A NEW TWO-VARIABLE GENERALIZATION OF THE CHROMATIC POLYNOMIAL FOR SIGNED GRAPHS

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by

Mela Maria Hardin

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CERTIFICATION OF APPROVAL

I certify that I have read *A NEW TWO-VARIABLE GENERALIZATION OF THE CHROMATIC POLYNOMIAL FOR SIGNED GRAPHS* by Mela Maria Hardin and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

Matthias Beck
Professor of Mathematics

Serkan Hoşten
Professor of Mathematics

Federico Ardila
Professor of Mathematics

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Mela Maria Hardin
San Francisco State University
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The function that counts the number of proper colorings of a graph is the chromatic polynomial. Such colorings can also be done with signed graphs, graphs consisting of an unsigned graph and a sign function labeling each edge and loop positive or negative. A signed graph has a chromatic polynomial with the same enumerative and algebraic properties as unsigned graphs. In 2003, Dohmen, Pönitz and Tittmann introduced a two-variable chromatic polynomial by adding improper colors. We extend this polynomial to signed graphs and explore some interesting properties that realize other graph concepts as special evaluation. Furthermore, we extend Stanley's reciprocity theorem for the chromatic polynomial and its connections to acyclic orientations to the two-variable chromatic polynomial for unsigned and signed graphs.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Introduction

1.1 Introduction

George David Birkhoff introduced the chromatic polynomial in 1912, defining it only for planar graphs in an attempt to prove the four color theorem [4]. In 1932 Hassler Whitney generalized Birkhoff's polynomial from the planar case to general graphs, expanding the study of chromatic polynomials from maps to graphs [15]. Then, in 1968 Read introduced the concept of chromatically equivalent graphs [13]. Today, chromatic polynomials are one of the central objects of algebraic graph theory.

It was not until 2003 that Dohmen, Pönitz, and Tittmann introduced the two-variable version of the chromatic polynomial by adding improper colors [6]. These improper colors can be referred to as "wildcard" colors. They show that the two-variable chromatic polynomial is a polynomial in k and l and generalizes the chromatic polynomial, the independence polynomial, and the matching polynomial of a graph. Five years later, in 2008, Hillar and Windfeldt extended their work to show how the Fibonacci and Lucas numbers (and other integer recurrences) arise naturally in the context of graph coloring, and in particular, how identities among these numbers can be generated from the different choices for decomposing a graph into smaller pieces [11]. We extend the works of Dohmen, Pönitz, and Tittmann as well as Hillar and Windfeldt to signed graphs and discover some interesting properties that realize other graph concepts such as independence and antibalance as special evaluation.

Signed graphs were first developed in the social sciences in the 1950's to model social situations [10]. They have applications in many fields because they give a relationship, positive or negative, between two nodes. In addition to social networks, signed graphs show up in knot theory to show positive or negative crossings [5], biology to model different classes of biological networks [8], physics to compute the ground state energy in Ising model, computer science to cluster data by similarities, and qualitative economics to model changes [1]. Thirty years later, in 1982, Thomas Zaslavsky introduced signed graph coloring and the method of deletion–contraction to signed graphs [17].

Finally, we prove a reciprocity theorem for two-variable chromatic polynomials. A famous theorem of Stanley's [14] states that when one evaluates the chromatic polynomial at negative integers, one obtains the function that counts pairs consisting of k-colorings and compatible acyclic orientations of the graph; in particular, the evaluation at -1 gives the number of acyclic orientations. For the two-variable chromatic polynomial, we obtain an extension of Stanley's result to (k, l)-colorings of unsigned graphs.

1.2 Goals

This thesis introduces and studies the two-variable chromatic polynomial for signed graphs. In Chapter 2 we introduce the basic language used in graph theory. Chapter 3 begins with an introduction to signed graphs.

The main results of this paper appear in Chapters 4 and 5. In Chapter 4, we introduce the two-variable chromatic polynomial for signed graphs and the recurrences that result from decompositions of particular graphs. We extend the results of [3] on Ehrhart–Macdonald reciprocity and its connections to graph coloring to the two-variable chromatic polynomials for unsigned and signed graphs in Chapter 5.

The motivation for this paper is to extend known theorems to further develop connections between geometric combinatorics, number theory, and graph theory.

Chapter 2

(Unsigned) Graphs

We begin with basic definitions used in graph theory along with some examples and figures. In Section 2.2 we introduce coloring and the chromatic polynomial for unsigned graphs. We begin discussing the two-variable chromatic polynomial for unsigned graphs in Section 2.3.

