A Generating-Function Approach for Reciprocity Formulae of Dedekind-like Sums

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Abstract

Apostol sums and Hall-Wilson-Zagier sums are generalizations of Dedekind sums, important finite arithmetic sums introduced by Dedekind in the 1880's with applications in various fields of mathematics and computer science. These sums satisfy numerous reciprocity laws that have been the object of much research. We introduce and prove an identity which serves as a generalization of Apostol and Rademacher reciprocity for the case of Apostol sums with three variables. We do this by showing how it is implied by the Hall-Wilson-Zagier reciprocity and also show how Apostol and Rademacher follow as special cases.

1 Introduction: Dedekind Sums, Their Reciprocity Laws and Generalizations

In the 19th century, Richard Dedekind introduced Dedekind sums in his study [2] of the η-function

$$\eta(z) := e^{\frac{\pi i z}{12}} \prod_{n\geq 1}(1 - e^{2\pi i n z}).$$

Bernoulli polynomials are defined to be the coefficients of the generating function

$$\frac{e^{uz}}{e^z - 1} = \sum_{k\geq 0} \frac{B_k(u)}{k!} z^{k-1}.$$

The first few Bernoulli polynomials are:

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x - \frac{1}{2} \\ B_2(x) &= x^2 - x + \frac{1}{6} \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x. \end{aligned}$$

The periodized Bernoulli polynomial $\overline{B}_n(x)$ is obtained by evaluating the Bernoulli polynomial at the fractional part of x and defining $\overline{B}_1(x) = 0$ if $x \in \mathbb{Z}$. The classic Dedekind sum is defined as follows:

Definition 1. *For $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$, the Dedekind sum is*

$$s(a, b) := \sum_{h=0}^{b-1} \overline{B}_1\left(\frac{ah}{b}\right) \overline{B}_1\left(\frac{h}{b}\right).$$

The following well-known identity will prove useful throughout.

Lemma 1 (Raabe's Formula [1])**.** *For $a \in \mathbb{N}$, $x \in \mathbb{R}$,*

$$\sum_{h \bmod a} \overline{B}_m\left(x + \frac{h}{a}\right) = a^{1-m}\overline{B}_m(ax).$$

Dedekind sums appear naturally in various fields of mathematics and computer science. For example, they arise in lattice-point enumeration in combinatorics and random-number generation in computer science. Dedekind sums satisfy various reciprocity laws that have been the subject of much research for more than a century.

1.1 Reciprocity Formulae

In this context, reciprocity refers to the addition of Dedekind sums with certain permutations in their arguments resulting in a relatively simple expression. Dedekind proved the following reciprocity law in [2].

Theorem 1 (Dedekind Reciprocity)**.** *For $a, b \in \mathbb{N}$, $\gcd(a, b) = 1$ the following holds:*

$$s(a, b) + s(b, a) = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) - \frac{1}{4}.$$

Because of the simplicity of the right side of the Dedekind reciprocity law, the law is useful in computing Dedekind sums with large periods. For this purpose, we also need the following lemma, which establishes the periodicity of the Dedekind sum.

Lemma 2 (Periodicity of the Dedekind Sum)**.** *For $a, b \in \mathbb{N}$ with* $\gcd(a, b) = 1$*,*

$$s(a, b) = s(a \ mod\ b, b)$$

Proof. Because $\overline{B}_1(x)$ is periodic with period 1,

$$\begin{aligned}
s(a \bmod b, b) &= \sum_{h \bmod b} \overline{B}_1\left(\frac{(a \bmod b)h}{b}\right)\overline{B}_1\left(\frac{h}{b}\right) \\
&= \sum_{h \bmod b} \overline{B}_1\left(\frac{ah}{b}\right)\overline{B}_1\left(\frac{h}{b}\right) \\
&= s(a, b).
\end{aligned}$$

□ □

Using Theorem 1 and Lemma 2, we can significantly reduce the steps required to compute a Dedekind sum. For example, consider the Dedekind sum $s(100, 147)$. By Theorem 1, we can express it as follows:

$$s(100, 147) = \frac{1}{12}\left(\frac{100}{147} + \frac{147}{100} + \frac{1}{(100)(147)}\right) - \frac{1}{4} - s(147, 100).$$

By simplifying and using Lemma 2 on the sum $s(147, 100)$, we now have

$$s(100, 147) = -\frac{1249}{17640} - s(47, 100).$$

Continuing in this manner, we obtain

$$s(100, 147) = \frac{577}{882}.$$

The use of Theorem 1 and Lemma 2 in this example reduces the number of computations from 147 to only 9.

In [6], Rademacher introduced a more general reciprocity law with three Dedekind sums.

Theorem 2 (Rademacher Reciprocity)**.** *For pairwise relatively prime $a, b, c \in \mathbb{N}$, the following holds:*

$$s(ab^{-1}, c) + s(bc^{-1}, a) + s(ca^{-1}, b) = \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) - \frac{1}{4}.$$

Here, the inverses denote the inverse of the integer mod the period of the corresponding sum.

Dedekind's reciprocity is a special case of Rademacher's relation. This is seen by setting $c = 1$ and by using the identity $s(a, b) = s(a^{-1}, b)$.

In [5], Pommersheim introduced a reciprocity law while studying Dedekind sums in the context of algebraic geometry.

