suppose that lim inf $\mathcal{G} =$ lim sup $\mathcal{G} = x$. Then given $\epsilon > 0$, we know there are U and L eventual upper and lower bounds for $\mathcal{G}$ respectively s.t. lim inf $\mathcal{G} - \frac{\epsilon}{2} < L$

and lim sup $\mathcal{G}+\frac{\epsilon}{2} > U$. Moreover, there are $a, a' \in A$ s.t. $a \sqsubseteq b$ implies $\mathcal{G}(b) < U+\frac{\epsilon}{2}$ and $a' \sqsubseteq b$ implies $\mathcal{G}(b) > L - \frac{\epsilon}{2}$. Therefore taking $c \in A$ s.t. $a, a' \sqsubseteq c$, we get that $c \sqsubseteq b$ implies $x-\epsilon < \mathcal{G}(b) < x+\epsilon$, ie $|\mathcal{G}(b)-x| < \epsilon$, by transitivity. Hence lim $\mathcal{G} = x$. □

Proposition D.9. *The limit of any convergent subnet of $\mathcal{G}$ is in the interval $[lim\ inf\,\mathcal{G}, lim\ sup\ \mathcal{G}]$.*

Proof. Let $\mathcal{G} \circ g : B \to \mathbb{R}$ be a convergent subnet of $\mathcal{G}$ with lim $\mathcal{G} \circ g = x \in \mathbb{R}$. Let U, L arbitrary eventual upper and lower bounds for $\mathcal{G}$ respectively and take any $\epsilon > 0$. Then there are $a_0, a_1 \in A$ s.t. $a_0 \sqsubseteq_A b$ implies $\mathcal{G}(b) < U + \frac{\epsilon}{2}$ and $a_1 \sqsubseteq_A b$ implies $\mathcal{G}(b) > L - \frac{\epsilon}{2}$. Since the net $\mathcal{G} \circ g$ converges to x, then there is some $i \in B$ s.t. $i \sqsubseteq_B j$ implies $x - \frac{\epsilon}{2} < \mathcal{G}(g(j)) < x + \frac{\epsilon}{2}$.

Since $g(B)$ is cofinal in A, then there are $i_0, i_1 \in B$ s.t. $a_0 \sqsubseteq_A g(i_0)$ and $a_1 \sqsubseteq_A g(i_1)$. Moreover, since B is a directed set, there are $j_0, j_1 \in B$ s.t. $i, i_0 \sqsubseteq_B j_0$ and $i, i_1 \sqsubseteq_B j_1$. In particular, we now have $x - \frac{\epsilon}{2} < \mathcal{G}(g(j_0))$ and $\mathcal{G}(g(j_1)) < x + \frac{\epsilon}{2}$. Furthermore, since $\mathcal{G} \circ g$ is a subnet, we have $a_0 \sqsubseteq_A g(i_0) \sqsubseteq_A g(j_0)$ and $a_1 \sqsubseteq_A g(i_1) \sqsubseteq_A g(j_1)$, in which case we now also have $\mathcal{G}(g(j_0)) < U + \frac{\epsilon}{2}$ and $\mathcal{G}(g(j_1)) > L - \frac{\epsilon}{2}$. Putting all this together, we have $x - \frac{\epsilon}{2} < \mathcal{G}(g(j_0)) < U + \frac{\epsilon}{2}$ and $L - \frac{\epsilon}{2} < \mathcal{G}(g(j_1)) < x + \frac{\epsilon}{2}$. Therefore $L - \epsilon < x < U + \epsilon$.

Since $\epsilon > 0$ was arbitary, then $L \leq x \leq U$. Since U, L were arbitrary eventual upper and lower bounds for $\mathcal{G}$ respectively, then lim inf $\mathcal{G} \leq x \leq$ lim sup $\mathcal{G}$ ie $x \in$ [lim inf $\mathcal{G}$, lim sup $\mathcal{G}$]. □

Appendix E: Summary of Main Results

Here we will summarize our main results concerning how sup lim sup f_β, lim $\bar{f}$, and lim $\hat{f}$ compare with one another.

E.1 The Positive Results

Given any non-empty tree T and any function $f : T \to \mathbb{R}$ we have:

- lim $\hat{f} \leq$ sup lim sup $f_\beta \leq$ lim $\bar{f}$
- If f^+ is monotone, then sup lim sup $f_\beta =$ lim $\bar{f}$.
- If f^* is monotone decreasing, then sup lim sup $f_\beta =$ lim $\bar{f}$.
- If $f \restriction \beta$ is monotone decreasing for each $\beta \in B(T)$, then sup lim sup $f_\beta =$ lim $\bar{f}$.

- If f^* is monotone increasing, then $\lim \hat{f} = \sup \limsup f_\beta = \lim \bar{f}$.

- If $f \restriction \beta$ is monotone increasing for each $\beta \in B(T)$, then $\lim \hat{f} = \sup \limsup f_\beta = \lim \bar{f}$.

- If for every $\beta \in B(T)$ there is some $x \in \beta$ s.t. $x \uparrow \subseteq \beta$, then $\lim \hat{f} = \sup \limsup f_\beta = \lim \bar{f}$.

E.2 The Negative Results

Consider the tree $(2^{<\omega}, \subset)$, and let (†) be any one of the following statements:

1. $\lim \hat{f} < \sup \limsup f_\beta < \lim \bar{f}$
2. f^+ is monotone decreasing and $\lim \hat{f} < \sup \limsup f_\beta$
3. f^+ is monotone increasing and $\lim \hat{f} < \sup \limsup f_\beta$
4. f^* is monotone decreasing and $\lim \hat{f} < \sup \limsup f_\beta$
5. $f \restriction \beta$ is monotone decreasing for each $\beta \in B(T)$ and $\lim \hat{f} < \sup \limsup f_\beta$

Then there is a function $f : 2^{<\omega} \to [0,1] \cap \mathbb{Q}$ s.t. (†) holds.

Bibliography

[1] Kenneth Kunen, *Set Theory*, Studies in Logic (London), 34, College Publications, London, 2011.

[2] Yiannis Moschovakis, *Notes on Set Theory*, Second edition, Undergraduate Texts in Mathematics, Springer, New York, 2006.