ENUMERATION OF GOLOMB RULERS

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by

Tu Pham

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CERTIFICATION OF APPROVAL

I certify that I have read *ENUMERATION OF GOLOMB RULERS* by Tu Pham and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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Generally a ruler is marked in equal increments, e.g., a 12 inch ruler has 12 markings, each 1 inch apart. In this paper we discuss a special type of ruler discovered by Soloman W. Golomb. We define a Golomb ruler to be a ruler of length d with n markings where the distance between any two markings is unique. This concept is useful in the field of electrical engineering.

We use the fact that a Golomb ruler can be represented as a lattice point in $\mathbb{R}^m$, where the k-th coordinate represent the k-th measure, to study it from a geometric point of view. It turns out that if we relax the definition of a Golomb ruler to allow real number markings then the set of all Golomb ruler of m measure and length d is a polytope in $\mathbb{R}^m$ equipped with a hyperplane arrangement. We prove results about the combinatorial structure of these special type of polytopes and derive from them structural theorems about the enumeration of Golomb rulers.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Chapter 1

Introduction

A Golomb ruler is a ruler with a fixed length and markings on them such that the distance between any two markings is unique. Golomb rulers are helpful in designing phased array radio antennas such as radio telescopes. They are also important in the field of radar and sonar signaling[5]. One can view a Golomb ruler of length t with n markings as an integer lattice point in $\mathbb{R}^{n-2}$. (Since the first marking is always 0 and the last marking will always be the length of the ruler, d, we really only have $n-2$ markings to work with.) If we consider all Golomb rulers of length t and n markings as lattice points $(x_1, x_2, \ldots, x_{n-2}) \in \mathbb{Z}^{n-2}$, they all satisfy the inequality $0 < x_1 < x_2 < x_3 < \cdots < x_{n-2} < t$. These inequalities form the hyperplane description of a simplex Δ in $\mathbb{R}^{n-2}$. But not every lattice point in this simplex is a Golomb ruler. Since the distance between any two markings is unique, $x_i - x_j \neq x_k - x_l$ for all $i, j, k, l \in \{0, 1, \ldots, n-1\}$ and $i > j$, $k > l$. The difference between any consecutive markings is called a **measure**. We refer to the differences between any two markings $x_i - x_j$ as a **measurement**. Therefore any two measurements in the Golomb ruler are not equal. We will see that a lattice point in the simplex Δ is a Golomb ruler if it avoids all the hyperplanes of the form $x_i - x_j = x_k - x_l$ for all $i > j \geq k > l$. A polytope that is equipped with a hyperplane arrangement is called an inside-out polytope [4]. Our goal is to count the Golomb ruler of a fixed length and number of markings. First we define the inside-out polytope that arise from Golomb rulers. In Chapter 3 we associate a mixed graph to a Golomb ruler and state our main result(Theorem 3.1). Then we apply Ehrhart theory to our inside-out polytope and state a reciprocity theorem(Theorem 4.5) in Chapter 4.

Chapter 2

Definitions and Background

2.1 Golomb Rulers

Definition 2.1. A **ruler** of length t and n markings is defined to be a set $\{x_0, x_1, \ldots, x_{n-1}\}$ where $x_i \in \mathbb{N}$ for $0 \leq i \leq n-1$ and $0 = x_0 < x_1 < \cdots < x_{n-1} < x_{n-1} = t$. Each x_j represents the placement of a marking therefore we will call this the **canonical representation** of the ruler.

There is a second way of representing a ruler which will be useful later on.

Definition 2.2. A ruler in the canonical representation $\{0, x_1, \ldots, x_{n-2}, t\}$ of length t with n markings can also be written as $\{z_1, z_2, \ldots, z_{n-1}\}$ where $z_j = x_j - x_{j-1}$ for $1 \leq j \leq n-1$. We call this form the **measure representation**. To distinguish between the two representations of a ruler we note that the first term of the canonical representation is always $x_0 = 0$ and the first term of the measure representation is always nonzero, i.e., $z_1 \neq 0$. It should also be noted that a measure representation with $n-1$ terms represents a ruler of n markings.

We will be studying the Golomb rulers in their measure representation. Therefore we will refer to a Golomb ruler of m measures and length t to be of the form $\{z_1, z_2, \ldots, z_m\}$ where $\sum_{i=1}^{n} z_i = t$.

Definition 2.3. $\mathbb{P}[m]$ is the set of proper consecutive subsets of $[m]$. This set has a lexicographical total ordering where if $|A| < |B|$ then $A <_{\mathbb{P}} B$. Else if $|A| = |B|$ then $\max A < \max B$ if and only if $A <_{\mathbb{P}} B$.

Example 2.1. $\mathbb{P}[4] = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}\}$ where we list the elements in increasing lexicographical order.

It is not hard to see that the cardinality is $|\mathbb{P}[m]| = m + (m-1) + \cdots + 3 + 2 = \frac{m(m+1)}{2} - 1$.

There are three different types of comparison between pair of measurements. The first type is that the two measurements are disjoint, i.e $x_i - x_j \neq x_k - x_l$ for $i > j > k > l$. The second type is where the two measurements are adjacent, $x_i - x_j = x_j - x_l$ where $i > j > l$. And the third is when the two measurements overlap, that is $x_i - x_k = x_j - x_l$ where $i > j > k > l$. But we observe that the comparison between the overlapping measurements $x_i - x_k = x_j - x_l$ is equivalent to the disjoint comparison of $x_i - x_j = x_k - x_l$ where $i > j > k > l$. Hence there are only two different comparisons we need to be concerned with and that is the disjoint and adjacent measurements. Therefore given a Golomb Ruler it must satisfy $x_i - x_j \neq x_k - x_l$ for all $i > j \geq k > l$.