Theorem 3 (Pommersheim Reciprocity)**.** *For $a_1, a_2, b_1, b_2 \in \mathbb{Z}_{>0}$ with $\gcd(a_1, a_2) = \gcd(b_1, b_2) = 1$ and $b_1^*, b_2^* \in \mathbb{Z}$ such that $b_1 b_1^* + b_2 b_2^* = 1$, let $x = a_2 b_2^* - a_1 b_1^*$, $y = a_1 b_2 + a_2 b_1$. Then*

$$s(a_1, a_2) + s(b_1, b_2) + s(x, y) = \frac{1}{12}\left(\frac{a_2}{b_2 x} + \frac{b_2}{a_2 x} + \frac{x}{a_2 b_2}\right) - \frac{1}{4}.$$

Rademacher reciprocity follows from Pommersheim reciprocity: First, let $a_2, b_2, y \in \mathbb{N}$ be arbitrary yet pairwise relatively prime. Then, choose a_1, b_1 such that $a_1 b_2 + a_2 b_2 = y$ and use the identity $s(a_1, a_2) = s(a_1^{-1}, a_2)$.

In [3], Kurt Girstmair showed the relationship between the three reciprocity laws above by proving that Pommersheim reciprocity follows from Dedekind reciprocity. By his work, we see that the three reciprocity theorems are actually logically equivalent.

1.2 Generalizations of Dedekind sums

Various generalizations of the classic Dedekind sum have been introduced. These generalizations have interested researchers from various fields of mathematics and computer science. One of the more fruitful ways of generalizing Dedekind sums involves considering higher-degree periodized Bernoulli polynomials. In [1], Tom Apostol introduced such a generalization by letting one of the Bernoulli polynomials in the classic Dedekind sum be arbitrary.

Definition 2 (Apostol Sum)**.** *For $a, b, p \in \mathbb{N}$ with $\gcd(a, b) = 1$,*

$$s_p(a, b) := \sum_{h=0}^{b-1} \overline{B}_1\left(\frac{h}{b}\right) \overline{B}_p\left(\frac{ah}{b}\right).$$

In [1], Apostol proved the following reciprocity law for the Apostol sums.

Theorem 4 (Apostol Reciprocity)**.** *For $a, b, p \in \mathbb{N}$ where p is odd,*

$$\begin{aligned}&(p+1)(ab^p s_p(a, b) + a^p b s_p(b, a)) \\ &\qquad = \sum_{s=0}^{p+1} \binom{p+1}{s} (-1)^s B_s a^s B_{p+1-s} b^{p+1-s} + pB_{p+1}.\end{aligned}$$

Here, B_s represents the s^{th} Bernoulli number obtained by evaluating the Bernoulli polynomials $B_s(x)$ at $x = 0$. As stated by Apostol, Theorem 1 may be written as

$$2(abs_1(a, b) + bas_1(b, a)) = B_2 a^2 - 2B_1^2 ab + B_2 b^2 + B_2,$$

from which it follows that Dedekind reciprocity is a special case of Apostol reciprocity when $p = 1$. Because of this, and because one can observe the same permutation in the arguments of the sums resulting in a relatively simple expression for these sums, Apostol's reciprocity can be seen as a direct analogue of Dedekind's law at the level of Apostol sums.

Our goal in this report is to find analogues of Rademacher and Pommersheim reciprocity at the level of Apostol sums using a generating-function approach. We seek reciprocity laws that can be reduced to the special cases of Rademacher and Pommersheim reciprocity for the case $p = 1$ of Apostol sums, much like Apostol reciprocity reduces to Dedekind reciprocity when $p = 1$.

It is natural to consider the more general Dedekind-like sum and use results from [4] at the generating-function level to seek patterns at the level of Apostol sums. In [4] Hall, Wilson, and Zagier introduced the HWZ sum.

Definition 3 (HWZ Sum)**.** *For relatively pairwise coprime $a_1, a_2, a_3 \in \mathbb{N}$ and $m, n \in \mathbb{N}$, the HWZ sum is defined by*

$$S_{m,n}(a_1, a_2, a_3) := \sum_{h=0}^{a_3-1} \overline{B}_m\left(a_1\frac{h}{a_3}\right)\overline{B}_n\left(a_2\frac{h}{a_3}\right).$$

Apostol sums are clearly a special case of these HWZ sums where $a_1 = 1$, $a_2 = a$, $a_3 = b$, $m = 1$, and $n = p$. Hall, Wilson, and Zagier also introduced the generating function of the HWZ sum.

Definition 4 (Hall-Wilson-Zagier Generating Function)**.** *For relatively pairwise coprime $a_1, a_2, a_3 \in \mathbb{N}$, the generating function of the HWZ sum is defined as*

$$\Omega\left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ y_1 & y_2 & \end{array}\right) := \sum_{m,n\geq 0} \frac{1}{m!n!} S_{m,n}(a_1, a_2, a_3)\left(\frac{y_1}{a_1}\right)^{m-1}\left(\frac{y_2}{a_2}\right)^{n-1}.$$

The terms in the coefficient (i.e., the factorials and the a_1^{1-m}, a_2^{1-n}) are introduced by Hall, Wilson, and Zagier to ensure the following reciprocity law, also proven in [4], at the level of generating functions:

Theorem 5 (HWZ Reciprocity)**.** *Letting $a_1, a_2, a_3 \in \mathbb{N}$, all pairwise coprime, and let y_1, y_2, y_3 be nonzero variables such that $y_1 + y_2 + y_3 = 0$. Then*

$$\Omega\left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ y_1 & y_2 & \end{array}\right) + \Omega\left(\begin{array}{ccc} (a_2 & a_3 & a_1 \\ y_2 & y_3 & \end{array}\right) + \Omega\left(\begin{array}{ccc} a_3 & a_1 & a_2 \\ y_3 & y_1 & \end{array}\right) = -\frac{1}{4}.$$

As in the case of Theorem 4, this reciprocity law also reduces to familiar reciprocity laws. As we will show in the next section, one can obtain Theorem 2 by looking at the constant term of the generating functions. Theorem 1 follows by setting $a_i = 1$ for one of the a_i's and looking at the constant term of the generating function.

When looking at the coefficients of higher powers of the generating function variables in HWZ reciprocity one can discover new identities (since each of these coefficients must equal 0 by Theorem 5). Our approach, described in the next section, consists of a method to systematically collect these coefficient identities and study them. In this report, we emphasize the Apostol sums contained in these identities.

2 The Generating-Function Approach

For pairwise coprime $a_1, a_2, a_3 \in \mathbb{N}$, we define the following generating function:

$$G(y_1, y_2) := \Omega\left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ y_1 & y_2 & \end{array}\right) + \Omega\left(\begin{array}{ccc} (a_2 & a_3 & a_1 \\ y_2 & y_3 & \end{array}\right) + \Omega\left(\begin{array}{ccc} a_3 & a_1 & a_2 \\ y_3 & y_1 & \end{array}\right)$$

which depends only on y_1 and y_2 after substituting $y_3 = -(y_1+y_2)$ from the condition in Theorem 5, which also implies that $G(y_1, y_2) = -\frac{1}{4}$. Furthermore, we introduce the notation

$$\begin{aligned}
\Omega(a_1, a_2, a_3) &= \sum_{m,n\geq 0} A_{m,n}\; y_1^{m-1} y_2^{n-1} \\
\Omega(a_2, a_3, a_1) &= \sum_{m,n\geq 0} B_{m,n}\; y_2^{m-1} y_3^{n-1} \\
\Omega(a_3, a_1, a_2) &= \sum_{m,n\geq 0} C_{m,n}\; y_3^{m-1} y_1^{n-1},
\end{aligned}$$

where

$$\begin{aligned}
A_{m,n} &= \frac{1}{m!}\frac{1}{n!} S_{m,n}(a_1, a_2, a_3) a_1^{1-m} a_2^{1-n} \\
B_{m,n} &= \frac{1}{m!}\frac{1}{n!} S_{m,n}(a_2, a_3, a_1) a_2^{1-m} a_3^{1-n} \\
C_{m,n} &= \frac{1}{m!}\frac{1}{n!} S_{m,n}(a_3, a_1, a_2) a_3^{1-m} a_1^{1-n}.
\end{aligned}$$

In the next sections we show how to expand $G(y_1, y_2)$ in total degrees of the variables y_1 and y_2. Thus, we can express G as

$$\begin{align}
G(y_1, y_2) &= \gamma_{0,0}\; y_1^{-1} y_2^{-1} \tag{1} \\
&+ \left(\gamma_{1,0}\; y_2^{-1} + \gamma_{0,1}\; y_1^{-1}\right) \nonumber \\
&+ \left(\gamma_{2,0}\; y_1 y_2^{-1} + \gamma_{1,1} + \gamma_{0,2} y_1^{-1} y_2\right) \tag{2} \\
&+ \left(\gamma_{3,0}\; y_1^2 y_2^{-1} + \gamma_{2,1} y_1 + \gamma_{1,2} y_2 + \gamma_{0,3} y_1^{-1} y_2^2\right) \nonumber \\
&+ \left(\gamma_{4,0}\; y_1^3 y_2^{-1} + \gamma_{3,1} y_1^2 + \gamma_{2,2}\; y_1 y_2 + \gamma_{1,3}\; y_2^2 + \gamma_{0,4}\; y_1^{-1} y_2^3\right) \tag{3} \\
&+ \cdots \nonumber
\end{align}$$

where $\gamma_{i,j}$ is a sum of terms of the form $A_{m,n}, B_{m,n}, C_{m,n}$. To be able to collect these coefficients, we require the following lemma.

Lemma 3. *The following identities hold for all $n \in \mathbb{N}$:*

$$\begin{aligned}
A_{n,0} &= B_{0,n} \\
B_{n,0} &= C_{0,n} \\
C_{n,0} &= A_{0,n}.
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
A_{n,0} &= \frac{a_1^{1-n}a_2}{n!} \sum_{h=0}^{a_3-1} \overline{B}_n\left(\frac{h}{a_3}\right) \\
&= B_n(0) \frac{a_1^{1-n}a_3^{1-n}a_2}{n!} \\
&= \frac{a_3^{1-n}a_2}{n!} \sum_{h=0}^{a_1-1} \overline{B}_n\left(\frac{h}{a_1}\right) \\
&= B_{0,n},
\end{aligned}$$

where the second and third equalities follow by Lemma 1. The other two identities can be proven in an analogous manner. □

The following lemma allows us to consider only those terms with even total degree because it implies that all terms with odd total degree vanish.