2.1 (Unsigned) Graphs

A **graph** G is a pair $G = (V, E)$ consisting of a set V and a set E. The elements of V are the **vertices** (or **nodes**) of the graph G. The set E is a subset of

$$U := \{\{v, w\}, \{v\} : v, w \in V\},$$

where $\{v, w\}$ is considered as a multiset. An element of E that is a one-element subset of V is either a **loop** with two coinciding vertices v denoted by $\{v, v\}$ or a **halfedge** with one vertex v denoted by $\{v\}$. An element $\{v, w\} \in E$ that is a two-element subset of V is a **link**. For the purpose of this thesis, we will not consider free loops.

A vertex v is **incident** with an edge e if $v \in e$; then e is an edge *at* v. The two vertices incident with an edge are its **endpoints**. An edge $\{v_1, v_2\}$ can be written as v_1v_2 (or v_2v_1). Two vertices v_1, v_2 of G are **adjacent**, or **neighbors** if v_1v_2 is an edge of G. Two edges $e \neq f$ are **adjacent** if they have an endpoint in common. A graph without halfedges is called an **ordinary graph**.

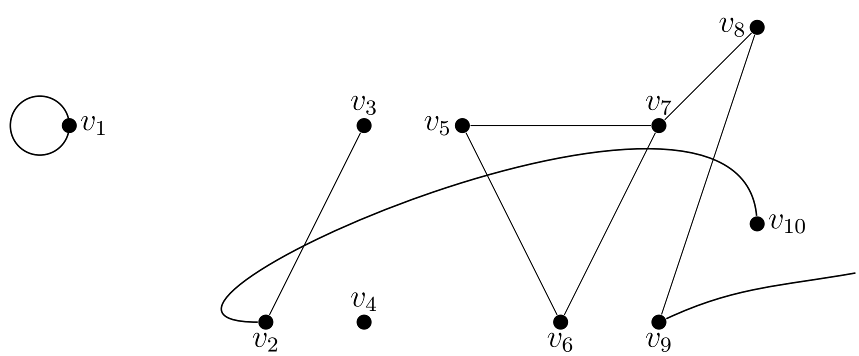

Figure 2.1: The graph G on vertex set $V = \{v_1, v_2, \ldots, v_{10}\}$ with edge set $E = \{v_1v_1, v_2v_3, v_2v_{10}, v_5v_6, v_6v_7, v_7v_5, v_7v_8, v_8v_9, \{v_9\}\}$. Notice that E contains a loop v_1v_1 and a halfedge $\{v_9\}$. The vertex v_4 is an isolated vertex.

2.1.1 Subgraphs

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $G \cap G' = \emptyset$, then G and G' are **disjoint**. If, however, $V' \subseteq V$ and $E' \subseteq E$, then G' is a **subgraph** of G denoted by $G' \subseteq G$.

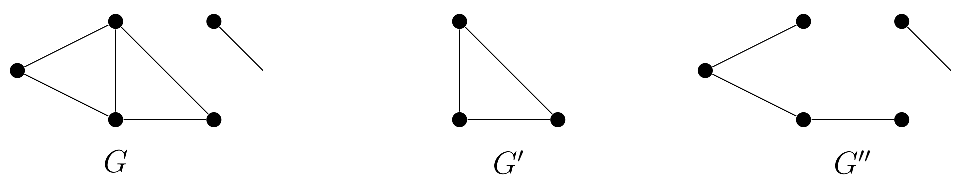

Figure 2.2: A graph G with subgraphs G' and G''.

2.1.2 The Degree of a Vertex

Let $G = (V, E)$ be a non-empty graph. The **degree** $d_G(v) = d(v)$ of a vertex is the number $|E(v)|$ of edges at v. A vertex of degree 0 is **isolated**.

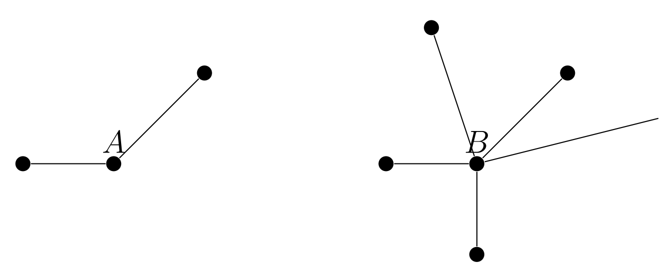

Figure 2.3: The degree of vertex A is 2, $d(A) = 2$. The degree of vertex B is 5, $d(B) = 5$.