Definition 2.4 (Canonical representation)**.** A **Golomb ruler** of n markings and length t is a ruler $\{x_0, x_1, x_2, \ldots, x_n\}$ with the property that $x_l - x_k \neq x_j - x_i$ for all $0 \leq i < j \leq k < l \leq n$ and $x_0 = 0$, $x_n = t$.

Definition 2.5 (Measure representation)**.** A **Golomb ruler** of m measures and length t is a ruler $\{z_1, z_2, \ldots, z_m\}$ with the property that $\sum_{u=\min U}^{\max U} z_u \neq \sum_{v=\min V}^{\max V} z_v$, for all disjoint pairs $U, V \in \mathbb{P}[m]$, and $\sum_{k=1}^{m} z_k = t$.

Proposition 2.1. *The canonical and measure definitions of a Golomb ruler are equivalent.*

Proof. To show the equivalence of both definitions we consider a Golomb ruler of n markings and therefore $m = n - 1$ measures and we recall that a measure $z_i = x_i - x_{i-1}$. Then we show a bijection between the canonical representation of Golomb rulers that satisfy the condition$x_l - x_k \neq x_j - x_i$ for all $0 \leq i < j \leq k < l \leq n - 1$ and $\sum_{u=\min U}^{\max U} z_u \neq \sum_{v=\min V}^{\max V} z_v$ for all disjoint pairs $U, V \in \mathbb{P}[m]$.

We first start with $x_l - x_k \neq x_j - x_i$ for all $0 \leq i < j \leq k < l \leq n$. By using the identity $z_i = x_i - x_{i-1}$ for all i, we get the equivalent condition $\sum_{a=k+1}^{l} z_a \neq \sum_{b=i+1}^{j} z_b$. Let us consider the proper consecutive sets $U = \{k+1, \ldots, l\}$ and $V = \{i+1, \ldots, j\}$. Since $i < j \leq k < l$ we know $U \neq \emptyset$ and $V \neq \emptyset$. Finally we conclude that U and V are disjoint since $\max V = j < k+1 = \min U$.

To show the other direction we first start with $\sum_{u=\min U}^{\max U} z_u \neq \sum_{v=\min V}^{\max V} z_v$ where $U, V \in \mathbb{P}[n]$ and $U \cap V = \emptyset$, then without loss of generality let $\min U > \max V$. Set $l = \max U$, $k = \min U - 1$, $j = \max V$, and $i = \min V - 1$. Therefore we get the inequality $i < j \leq k < l$, and hence $x_l - x_k \neq x_j - x_i$. $\square$

Example 2.2. Figure 2.1 shows a Golomb ruler with 4 markings of length 6 that has markings at $\{0, 1, 4, 6\}$. The measure representation is $\{1, 3, 2\}$. Since all integer measurements less than or equal to its length are obtainable, we call this a **perfect Golomb ruler**. Sadly this is the longest of its kind.

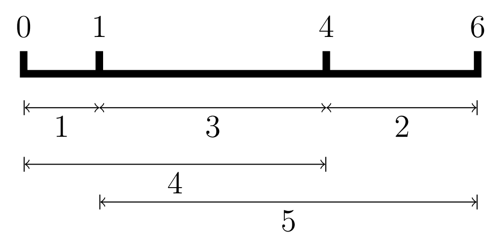


Figure 2.1: A perfect Golomb ruler of length 6.

By reversing the ordering of the markings we also get a Golomb ruler, hence Golomb rulers come in pairs. For every Golomb ruler $G = \{x_0, x_1, \ldots, x_{n-1}\}$, we define its conjugate $\overline{G} := \{x_{n-1} - x_{n-1}, x_{n-1} - x_{n-2}, x_{n-1} - x_{n-3)}, \ldots, x_{n-1} - x_0\}$. In terms of the measure representation, given $G = \{z_1, z_2, \ldots, z_m\}$, then $\overline{G} = \{z_m, z_{m-1}, \ldots, z_2, z_1\}$.

2.2 Inside-out Polytopes

Definition 2.6. A **polytope** is the convex hull of finitely many points in $\mathbb{R}^n$. Given a set of points $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$, we define a polytope $\mathcal{P}$ to be

$$\mathcal{P} = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k \,:\, \lambda_i \geq 0 \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1\}. \tag{2.1}$$

A polytope can also be described in terms of hyperplanes:

$$\mathcal{P} = \{x \in \mathbb{R}^m : Ax \leq b\}, \tag{2.2}$$

where A is an $n \times m$ matrix and $b \in \mathbb{R}^n$.

Definition 2.7. $\mathcal{P}^\circ := \{x \in \mathbb{R}^m : Ax < b\}$ is the interior of $\mathcal{P}$.

Example 2.3. $\mathcal{P}^\circ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, 0 < x_1, x_2, x_3 < 1\}$ is the open unit cube in $\mathbb{R}^3$.

Definition 2.8. A **hyperplane** in $\mathbb{R}^m$ is an $(m-1)$-dimensional affine subspace. A collection of hyperplanes is called a **hyperplane arrangement**.

Example 2.4. All points $(x_1, x_2, x_3) \in \mathbb{R}^3$ that satisfy the equation $x_1 = x_2$ form a hyperplane. The collection of hyperplanes $\mathcal{H} = \{x_i = x_j \,:\, \text{ for all } i \neq j\}$ is a hyperplane arrangement where $|\mathcal{H}| = \binom{3}{2} = 3$. This is a special hyperplane arrangement called the **braid arrangement**. In general we define the braid arrangement in $\mathbb{R}^m$ to be $B_m := \{x_i = x_j \,:\, 1 \leq i < j \leq m\}$

Definition 2.9. [4] A polytope $\mathcal{P}$ together with a hyperplane arrangement $\mathcal{H}$ is called an **inside-out polytope** and we denote it as $\mathcal{Q} = (\mathcal{P}, \mathcal{H})$. We say that the inside-out polytope $\mathcal{Q}$ is open if $\mathcal{P}$ is open.