Lemma 4. *For $a, b, c, m, n \in \mathbb{N}$ with $m + n$ odd we have that*

$$S_{m,n}(a, b, c) = 0$$

Proof. First, consider the Fourier series expansion of $\overline{B}_k(x)$ evaluated at $-x$**??**:

$$\begin{aligned}
\overline{B}_k(-x) &= \frac{-k!}{(2\pi i)^k} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{e^{-2\pi i m x}}{m^k} \\
&= \frac{-k!}{(2\pi i)^k} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi i m x}}{(-m)^k} \\
&= (-1)^k \overline{B}_k(x).
\end{aligned}$$

where in the second equality we substituted m by $-m$ since the index set includes all nonzero integers. Hence, $\overline{B}_k(x)$ is an odd function if k is odd and an even function otherwise.

Now we consider $S_{m,n}(a, b, c)$ and note

$$\begin{aligned}
S_{m,n}(a, b, c) &= \sum_{k \bmod c} \overline{B}_m\left(\frac{ak}{c}\right) \overline{B}_n\left(\frac{bk}{c}\right) \\
&= \sum_{k \bmod c} \overline{B}_m\left(-\frac{ak}{c}\right) \overline{B}_n\left(-\frac{bk}{c}\right) \\
&= (-1)^{m+n} \sum_{k \bmod c} \overline{B}_m\left(\frac{ak}{c}\right) \overline{B}_n\left(\frac{bk}{c}\right) \\
&= (-1)^{m+n} S_{m,n}(a, b, c),
\end{aligned}$$

where the second equality follows from noting that $\{1-c, 2-c, ..., 0\}$ is a complete residue system mod c and the third equality follows from the above observation on the parity of $\overline{B}_k(x)$. Hence, $S_{m,n}(a, b, c) = -S_{m,n}(a, b, c)$ when $m + n$ is odd, which implies that $S_{m,n}(a, b, c) = 0$, as claimed. □

We proceed to show how to collect the terms in the generating function G and how some of its coefficients $\gamma_{i,j}$ correspond to identities involving Apostol sums.

2.1 Total Degree -2

First we look at the terms in (1) with total degree -2. This restricts our terms to those where $m+n=0$ in the generating function. Using the definition of our expanded generating function, the terms with total degree 0 look like

$$A_{0,0}\ y_1^{-1}y_2^{-1} + B_{0,0}\ y_2^{-1}y_3^{-1} + C_{0,0}\ y_3^{-1}y_1^{-1}.$$

Using Lemma 3 and factoring, this can be written as

$$B_{0,0}\ (y_1^{-1}y_2^{-1} + y_2^{-1}y_3^{-1} + y_3^{-1}y_1^{-1}).$$

By finding a common denominator, we get

$$B_{0,0}\ \left(\frac{y_1+y_2+y_3}{y_1y_2y_3}\right),$$

which equals 0 when we use the assumption on the generating function variables $y_1+y_2+y_3=0$. Hence, $\gamma_{0,0}=0$.

2.2 Total Degree 0

Next, we work to express the terms that have total degree 0 in (2). Therefore, we will compute the terms that satisfy the condition $m+n=2$. Total degree 0 is especially important because it provides us with the constant term of the generating funtion. Using the definition of our expanded generating function, the terms with total degree 0 are

$$\begin{array}{r} A_{2,0}\ y_1y_2^{-1} + A_{1,1} + A_{0,2}\ y_1^{-1}y_2 \\ +B_{2,0}\ y_2y_3^{-1} + B_{1,1} + B_{0,2}\ y_2^{-1}y_3 \\ +C_{2,0}\ y_3y_1^{-1} + C_{1,1} + C_{0,2}\ y_3^{-1}y_1. \end{array}$$

Using Lemma 3 and simplifying, we can write this as

$$\begin{aligned} A_{2,0}(y_1y_2^{-1} + y_2^{-1}y_3) + B_{2,0}(y_2y_3^{-1} + y_3^{-1}y_1) + C_{2,0}(y_1^{-1}y_2 + y_3y_1^{-1}) \\ +A_{1,1} + B_{1,1} + C_{1,1}. \end{aligned} \tag{4}$$

Now, note that the condition on the generating function variables $y_1+y_2+y_3=0$ implies

$$\begin{aligned} y_1y_3^{-1} + y_2y_3^{-1} + 1 &= 0 \\ y_2y_1^{-1} + y_3y_1^{-1} + 1 &= 0 \\ y_1y_2^{-1} + y_3y_2^{-1} + 1 &= 0, \end{aligned}$$

which turns (4) into

$$A_{1,1} + B_{1,1} + C_{1,1} - A_{2,0} - B_{2,0} - C_{2,0}. \tag{5}$$

Notice that this sum is the constant term of the generating function. We know that the expansion of the entire generating function equals $-\frac{1}{4}$. Thus, we conclude the identity

$$A_{1,1} + B_{1,1} + C_{1,1} - A_{2,0} - B_{2,0} - C_{2,0} = -\frac{1}{4}. \tag{6}$$

Since a_1, a_2, a_3 are pairwise relatively prime we can multiply the index of any of the sums $S_{m,n}$ by the inverse of any of a_i modulo its denominator. Using this fact and applying Lemma 1 to the $A_{2,0}, B_{2,0}, C_{2,0}$ terms, (6) becomes

$$\begin{aligned} &S_{1,1}(a_1a_2^{-1}, 1, a_3) + S_{1,1}(a_2a_3^{-1}, 1, a_1) + S_{1,1}(a_3a_1^{-1}, 1, a_2) \\ &\qquad = \frac{1}{12}\left(\frac{a_1}{a_2a_3} + \frac{a_2}{a_1a_3} + \frac{a_3}{a_1a_2}\right) - \frac{1}{4}, \end{aligned}$$

which is Rademacher reciprocity. That is, the constant term of the generating-function identity corresponds to Rademacher reciprocity as claimed in the introduction.