2.1.3 Paths and Cycles

A **path** is a non-empty graph $P = (V, E)$ of the form

$$V = \{v_0, v_1, \ldots, v_k\} \quad E = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\},$$

where the v_i are all distinct. The vertices v_0 and v_k are **linked** by P and are called its **ends**; the vertices $v_1, \ldots, v_{k-1}$ are the **inner** vertices of P. The number of edges of a path is its **length**. A path graph contains only vertices of degrees two and one. We refer to a path by the natural sequence of its vertices, $P = v_0v_1 \ldots v_k$ and say that P is a path *from* v_0 *to* v_k.

Figure 2.4: A path $P = v_1v_2v_3v_4$ from v_1 to v_4 in G.

A **cycle** is a non-empty graph $C := P \cup \{v_{k-1}v_0\}$, where $P = v_0 \ldots v_{k-1}$ is a path and $k \geq 3$. In a cycle, the number of vertices equals the number of edges and each vertex has degree two. The **length** of a cycle is its number of edges (or vertices); a cycle of length k is called a *k-cycle*.

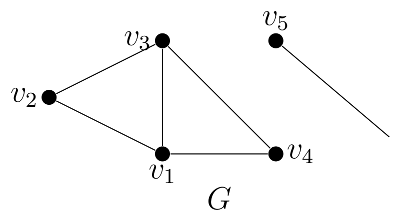

Figure 2.5: A graph G with cycles $C_1 = v_1v_2v_3v_1$, $C_2 = v_1v_3v_4v_1$, and $C_3 = v_1v_2v_3v_4v_1$.

A non-empty graph G is **connected** if any two of its vertices are linked by a path in G. A maximal connected subgraph of G is called a **component** of G.

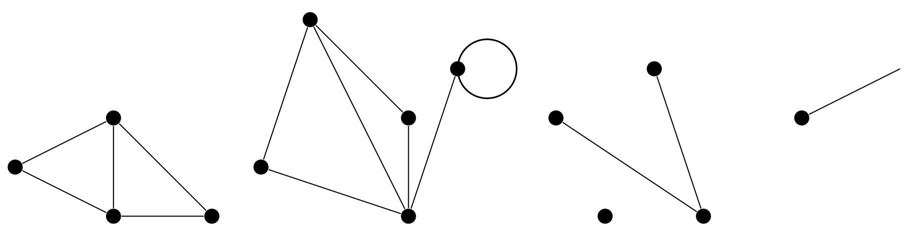

Figure 2.6: A graph G with five components. G contains six cycles.

2.1.4 Contraction

Let $e = v_1v_2$ be an edge of a graph $G = (V, E)$. By G/e we denote the graph obtained from G by *contracting* the edge e into a new vertex v_e, which becomes adjacent to all the former neighbors of v_1 and of v_2. G/e is a graph (V', E') with vertex set $V' := (V - \{v_1, v_2\}) \cup \{v_e\}$, where v_e is the 'new' vertex and the edge set

$E' := \{w_1w_2 \in E | \{w_1, w_2\} \cap \{v_1, v_2\} = \emptyset\} \cup \{v_ew_2 | v_1w_2 \in E - \{e\} \text{ or } v_2w_2 \in E - \{e\}\}$.

In other words, a **contraction** of an edge e of G is the graph G/e obtained by removing e and identifying as equal the two vertices sharing this edge.

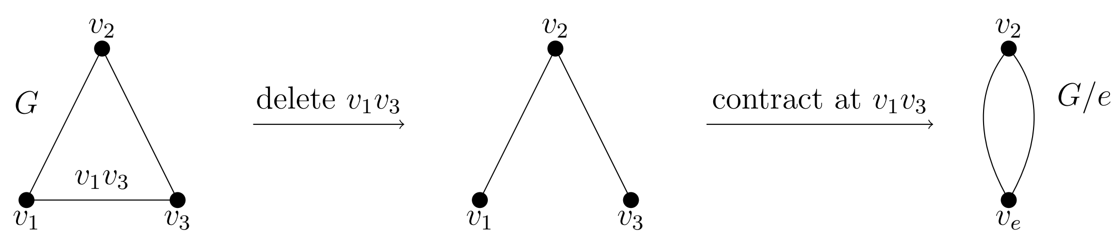

Figure 2.7: Deletion-contraction of a graph $G = (V = \{v_1, v_2, v_3\}, E = \{v_1v_2, v_2v_3, v_1v_3\})$ to graph $G/e = (V' = \{v_2, v_e\}, E' = \{v_2v_e\})$.