Example 2.5. Let $\mathcal{P}^\circ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1, x_2, x_3 < 1\}$
$\mathcal{H} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1\}$
Then $\mathcal{Q} = (\mathcal{P}^\circ, \mathcal{H})$ is an inside-out polytope. $\mathcal{P}^\circ$ is a unit cube in $\mathbb{R}^3$ with the hyperplane $\mathcal{H}$ which dissects the cube in half.

Definition 2.10. A **lattice point** in $\mathbb{R}^d$ is a point with all integer coordinates.

Definition 2.11. $R(m,t) := \{(z_1, z_2, \ldots, z_m) \in \mathbb{R}^m : z_i > 0$ for all $1 \leq i \leq m$ and $\sum_{i=1}^m z_i = t\}$ is an open polytope where every lattice point is a ruler of length t and m measures. For convenience we denote $R(m) := R(m,1)$.

Definition 2.12. Given an m-dimensional polytope $\mathcal{P}$, a point $v \in \mathcal{P}$ is a **vertex** if there exist a vector $a \in \mathbb{R}^m$ such that $a^\intercal v > a^\intercal w$ for all $w \in \mathcal{P} \backslash \{v\}$. We can think of the vertices as the "extremal points" of a polytope.

Definition 2.13. A **simplex** is an m-dimensional polytope with $m + 1$ vertices.

Since $R(m,t)$ lives on the plane $\sum_i^m z_i = t$ in $\mathbb{R}^m$ it is an $(m-1)$-dimensional polytope whose vertices are te_j for $1 \leq j \leq m$, where e_j is the j-th coordinate unit vector. Hence $R(m,t)$ has m vertices, therefore by definition it is an $(m-1)$-dimensional simplex living in $\mathbb{R}^m$.

Every lattice point in $R(m,d)$ is a ruler but which point is a Golomb ruler? To answer this question we recall what it means to be a Golomb ruler. Any point $(z_1, z_2, \ldots, z_m) \in R(m,t)$ such that $\sum_{u=\min U}^{\max U} z_u = \sum_{v=\min V}^{\max V} z_v$ for some disjoint pair $U, V \in \mathbb{P}[m]$ is not a Golomb ruler. The collection of all hyperplanes $\sum_{u=\min U}^{\max U} z_u = \sum_{v=\min V}^{\max V} z_v$ for all disjoint pairs $U, V \in \mathbb{P}[m]$ is a hyperplane arrangement.

Definition 2.14. Define $\mathcal{H}(m) := \{\sum_{u=\min U}^{\max U} z_u = \sum_{v=\min V}^{\max V} z_v \ : U, V \in \mathbb{P}[m], U \cap V = \emptyset\}$. We call this the **Golomb arrangement**.

From this definition we see that the braid arrangement B_m is contained in the Golomb arrangement $\mathcal{H}(m)$.

Definition 2.15. $\mathcal{G}(m,t) := (R(m,t), \mathcal{H}(m))$ is an inside-out polytope where every lattice point in $R(m,t) \backslash \mathcal{H}(m)$ corresponds to a Golomb ruler of length t and m measures. For convenience we will denote $\mathcal{G}(m) := \mathcal{G}(m,1)$.

Proposition 2.2. $|\mathcal{H}(m)| = \binom{m+1}{4} + \binom{m+1}{3}$.

Proof. We examine all the possible cases of the hyperplanes in $\mathcal{H}(m)$. Every hyperplane has the form $\sum_{u=\min U}^{\max U} z_u = \sum_{v=\min V}^{\max V} z_v$ for some disjoint set $U, V \in \mathbb{P}[m]$. In Proposition 2.1 we showed that there is a bijection between the set of pairs $U, V \in \mathcal{P}[m]$ where $U \cap V = \emptyset$ and the set

$$\{x_l - x_k \neq x_j - x_i \ : \ \text{for all } 0 \leq i < j \leq k < l \leq m\}.$$

Therefore it suffices to show that the cardinality of the set $\{x_l - x_k \neq x_j - x_i :$ for all $0 \leq i < j \leq k < l \leq m\}$ is equal to $\binom{m+1}{4} + \binom{m+1}{3}$. This is equivalent to counting how many i, j, k, and l there are such that $0 \leq i < j \leq k < l \leq n$.

Case 1: When $j < k$ we have $0 \leq i < j < k < l \leq n$. Hence there are exactly $\binom{n}{4}$ ways of choosing i, j, k, and l such that $0 \leq i < j < k < l \leq n$.

Case 2: When $j = k$ we have $i < j < l$ and $0 \leq i, j, l \leq n$. The number of ways to choose i, j, l such that $0 \leq l < j < i \leq n$ is exactly $\binom{n}{3}$. Hence $|\mathcal{H}(n)| = \binom{n}{4} + \binom{n}{3}$. □

Corollary 2.3. *The number of disjoint pairs* (U, V) *where* $U, V \in \mathbb{P}[m]$ *is equal to* $\binom{m+1}{4} + \binom{m+1}{3}$.

Definition 2.16. Let $L_{\mathcal{G}(m)}(t) := \#\{(R(m,t) \backslash \mathcal{H}(m)) \cap \mathbb{Z}^m\}$, the number of Golomb rulers with m measures and length t.