2.3 Total Degree 2

Consider the terms with total degree 2. This restricts our attention to $m, n \in \mathbb{Z}_{\geq 0}$ such that $m + n = 4$. Note that the terms with total degree 2 in (3) include the sums $S_{1,3}, S_{3,1}$, which correspond to Apostol sums where $p = 3$. Collecting terms yields

$$\begin{aligned} &A_{4,0}\ y_1^3y_2^{-1} + A_{3,1}\ y_1^2 + A_{2,2}\ y_1y_2 + A_{1,3}\ y_2^2 + A_{0,4}\ y_1^{-1}y_2^3 \\ +\quad &B_{4,0}\ y_2^3y_3^{-1} + B_{3,1}\ y_2^2 + B_{2,2}\ y_2y_3 + B_{1,3}\ y_3^2 + B_{0,4}\ y_2^{-1}y_3^3 \\ +\quad &C_{4,0}\ y_3^3y_1^{-1} + C_{3,1}\ y_3^2 + C_{2,2}\ y_3y_1 + C_{1,3}\ y_1^2 + C_{0,4}\ y_3^{-1}y_1^3. \end{aligned}$$

Using Lemma 3, we can rearrange these to obtain

$$\begin{aligned} &A_{4,0}(y_1^3y_2^{-1} + y_2^{-1}y_3^3) + B_{4,0}(y_2^3y_3^{-1} + y_3^{-1}y_1^3) + C_{4,0}(y_1^{-1}y_2^3 + y_3^3y_1^{-1}) \\ +\quad &(A_{3,1} + C_{1,3})y_1^2 + (B_{3,1} + A_{1,3})y_2^2 + (C_{3,1} + B_{1,3})y_3^2 \\ +\quad &A_{2,2}\ y_1y_2 + B_{2,2}\ y_2y_3 + C_{2,2}\ y_3y_1. \end{aligned}$$

Using the condition on the generating function variables $y_3 = -(y_1 + y_2)$ and simplifying we obtain

$$\begin{aligned} -\quad &A_{4,0}(3y_1^2 + 3y_1y_2 + y_2^2) + B_{4,0}(-y_1^2 + y_1y_2 - y_2^2) - C_{4,0}(3y_2^2 + 3y_1y_2 + y_1^2) \\ +\quad &(A_{3,1} + C_{1,3})y_1^2 + (B_{3,1} + A_{1,3})y_2^2 + (C_{3,1} + B_{1,3})(y_1^2 + 2y_1y_2 + y_2^2) \\ +\quad &A_{2,2}\ y_1y_2 - B_{2,2}(y_2^2 + y_1y_2) - C_{2,2}(y_1^2 + y_1y_2). \end{aligned}$$

Collecting terms, we obtain three coefficients with total degree 2 in the generating function:

$$
\begin{aligned}
& (-3A_{4,0} - B_{4,0} - C_{4,0} + A_{3,1} + C_{1,3} + B_{1,3} + C_{3,1} - C_{2,2})y_1^2 \\
+ \; & (-3A_{4,0} + B_{4,0} - 3C_{4,0} + 2B_{1,3} + 2C_{3,1} + A_{2,2} - B_{2,2} - C_{2,2})y_1y_2 \\
+ \; & (-A_{4,0} - B_{4,0} - 3C_{4,0} + B_{3,1} + A_{1,3} + B_{1,3} + C_{3,1} - B_{2,2})y_2^2.
\end{aligned}
$$

Because any higher powers in the generating function have coefficient 0, each of these coefficients must equal 0, which yields three identities:

$$A_{3,1} + C_{1,3} + B_{1,3} + C_{3,1} = 3A_{4,0} + B_{4,0} + C_{4,0} + C_{2,2} \tag{7}$$

$$2B_{1,3} + 2C_{3,1} = 3A_{4,0} - B_{4,0} + 3C_{4,0} - A_{2,2} + B_{2,2} + C_{2,2} \tag{8}$$

$$A_{1,3} + B_{3,1} + B_{1,3} + C_{3,1} = A_{4,0} + B_{4,0} + 3C_{4,0} + B_{2,2}. \tag{9}$$

To interpret these identities, we apply Lemma 1 to get

$$A_{4,0} = \frac{B_4}{24}\frac{a_2}{a_1^3a_3^3}\ ,\ B_{4,0} = \frac{B_4}{24}\frac{a_3}{a_1^3a_2^3}\ ,\ C_{4,0} = \frac{B_4}{24}\frac{a_1}{a_2^3a_3^3}.$$