2.1.5 Independent Set

An **independent set** of a graph G, denoted by I, is a subset of V such that no two vertices are adjacent, i.e., it is a set I of vertices such that for every two vertices $v_i, v_j \in I$, v_iv_j is not an edge. Equivalently, each edge in G has at most one endpoint in I. The size of an independent set is the number of vertices it contains.

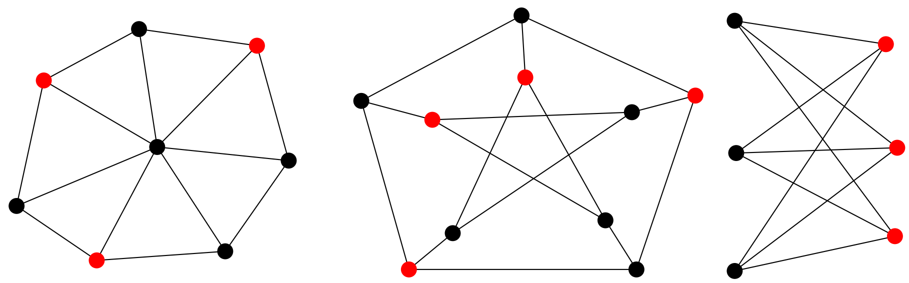

Figure 2.8: The red vertices represent elements of an independent set for each graph.

The **independence number** or the **maximal independence set**, denoted by α, is the cardinality of the largest independent set, i.e., such that adding any other vertex to the this set forces the set to contain an edge with two endpoints. Formally,

$$\alpha := \max\{|U| : U \subset V \text{ independent}\}$$

for a graph G, where $|U|$ denotes the cardinality of the set U.

2.1.6 Independence Polynomial

Let a_i be the number of independent vertex sets of cardinality i in a graph G. The **independence polynomial** of G is then defined by

$$I(G; x) := \sum_{i=0}^{n} a_i x^{n-i},$$

where n denotes the number of vertices in G.

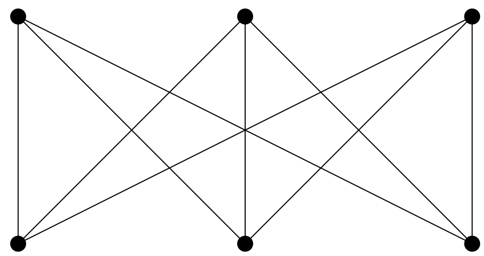

Figure 2.9: $K_{3,3}$ used in Example 2.1

Example 2.1.

$$\begin{aligned} I(G;x) &= \sum_{i=0}^{6} a_i x^{n-i} \\ &= x^6 + 6x^5 + 6x^4 + 2x^3 \end{aligned}$$

2.2 Coloring and the Chromatic Polynomial

A **(vertex) coloring** of a graph $G = (V, E)$ is a map

$$\varphi : V \longrightarrow \{1, 2, \ldots, k\},$$

where $\{1, 2, \ldots, k\}$ is the set of available colors. A **proper** vertex k-coloring of a graph G is a map $\varphi : V \longrightarrow \{1, 2, \ldots, k\}$ such that $\varphi(v_1) \neq \varphi(v_2)$ whenever v_1 and v_2 are adjacent. Otherwise, we say that the map φ is **improper**.

In order to compute the number of ways to properly k-color a graph G, we compute the chromatic polynomial. The **chromatic polynomial** of G is the function

$$P(G;k) := \#(\text{proper } k\text{-colorings of } G).$$

One way to compute the chromatic polynomial of G is to use the method of "deletion and contraction", which involves the recursive combination of chromatic polynomials for smaller graphs.

Theorem 2.1 (Read [13])**.** *$P(G;k) = P(G-e;k) - P(G/e;k)$.*

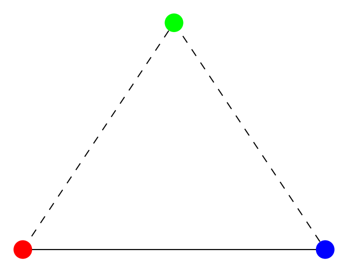

Figure 2.10: A graph G used for Example 2.2

Example 2.2. $P(K_3;k) = (k)(k-1)(k-2)$. (See Figure 2.10.)