Example 2.6. In Figure 2.2 the inside-out polytope $\mathcal{G}(3, 6)$, where each lattice point represents a Golomb ruler of length 6 with 3 measures, has $\binom{4}{4} + \binom{4}{3} = 5$ hyperplanes:

$$\mathcal{H}(3) = \{z_1 = z_2, z_2 = z_3, z_1 = z_3, z_1 + z_2 = z_3, z_1 = z_2 + z_3\}.$$

Since $(1, 3, 2)$ and $(2, 3, 1)$ are the only two lattice points in $\mathcal{G}(3, 6)$, there are only two Golomb rulers of length 6 with 3 measures. The lattice point $(1, 3, 2) \in \mathcal{G}(3, 6)$ corresponds to the Golomb ruler $\{1, 3, 2\}$ from Example 2.2.

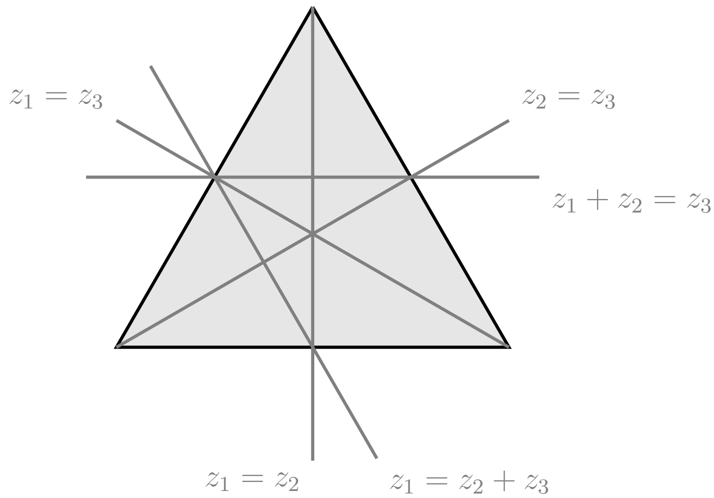


Figure 2.2: $\mathcal{G}(3,6) \subseteq \{(z_1, z_2, z_3) \in \mathbb{R}^3 \,:\, z_1 + z_2 + z_3 = 1\}$.

Definition 2.17. We define $g(m) := \min\{t \in \mathbb{Z} \,:\, L_{\mathcal{G}(m)}(t) > 0\}$
The lattice points in $(R(m, g(m)) \backslash \mathcal{H}(m)) \cap \mathbb{Z}^n$ are called **optimal Golomb rulers** with m measures.

The function $g(m)$ tells us the length of optimal Golomb rulers of m measures. For example, the shortest length Golomb ruler of 5 measures is 17 and there are 4 of them. The current record for finding $g(m)$ is $g(25) = 492$ [1]. An optimal Golomb ruler of 25 measures in its canonical representation is

$$\{0, 1, 33, 83, 104, 110, 124, 163, 185, 200, 203, 249, 251,$$

$$258, 314, 318, 343, 356, 386, 430, 440, 456, 464, 475, 487, 492\}.$$

It is still not known if $L_{\mathcal{G}(25)}(492) > 1$. To find $g(26)$ is equivalent to finding the smallest dilation of $\mathcal{G}(26)$ such that there is at least one lattice point. Our main focus will be to study $\mathcal{G}(m, 1)$ and into how many regions $\mathcal{H}(m)$ divides the open polytope $R(m, 1)$.

Chapter 3

The Regions of $\mathcal{G}(m)$

We know that every lattice point in $R(m,t)\backslash\mathcal{H}(m)$ represents a Golomb ruler of length t and m measures but we do not know what each region represents. Our goal will be to count the number of regions of $\mathcal{G}(m,t)$ and interpret what it means. Since dilation of an inside-out polytope does not change the number of regions, we will study the regions of $\mathcal{G}(m)$ and denote this count by $\mathcal{R}(m)$.

We start with the trivial example $\mathcal{R}(2)$ which counts the number of regions of $\mathcal{G}(2)$. This is the inside-out polytope that represents the Golomb ruler with 2 measures which is one-dimensional and lives in $\mathbb{R}^2$ with one hyperplane, namely $z_1 = z_2$. Therefore $\mathcal{R}(2) = 2$. Next we examine Golomb rulers with 3 measures, $\{z_1, z_2, z_3\}$. First we will look at the number of regions in $\mathcal{G}(3)$ and then introduce the mixed graph Γ_3.

3.1 The Inside-out Polytope $\mathcal{G}(m)$

$\mathcal{G}(m)$ is an $(m-1)$-dimensional inside-out polytope that lives in $\mathbb{R}^m$. Recall that the braid arrangement B_m is a subset of $\mathcal{H}(m)$. This implies that the number of regions of $\mathcal{G}(m)$ is greater than the number of regions of B_m, which is equal to $m!$. Hence we immediately have a lower bound on the number of regions of $\mathcal{G}(m)$. Recall that $\mathcal{H}(m)$ has $\binom{m+1}{4} + \binom{m+1}{3}$ hyperplanes. Figure 3.1 shows that $\mathcal{R}(3) = 10$.

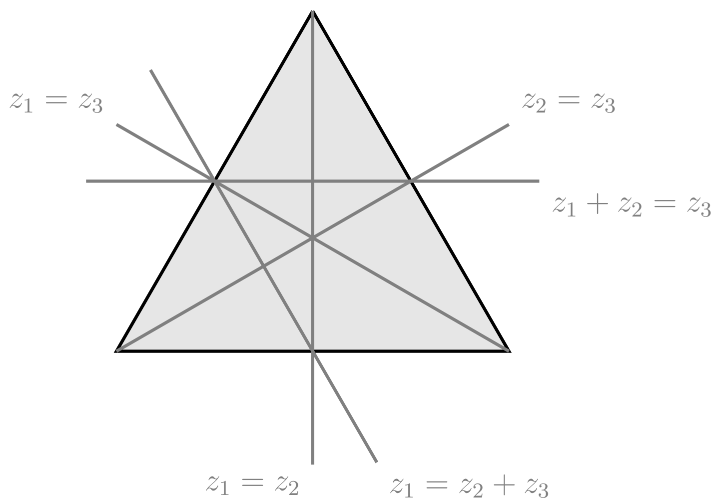


Figure 3.1: The number of regions of $\mathcal{G}(3)$.