We can also express the following terms using Apostol sums:

$$
\begin{aligned}
A_{1,3} &= \frac{1}{6a_2^2}s_3(a_1^{-1}a_2, a_3) \quad , \quad A_{3,1} = \frac{1}{6a_1^2}s_3(a_1a_2^{-1}, a_3) \\
B_{1,3} &= \frac{1}{6a_3^2}s_3(a_2^{-1}a_3, a_1) \quad , \quad B_{3,1} = \frac{1}{6a_2^2}s_3(a_2a_3^{-1}, a_1) \\
C_{1,3} &= \frac{1}{6a_1^2}s_3(a_3^{-1}a_1, a_2) \quad , \quad C_{3,1} = \frac{1}{6a_3^2}s_3(a_3a_1^{-1}, a_2).
\end{aligned}
$$

With the above, we can now express (7) as:

$$
\begin{aligned}
& \frac{1}{6a_1^2}s_3(a_1a_2^{-1}, a_3) + \frac{1}{6a_1^2}s_3(a_3^{-1}a_1, a_2) + \frac{1}{6a_3^2}s_3(a_2^{-1}a_3, a_1) + \frac{1}{6a_3^2}s_3(a_3a_1^{-1}, a_2) \\
&\qquad = \frac{1}{24}\left(3B_4\frac{a_2}{a_1^3a_3^3} + B_4\frac{a_3}{a_1^3a_2^3} + B_4\frac{a_1}{a_2^3a_3^3}\right) + \frac{1}{4a_1a_3}S_{2,2}(a_3, a_1, a_2).
\end{aligned}
$$

Observe there is a Rademacher-like permutation in the arguments of the Apostol sums in the above identity. However, the identity involves four sums instead of the three terms in Rademacher reciprocity for the classic Dedekind sum. We conjecture this identity to be a Rademacher analogue for Apostol sums with $p = 3$, though the sum $S_{2,2}$ in the right-hand side remains to be simplified in closed form.

Furthermore, in this form, (7) can be specialized to obtain Apostol reciprocity for $p = 3$. For this, we set $a_2 = 1$, which yields

$$
\begin{aligned}
& \frac{1}{6a_1^2}s_3(a_1, a_3) + \frac{1}{6a_1^2}s_3(a_3^{-1}a_1, 1) + \frac{1}{6a_3^2}s_3(a_3, a_1) + \frac{1}{6a_3^2}s_3(a_3a_1^{-1}, 1) \\
&\qquad = \frac{1}{24}\left(3B_4\frac{1}{a_1^3a_3^3} + B_4\frac{a_1}{a_3^3} + B_4\frac{a_3}{a_1^3}\right) + \frac{1}{4a_1a_3}S_{2,2}(a_3, a_1, 1).
\end{aligned}
$$

Noting that $s_3(a_3^{-1}a_1, 1), s_3(a_3a_1^{-1}, 1) = B_1B_3 = 0$ and that $S_{2,2}(a_3, a_1, 1) = B_2^2$ and rearranging terms, we obtain

$$4\left(a_1a_3^3s_3(a_1, a_3) + a_1^3a_3s_3(a_3, a_1)\right) = B_4a_1^4 + 6B_2^2a_1^2a_3^2 + B_4a_3^4 + 3B_4,$$

which corresponds to Apostol reciprocity for $p = 3$.

Above, we used (7), but we have two additional identities that can be used.
We proceed in an analogous manner with (9). We first express it in terms of Apostol sums:

$$\frac{1}{6a_2^2}s_3(a_2a_3^{-1}, a_1) + \frac{1}{6a_2^2}s_3(a_1^{-1}a_2, a_3) + \frac{1}{6a_3^2}s_3(a_2^{-1}a_3, a_1) + \frac{1}{6a_3^2}s_3(a_3a_1^{-1}, a_2)$$
$$= \frac{1}{24}\left(B_4\frac{a_2}{a_1^3a_3^3} + 3B_4\frac{a_1}{a_2^3a_3^3} + B_4\frac{a_3}{a_1^3a_2^3}\right) + \frac{1}{4a_2a_3}S_{2,2}(a_2, a_3, a_1).$$

Setting $a_1 = 1$, we obtain Apostol reciprocity but with the arguments of the sums permuted:

$$4\left(a_2a_3^3s_3(a_2, a_3) + a_2^3a_3s_3(a_3, a_2)\right) = B_4a_2^4 + 6B_2^2a_2^2a_3^2 + B_4a_3^4 + 3B_4.$$

The above results suggest that (8) yields the remaining cyclic permutation of Apostol reciprocity for $p = 3$ after setting $a_3 = 1$. This can be seen by replacing identity (8) with the linear combination (7) + (9) − (8). In the following section, we discuss our planned work regarding (8) and the general case of even total degree.