3.2 The Mixed Graph Γ_m

In a graph we say that an edge that connects vertices v_1 and v_2 is **directed** if it has an orientation $v_1 \to v_2$. An undirected edge is just an edge with no orientation.

Definition 3.1. A **mixed graph** Γ is a graph with the set of vertices $V(\Gamma) = \{v_1, \ldots, v_k\}$ and edges $E(\Gamma) = \{e_1, \ldots, e_q\}$ where some edges are directed and some are undirected.

Definition 3.2. We define Γ_m to be the mixed graph with vertex set $\mathbb{P}[m]$ where if $A \subset B$ then we have a directed edge $A \to B \in E(\Gamma_m)$ and if $A \not\subset B$ then we have the undirected edge $AB \in E(\Gamma_m)$. We call Γ_m a **Golomb graph** .

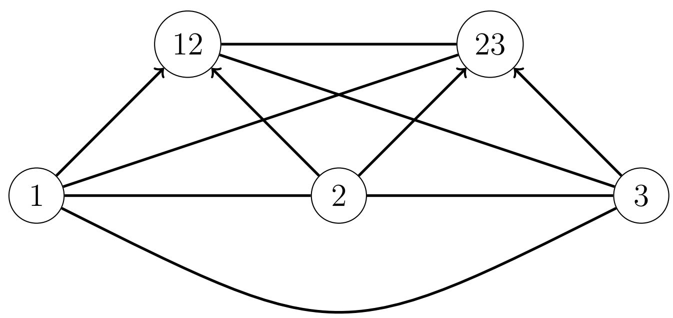
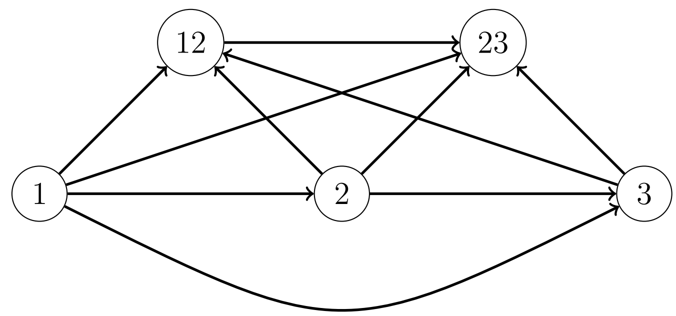


Figure 3.2: The mixed graph Γ_3.

Example 3.1. Since $\mathbb{P}[3] = \{1, 2, 3, \{1, 2\}, \{2, 3\}\}$, Γ_3 is a mixed graph with 5 vertices and 10 edges, 4 of which are directed. See Figure 3.2.

Definition 3.3. An **acyclic orientation** of Γ_m is an orientation of all the edges such that there does not exist a coherently oriented cycle, i.e., there does not exist $E_1, E_2, \ldots, E_k$ such that $E_1 \to E_2 \to \cdots \to E_k \to E_1$. Since a directed edge is already oriented, it cannot change.

Example 3.2. An acyclic orientation of Γ_3:



For each oriented edge $A \to B$ in Example 3.2, we consider the halfspace $\sum_{a=\min A}^{\max A} z_a < \sum_{b=\min B}^{\max B} z_b$ in $\mathcal{G}(3)$. Then each acyclic orientation of Γ_3 corresponds to a region in $\mathcal{G}(3)$. This is true in general for Γ_m and $\mathcal{G}(m)$ and we will state it formally with the following theorem.

Theorem 3.1. *The number of regions of $\mathcal{G}(m)$ is equal to the number of acyclic orientations of Γ_m with the condition that for every $A, B \in \mathbb{P}[m]$ where $A = U \cup W$ and $B = V \cup W$ and U, V, and W are nonempty disjoint sets, then $A \to B$ if and only if $U \to V$. We denote this condition as $[C1]$.*

Proof. We start with a region in $\mathcal{G}(m)$ which is defined by halfspaces bounded by the hyperplane arrangement in $\mathcal{G}(m)$. We map each halfspace to an orientation of an edge between two disjoints sets: for each halfspace $\sum_{u=\min U}^{\max U} z_u < \sum_{v=\min V}^{\max V} z_v$ we take the undirected edge $UV \in E(\Gamma_m)$ and give it the orientation $U \to V$. For any edge AB such that $A = U \cup W$ and $B = V \cup W$ where U, V, W are disjoint nonempty sets, we set the orientation $A \to B$ if and only if $U \to V$. Therefore condition $[C1]$

is satisfied by our construction and every edge in Γ_m is oriented. Next we check that this is an acyclic orientation. Suppose there is a cycle $A_1 \to A_2 \to \cdots \to A_k \to A_1$ for $A_1, A_2, \ldots, A_k \in V(\Gamma_m)$. For each $A_i \to A_{i+1}$ we let $W = A_i \cap A_{i+1}$ and therefore $A_i = U \cup W$ and $A_{i+1} = V \cup W$ for some U, V where $U \cap V = \emptyset$. Our construction implies $U \to V$ and therefore $\sum_{u=\min U}^{\max U} z_u < \sum_{v=\min V}^{\max V} z_v$. Hence we also have $\sum_{u=\min U}^{\max U} z_u + \sum_{w=\min W}^{\max W} z_w < \sum_{v=\min V}^{\max V} z_v + \sum_{w=\min W}^{\max W} z_w$, which is equivalent to

$$\sum_{a=\min A_i}^{\max A_i} z_a < \sum_{b=\min A_{i+1}}^{\max A_{i+1}} z_b.$$

But this is a contradiction since this would imply $\sum_{a=\min A_1}^{\max A_1} z_a < \sum_{a=\min A_2}^{\max A_2} z_a < \cdots < \sum_{a=\min A_1}^{\max A_1} z_a$.

Given an acyclic orientation of Γ_m with the condition $[C1]$, we want to map this to a region in $\mathcal{G}(m)$. To do so, we consider the space $\mathbb{R}^{\mathbb{P}[m]} := \{(z_{A_1}, z_{A_2}, \ldots, z_{A_p}) \in \mathbb{R}^p \,:\, A_k <_{\mathbb{P}} A_{k+1} \text{ for all } k\}$ where $p = |\mathbb{P}[m]| = \frac{m(m+1)}{2} - 1$ and we order the coordinates according to the lexicographical total ordering of $\mathbb{P}[m]$. Then we map the acyclic orientation of Γ_m to the corresponding region of the braid arrangement in $\mathbb{R}^{\mathbb{P}[m]}$ where we define it to be $\mathbb{B}_m := \{z_A = z_B \,:\, \{1\} \leq_{\mathbb{P}} A <_{\mathbb{P}} B \leq_{\mathbb{P}} \{2, 3, \ldots, m\}\}$. We restrict the space to $\mathbb{R}^{\mathbb{P}[m]} \cap \{\sum_{k=1}^{m} z_{\{k\}} = 1\}$. Since the regions of the braid arrangement are uniquely determined by an ordering of the coordinates, if we restrict our space to $\sum_{A \in \mathbb{P}[m]} z_A = 1$ we are still able to attain all possible orderings of the coordinates; therefore the number of regions is preserved. Each edge AB with the orientation $A \to B$ we map to the ordering $z_A < z_B$. Since the orientation of Γ_m is acyclic, this gives rise to a region of $\mathbb{B}_m$ in our space $\mathbb{R}^{\mathbb{P}[m]} \cap \{\sum_{k=1}^{m} z_{\{k\}} = 1\}$. Next we consider the projection from $\mathbb{R}^{\mathbb{P}[m]} \cap \{\sum_{k=1}^{m} z_{\{k\}} = 1\} \mapsto \mathbb{R}^m$ where we map $z_A \mapsto \sum_{a=\min A}^{\max A} z_a$ for all $A \in \mathbb{P}[m]$. This projection is a linear transformation from $\mathbb{R}^{\mathbb{P}[m]} \cap \{\sum_{k=1}^{m} z_{\{k\}} = 1\} \mapsto \mathbb{R}^m$ with the matrix representation

$$M_m = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{A_1} & c_{A_2} & c_{A_3} & \dots & c_{A_p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\text{where } c_{A_k} = \sum_{j=\min A_k}^{\max A_k} e_j.$$

(Recall that e_j is the j-th unit vector.)

Example 3.3. When $n = 4$ we have

$$M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

We can easily see that M_m has full rank, this means that the null space is trivial. Therefore a region of $\mathbb{R}^{\mathbb{P}[m]} \cap \{\sum_{k=1}^{m} z_{\{k\}} = 1\}$ projected down to $\mathbb{R}^m$ by M_m is still a region.

Next we show that our projected region is exactly a region of $\mathcal{G}(m)$. Since the region in $\mathbb{R}^{\mathbb{P}[m]} \cap \{\sum_{k=1}^{m} z_{\{k\}} = 1\}$ lives on the hyperplane $\sum_{k=1}^{m} z_{\{k\}} = 1$ and M_m maps $z_{\{k\}} \mapsto z_k$, our projected region lives on the hyperplane $\sum_{k=1}^{m} z_k = 1$. Recall that if $A \subset B$ then we have the directed edge $A \mapsto B$. Each oriented edge $\{k+1\} \mapsto \{k, k+1\}$ corresponds to the halfspace $z_{\{k+1\}} < z_{\{k,k+1\}}$ in $\mathbb{R}^{\mathbb{P}[m]} \cap \{\sum_{k=1}^{m} z_{\{k\}} = 1\}$, which is projected to $z_{(k+1)} < z_k + z_{(k+1)}$ in $\mathbb{R}^m$ for all $1 \leq k \leq m$. This implies that $z_k > 0$ for all $1 \leq k \leq m$. Therefore each projected region lives in the open polytope $R(m)$. Recall that a region in the braid arrangement $\mathbb{B}_m$ is defined by an ordering of the coordinates z_A for $A \in \mathbb{P}[m]$. Through this projection each ordering $z_A < z_B$ is projected to the halfspace $\sum_{a=\min A}^{\max A} z_a < \sum_{b=\min B}^{\max B} z_b$. For our acyclic orientation of Γ_m we have the condition $[C1]$. We will now show why this condition is necessary for the projection M_m. For any pair $A = U \cup W$ and $B = V \cup W$, without loss of generality we assume $A \to B$, therefore $U \to V$. This would be mapped to $z_A < z_B$ and $z_U < z_V$ in B_m. Let us denote $A = \{u, u+1, \ldots, u+r, w, w+1, \ldots, w+t\}$ and $B = \{v, v+1, \ldots, v+s, w, w+1, \ldots, w+t\}$. Then through the projection of M_m we have the mapping

$$z_A < z_B \mapsto \sum_{a=u}^{u+r} z_a + \sum_{c=w}^{w+t} z_c < \sum_{b=v}^{v+s} z_b + \sum_{c=w}^{w+t} z_c$$

$$z_U < z_V \mapsto \sum_{a=u}^{u+r} z_a < \sum_{b=v}^{v+s} z_b$$

which agree with each other. Therefore we only need to consider disjoint sets U and V. This means the projected region is precisely described by the hyperplane arrangement $\mathcal{H}(m)$. Hence we conclude the projected region is a region of $\mathcal{G}(m)$. $\square$

By using a program to compute the number of regions of an inside-out polytope written by Andrew Van Herick[3], we were able to compute the number of regions of $\mathcal{G}(m)$ where m ranges from 1 to 6.

m	$\mathcal{R}(m)$
1	1
2	2
3	10
4	114
5	2608
6	107498

Table 3.1: The number of regions of $\mathcal{G}(m)$.