2.4 Total Degree p Even

Proceeding as we did to collect terms for total degree 0 and 2, we can collect terms for any even total degree p. When $p = 0, 2, 4, 6, 8$ we obtain the following coefficients of the y_2^p term:

$$\begin{aligned}
p = 0: &\quad A_{1,1} + C_{1,1} - B_{0,2} + B_{1,1} - B_{2,0} - C_{2,0}\\
p = 2: &\quad A_{1,3} + C_{3,1} - B_{0,4} + B_{1,3} - B_{2,2} + B_{3,1} - B_{4,0} - 3C_{4,0}\\
p = 4: &\quad A_{1,5} + C_{5,1} - B_{0,6} + B_{1,5} - B_{2,4} + B_{3,3} - B_{4,2} + B_{5,1} - B_{6,0} - 5C_{6,0}\\
p = 6: &\quad A_{1,7} + C_{7,1} - B_{0,8} + B_{1,7} - B_{2,6} + B_{3,5} - B_{4,4} + B_{5,3} - B_{6,2} + B_{7,1}\\
&\quad - B_{8,0} - 7C_{8,0}\\
p = 8: &\quad A_{1,9} + C_{9,1} - B_{0,10} + B_{1,9} - B_{2,8} + B_{3,7} - B_{4,6} + B_{5,5} - B_{6,4} + B_{7,3}\\
&\quad - B_{8,2} + B_{9,1} - B_{10,0} - 9C_{10,0}.
\end{aligned}$$

The above pattern continues for even degree which suggests the following general expression for the coefficient of y_2^p for arbitrary even p:

Theorem 6. *For even p, the coefficient of the y_2^p variable in the generating function $G(y_1, y_2)$ is*

$$A_{1,p+1} + C_{p+1,1} + \sum_{i=0}^{p+2}(-1)^{i+1}B_{i,p+2-i} - (p+1)C_{p+2,0}.$$

Also, this coefficient satisfies the identity

$$A_{1,p+1} + C_{p+1,1} + \sum_{i=0}^{p+2}(-1)^{i+1}B_{i,p+2-i} - (p+1)A_{0,p+2} = \begin{cases} -\frac{1}{4} & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$$

Proof. Note that

$$\begin{aligned}
G(y_1, y_2) &= \Omega(a_1, a_2, a_3) + \Omega(a_2, a_3, a_1) + \Omega(a_3, a_1, a_2) \\
&= \sum_{m,n\geq 0} A_{m,n} y_1^{m-1} y_2^{n-1} + \sum_{m,n\geq 0} B_{m,n} y_2^{m-1} y_3^{n-1} + \sum_{m,n\geq 0} C_{m,n} y_3^{m-1} y_1^{n-1} \\
&= \sum_{m,n\geq 0} A_{m,n} y_1^{m-1} y_2^{n-1} + \sum_{m,n\geq 0} B_{m,n} y_2^{m-1}(-y_1 - y_2)^{n-1} + \sum_{m,n\geq 0} C_{m,n}(-y_1 - y_2)^{m-1} y_1^{n-1} \\
&= \sum_{m,n\geq 0} A_{m,n} y_1^{m-1} y_2^{n-1} + \sum_{m,n\geq 0} B_{m,n}(-1)^{n-1} y_2^{m-1} \sum_{j=0}^{n-1} \binom{n-1}{j} y_1^{n-1-j} y_2^{j} \\
&\qquad + \sum_{m,n\geq 0} C_{m,n}(-1)^{m-1} y_1^{n-1} \sum_{j=0}^{m-1} \binom{m-1}{j} y_1^{j} y_2^{m-1-j} \\
&= \sum_{m,n\geq 0} A_{m,n} y_1^{m-1} y_2^{n-1} + \sum_{m,n\geq 0} B_{m,n}(-1)^{n-1} \sum_{j=0}^{n-1} \binom{n-1}{j} y_1^{n-1-j} y_2^{m-1+j} \\
&\qquad + \sum_{m,n\geq 0} C_{m,n}(-1)^{m-1} y_1^{n-1} \sum_{j=0}^{m-1} \binom{m-1}{j} y_1^{n-1+j} y_2^{m-1-j},
\end{aligned}$$

where in the third equality we used the condition $y_3 = -y_1 - y_2$ and in the fourth equality we used the Binomial Theorem. We now seek to extract the coefficient of y_2^p from the three summations corresponding to each Ω.

In the first Ω, we want

$$\begin{aligned}
m - 1 = 0 \quad &\Rightarrow \quad m = 1 \\
n - 1 = p \quad &\Rightarrow \quad n = p + 1
\end{aligned}$$

which implies that the coefficient of y_2^p in this summation is $A_{1,p+1}$.

In the second Ω, we want

$$\begin{aligned}
m - 1 + j &= p \\
n - 1 - j &= 0
\end{aligned}$$

which in turn implies that

$$\begin{aligned} j &= n-1 \\ m+n &= p+2. \end{aligned}$$

Thus, the coefficient of y_2^p in the second summation is

$$\sum_{\substack{m,n\geq 0\\ m+n=p+2}} (-1)^{n-1}B_{m,n} = \sum_{i=0}^{p+2}(-1)^{p+1-i}B_{i,p+2-i} = \sum_{i=0}^{p+2}(-1)^{i+1}B_{i,p+2-i},$$

where the first equality follows by noting that we can include all the pairs $(m,n) \in \mathbb{Z}^2_{\geq 0}$ such that $m+n = p+2$ with the i, $p+2-i$ subscripts.