Chapter 4

A Reciprocity Theorem for the Golomb Counting Function

Ehrhart theory is a powerful tool used to enumerate lattice points in rational polytopes, polytopes with rational vertices. We assume that every polytope defined is rational unless stated otherwise. For an m-dimensional polytope $\mathcal{P}$ we define the function $L_{\mathcal{P}}(t) := |t\mathcal{P} \cap \mathbb{Z}^m|$ as the Ehrhart counting function for the number of lattice points in $t\mathcal{P}$, the t-th dilate of $\mathcal{P}$. Ehrhart theory tells us that $L_{\mathcal{P}}(t)$ is a quasipolynomial whose period is equal to the denominator of the polytope (defined below). Ehrhart theory can also be applied to inside-out polytopes with a slight modification to the theorems. At the end we will state a reciprocity theorem for $L_{\mathcal{G}(m)}(t)$, the Golomb ruler counting function.

Definition 4.1. We define the **denominator** of a point $(a_1, \ldots, a_m) \in \mathbb{Q}^m$ as the least common multiple of all the denominators of the coordinates.

Example 4.1. The denominator of $(\frac{1}{2}, \frac{2}{3}, \frac{5}{7})$ is $\mathrm{LCM}(2, 3, 7) = 42$.

Definition 4.2. For a polytope $\mathcal{P}$ with vertices $v_1, \ldots, v_n$, we define the **denominator** of $\mathcal{P}$ as the least common multiple of the denominators of all vertices. We say that a polytope is an integral polytope if the denominator is 1, i.e., all the vertices are lattice points.

Definition 4.3. A function $q(t) := c_n(t)t^n + c_{n-1}(t)t^{n-1} + \cdots + c_1(t)t + c_0(t)$ where each coefficient $c_k(t)$ is a periodic function is called a **quasipolynomial** of period p

Theorem 4.1. *[2] Given an m-dimensional polytope $\mathcal{P}$ with denominator r, then $L_{\mathcal{P}}(t)$ is a quasipolynomial in t with period dividing r. We call $L_{\mathcal{P}}(t)$ the Ehrhart quasipolynomial of $\mathcal{P}$. If $\mathcal{P}$ is an integral polytope then $L_{\mathcal{P}}(t)$ is an Ehrhart polynomial with the constant term 1. Further,*

$$L_{\mathcal{P}}(-t) = (-1)^m L_{\mathcal{P}^\circ}(t)$$

where $\mathcal{P}^\circ$ is the interior of $\mathcal{P}$.

Erhart theory can also be applied to inside-out polytopes. We do so by observing that an open inside-out polytope can be viewed as an open polytope that is divided up by a hyperplane arrangement into multiple open polytopes. Therefore we can also view an open inside-out polytope as a union of all these sub-divided open polytopes. Given an inside-out polytope $Q = (\mathcal{P}, \mathcal{H})$ with q regions, we apply Ehrhart theory to the open inside-out polytope $Q^\circ = (\mathcal{P}^\circ, \mathcal{H})$ which has q open regions. Then $\mathcal{P}^\circ \backslash \mathcal{H} = \cup_{i=1}^{q} \mathcal{P}_i^\circ$ where each $\mathcal{P}_i^\circ$ is a region of Q°. Similarly we define $L_{\mathcal{Q}^\circ}(t)$ to be the counting function for the lattice points in the open inside-out polytope $\mathcal{Q}^\circ$. Thus

$$L_{Q^\circ}(t) = \sum_{i=1}^{q} L_{\mathcal{P}_i^\circ}(t),$$

and so $L_{Q^\circ}(t)$ is also a quasipolynomial but now its constant term is $\pm q$, which is exactly the number of regions of Q. We now state it as a theorem.

Definition 4.4. Let $\mathcal{Q} = (\mathcal{P}, \mathcal{H})$ be an inside-out polytope with q regions, where the regions are denoted as $\mathcal{P}_i$ for $1 \leq i \leq q$. Then we define the denominator of $\mathcal{Q}$ to be the least common multiple of all the vertices of the regions $\mathcal{P}_i$.

Theorem 4.2. *[4] If $\mathcal{Q}$ is an inside-out polytope, then $L_{\mathcal{Q}^\circ}(t)$ is a quasipolynomial in t, where the period is the denominator of $\mathcal{Q}^\circ$ and the constant term is up to a sign the number of regions of $\mathcal{Q}$.*

Corollary 4.3. $L_{\mathcal{G}(m)}(0) = \#$ *of acyclic orientation of* Γ_m *with the condition* $[C1]$.

Proof. By Theorem 4.2 we get $L_{\mathcal{G}(m)}(0) = \#$ of regions in $\mathcal{G}(m)$. Then by Theorem 3.1 we get the number of regions of $\mathcal{G}(m)$ is equal to the number of acyclic orientation of the mixed graph Γ_m with the condition $[C1]$. $\square$

Suppose $\mathcal{G}(m)$ has q regions; denoted by $\mathcal{P}_i^\circ$ for $1 \le i \le q$. Since $L_{\mathcal{G}(m)}(t)$ can be viewed as the sum of the Ehrhart quasipolynomial of all the regions, then by Theorem 4.1 we get

$$L_{\mathcal{G}(m)}(-t) = \sum_{i=1}^{q} L_{\mathcal{P}_i^\circ}(-t) = (-1)^{m-1} \sum_{i=1}^{q} L_{\mathcal{P}_i}(t), \tag{4.1}$$

where each $\mathcal{P}_i$ represents the closure of $\mathcal{P}_i^\circ$. Therefore $L_{\mathcal{G}(m)}(-t)$ counts not only the interior points but also the points on the boundary of each region, which means if a lattice point lies on the boundary of k regions then it will be counted k times. Hence we get the following corollary.

Corollary 4.4. *By evaluating $L_{\mathcal{G}(m)}(t)$ at $-t$ we get*
$L_{\mathcal{G}(m)}(-t) = \#$ *of lattice points , each counted with multiplicity of the number of closed region of $\mathcal{G}(m)$ it lies in.*

Definition 4.5. We define a **real Golomb ruler** to be a ruler $\{z_1, \ldots, z_m\}$ of m measures and length t where $z_i \in \mathbb{R}_+$ for $1 \le i \le m$ satisfing the condition of a Golomb ruler, i.e., $\sum_{u=\min U}^{\max U} z_u \neq \sum_{v=\min V}^{\max V} z_v$, for all disjoint pairs $U, V \in \mathbb{P}[m]$, and $\sum_{k=1}^{m} z_k = t$.

.

Hence by construction, we can view the inside-out polytope $\mathcal{G}(m,t)$ as the set of all real Golomb rulers of length t and m measures.

Definition 4.6. Given a real Golomb ruler of m measures, $\{z_1, z_2, \ldots, z_m\}$, we give an orientation to the Golomb graph Γ_m where $A \to B$ if and only if $\sum_{a=\min A}^{\max A} z_a < \sum_{b=\min B}^{\max B} z_b$. Then we say that two real Golomb rulers of m measures are combinatorially equivalent if and only if they give rise to the same orientation of Γ_m.

Hence each region in $\mathcal{G}(m)$ corresponds to one combinatorial type of real Golomb rulers of m measures.

Definition 4.7. For a sufficiently small ε, we define the function $\nu_{\mathcal{G}(m)}(z)$ to count the number of combinatorial type of real Golomb rulers in a neighborhood of radius ε from z, where z is a point in $\mathbb{R}^m$.

Definition 4.8. We define $\overline{R}(m)$ to be the closed polytope of $R(m)$.

$$\overline{R}(m) := \{(z_1, \ldots, z_m) \in \mathbb{R}^m \,:\, z_k \geq 0 \text{ for all } 1 \leq k \leq m \text{ and } \sum_{i=1}^{m} z_i = 1\}$$

Theorem 4.5. $L_{\mathcal{G}(m)}(-t) = \sum_{z \in \overline{R}(m) \cap \mathbb{Z}^m} \nu_{\mathcal{G}(m)}(z)$.

Proof. From (4.1)we know

$$L_{\mathcal{G}(m)}(-t) = \sum_{i=1}^{q} L_{\mathcal{P}_i^\circ}(-t) = (-1)^m \sum_{i=1}^{q} L_{\mathcal{P}_i}(t).$$

Thus $L_{\mathcal{G}(m)}(-t)$ is counting all the lattice points in $\overline{R}(m)$ but with multiplicity k if a point lies on k closed regions. To rephrase it more clearly, if $z \in \overline{R}(m) \cap \mathbb{Z}^m$, then it is counted k times if it lies on k closed regions of $\mathcal{G}(m,t)$. It suffices to show that the function $\nu_{\mathcal{G}(m)}(z)$ counts the number of closed regions that z lies on. If z is an interior point of a region, say $\mathcal{P}_j$, then for a sufficiently small enough ε, a ball of radius ε will be contained in $\mathcal{P}_j$ and therefore $\nu_{\mathcal{G}(m)}(z) = 1$. If z is on the boundary of k closed regions, say $\mathcal{P}_i$ for $1 \leq i \leq k$, then for any $\varepsilon > 0$, the ball of radius ε will always contain points from regions $\mathcal{P}_i$ for $1 \leq i \leq k$. Hence $\nu_{\mathcal{G}(m)}(z) = k$. $\square$

Appendix A: Notations

$\mathbb{P}[m]$	The set of all proper consecutive subsets of $[m]$.
$R(m,t)$	The $(m-1)$-dimensional open polytope in $\mathbb{R}^m$ where every lattice point correspond to a ruler of length t and m measures. The polytope is defined to be the set of points $(z_1,\ldots,z_m) \in \mathbb{R}^m_{>0}$ such that $\sum_{k=1}^{m} z_k = t$.
$\mathcal{H}(m)$	The hyperplane arrangement of hyperplanes $\sum_{u=\min U}^{\max U} z_u = \sum_{v=\min V}^{\max V} z_v$ where $U,V \in \mathbb{P}[m]$ and $U \cap V = \emptyset$.
$\mathcal{G}(m,t)$	The inside-out polytope $\mathcal{G}(m,t) := (R(m,t), \mathcal{H}(m))$.
$\mathcal{G}(m)$	$\mathcal{G}(m) := \mathcal{G}(m,1)$.
Γ_m	The Golomb graph, this is a mixed graph with the vertex set $\mathbb{P}[m]$ where if $A \subset B$ then we have a directed edge $A \to B \in E(\Gamma_m)$, else we have the undirected edge $AB \in E(\Gamma_m)$.
$\mathcal{R}(m)$	counts the number of regions of $\mathcal{G}(m)$.

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