In the third Ω, we require

$$\begin{aligned} m-1-j &= p \\ n-1+j &= 0, \end{aligned}$$

which implies that

$$\begin{aligned} j &= m-1-p \\ m+n &= p+2. \end{aligned}$$

We thus have the coefficient of y_2^p as

$$\begin{aligned} &\sum_{\substack{m,n\geq 0\\ m+n=p+2}} \binom{m-1}{m-1-p}(-1)^{m-1}C_{m,n} \\ &= \sum_{i=0}^{p+2}\binom{i-1}{i-1-p}(-1)^{i-1}C_{i,p+2-i} \\ &= C_{p+1,1}-(p+1)C_{p+2,0}+\sum_{i=0}^{p}\binom{i-1}{i-1-p}(-1)^{i-1}C_{i,p+2-i} \\ &= C_{p+1,1}-(p+1)C_{p+2,0}, \end{aligned}$$

where the first equality follows by the same argument as in the previous case, the second equality follows by noting that p is required to be even and the last equality follows since $i-1-p<0$ for $i=0,1,...,p$, making $\binom{i-1}{i-1-p}=0$.

Hence, collecting the terms from each of the three $\Omega' s$, we have that the coefficient of y_2^p in $G(y_1,y_2)$ is

$$A_{1,p+1}+C_{p+1,1}+\sum_{i=0}^{p+2}(-1)^{i+1}B_{i,p+2-i}-(p+1)C_{p+2,0}$$

as claimed. The coefficient identity follows from Theorem 5. $\square$

The identity in Theorem 6 can be rearranged to obtain an identity analogous to Theorem 2 for Apostol sums. For this, we first extract the terms corresponding to $i = p+1, p+2$ from the summation to obtain:

$$A_{1,p+1}+B_{p+1,1}+C_{p+1,1}+\sum_{i=0}^{p}(-1)^{i+1}B_{i,p+2-i}-B_{p+2,0}-(p+1)A_{0,p+2}=\begin{cases}-\frac{1}{4} & \text{if } p=0,\\ 0 & \text{if } p>0.\end{cases}$$

Now note that the sum $A_{1,p+1} + B_{p+1,1} + C_{p+1,1}$ corresponds to the sum

$$\frac{1}{a_2^p(p+1)!}s_{p+1}(a_2a_1^{-1}, a_3) + \frac{1}{a_2^p(p+1)!}s_{p+1}(a_2a_3^{-1}, a_1) + \frac{1}{a_3^p(p+1)!}s_{p+1}(a_3a_2^{-1}, a_2),$$

where we observe Apotol sums with a permutation of the arguments of the sums similar, though not identical, to that in Theorem 2. Despite the difference in the way the arguments in the sum are permuted, the rearranged identity seems to be a generalization of Theorem 2 since when $p = 0$, the identity reduces to Theorem 2. For this, note that when $p = 0$ the identity becomes

$$A_{1,1} + B_{1,1} + C_{1,1} - B_{0,2} - B_{2,0} - A_{0,2} = -\frac{1}{4}.$$

Using Lemma 3, we express this as

$$A_{1,1} + B_{1,1} + C_{1,1} - A_{2,0} - B_{2,0} - C_{2,0} = -\frac{1}{4},$$

which is Theorem 2 as shown before when considering total degree 0.

We recall Theorem 4, which we express as

$$\begin{aligned}&(p+2)(ab^{p+1}s_{p+1}(a,b) + a^{p+1}bs_{p+1}(b,a))\\ &\quad+\sum_{i=0}^{p+2}\binom{p+2}{i}(-1)^{i+1}a^ib^{p+2-i}B_iB_{p+2-i} - (p+1)B_{p+2} = 0\end{aligned}$$

where B_k is the the k^{th} Bernoulli number and the theorem holds for even $p > 0$. Under the same condition on p we set $a_1 = c$, $a_2 = a$, $a_3 = b$, multiply Theorem 6 by $a^{p+1}b^{p+1}(p+2)!$ and use Lemma 1 on the last term to express the identity from Theorem 6 as

$$\begin{aligned}&(p+2)\left(ab^{p+1}s_{p+1}(ac^{-1}, b) + a^{p+1}bs_{p+1}(bc^{-1}, a)\right)\\ &\quad+\sum_{i=0}^{p+2}\binom{p+2}{i}(-1)^{i+1}a^{p+2-i}b^iS_{i,p+2-i}(a,b,c) - c(p+1)B_{p+2} = 0,\end{aligned}$$

where again B_k is again the k^{th} Bernoulli number. Hence, we can also view the identity from Theorem 6 as a generalization of Theorem 4 for three variables instead of two. It is interesting to note that the role the Bernoulli numbers play in the summation of Theorem 4 is taken up by the HWZ sum $S_{i,p+2-i}(a,b,c)$ in Theorem 6. In fact, setting $c = 1$ reduces Theorem 6 to Theorem 4 since

$$S_{i,p+2-i}(a,b,1) = \sum_{h=0}^{0} \overline{B}_i(ah)\overline{B}_{p+2-i}(bh) = B_i B_{p+2-i}$$

The above results provide evidence suggesting Theorem 6 is an analogue of Theorem 2 for Apostol sums and provide a connection between Theorems 4, 5, and 2.

3 Outlook

Extensions of this research include, but are not limited to, finding another reciprocity law and corresponding generating function. We would like to find the right analogue of Pommersheim reciprocity, Theorem 3, at the level of the Apostol sums. This would allow us to use the new reciprocity to imply Theorem 6 and Pommersheim reciprocity, Theorem 3. Furthermore, we may apply this same idea at the generating function level. A new generating function of HWZ sums could then imply the new reciprocity law and the HWZ generating function, Theorem 5.

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