Theorem 2.2. *For any graph G, the chromatic function $P(G;k)$ is a polynomial.*

Proof. We will do induction on the size of E, the edges of G. The fewest number of edges for G is zero. Let $E = \emptyset$. Then we have $P(G;k) = k^n$, where n is the number of vertices. This is clearly a polynomial. Now, assume that if we have G', a graph with fewer than m edges, then $P(G'; k)$ is a polynomial. Let G be a graph with m edges. By Theorem 2.1, we have $P(G; k) = P(G - e; k) - P(G/e; k)$. The graph $G - e$ has one fewer edge, namely edge e. G/e may have more than 1 fewer edges than G, depending on whether the endpoints of e had common neighbors. Therefore, by the induction hypothesis, $P(G - e; k)$ and $P(G/e; k)$ are polynomials. Thus,

$$P(G; k) = P(G - e; k) - P(G/e; k)$$

is a polynomial. □

2.3 Two-variable Chromatic Polynomial for Unsigned Graphs

In 2003, Dohmen, Pönitz, and Tittmann introduced a two-variable chromatic polynomial in k and l that generalizes the chromatic polynomial and the independence polynomial [6]. Then, in 2008 Hillar and Windfeldt showed how the Fibonacci and Lucas numbers (and other integer recurrences) arise naturally using the two-variable chromatic polynomial [11]. Our extension of both work to signed graphs is contained in Chapter 4. Here, we introduce and highlight important materials that we use in the chapters ahead.

Given nonnegative integers k and l, we define the set

$$\{\underbrace{1, \ldots, k}_{\mathcal{K}}, \underbrace{(k+1), \ldots, (k+l)}_{\mathcal{L}}\}$$

to be the set of $(k+l)$ available colors with two disjoint subsets $\mathcal{K}$ and $\mathcal{L}$.

A (k,l)-**coloring** of a graph G is a map $\varphi : V \longrightarrow \mathcal{K} \cup \mathcal{L}$. The map φ is called **proper** if whenever vertex v is adjacent to vertex w and $\varphi(v), \varphi(w) \in \mathcal{K}$, then $\varphi(v) \neq \varphi(w)$. Otherwise, we say that the coloring is **improper**.
As we will see in the following theorem, the **generalized two-variable chromatic polynomial of** G is the function

$$P(G;k,l) := \#(\text{proper } (k,l)\text{-colorings of } G).$$

Theorem 2.3 (Dohmen, Pönitz, & Tittmann [6])**.** *Let $G = (V,E)$ be a graph and W be a subset of V. Then*

$$P(G;k,l) = \sum_{W \subseteq V} l^{|W|} P(G-W;k).$$

Corollary 2.4 (Dohmen, Pönitz, & Tittmann [6])**.** *Let a_i be the number of independent vertex sets of cardinality i in $G = (V,E)$ and n denotes the number of vertices in G. Then the independence polynomial of G, defined by $I(G;k) := \sum_{i=0}^{n} a_i x^{n-i}$, satisfies*

$$I(G,k) = P(G;1,l).$$

In particular, $P(G;1,1)$ is the number of independent vertex sets of G.

Theorem 2.5 (Dohmen, Pönitz, & Tittmann [6]). *The generalized chromatic polynomial $P(G; k, l)$ satisfies*

$$\frac{\partial}{\partial l} P(G; k, l) = \sum_{v \in V} P(G - v; k, l).$$

We next state a rule to calculate the polynomial $P(G; k, l)$ recursively via deletion–contraction.

Lemma 2.6 (Hillar & Windfeldt [11]). *Let e be an edge in G, and let v be the vertex to which e contracts in G/e. Then,*

$$P(G; k, l) = P(G - e; k, l) - P(G/e; k, l) + l\, P([G/e] - v; k, l).$$

Chapter 3

Signed Graphs

Now that we have discussed unsigned graphs, we continue with signed graphs. In this chapter we first introduce some definitions and operations on signed graphs that are different from unsigned graphs. In Subsection 3.1.2 we explain, step-by-step, how to delete and contract a signed graph along with an example figure. We also highlight theorems, propositions, and corollaries important to coloring a signed graph and computing its chromatic polynomial. Section 3.2 is the beginning of proper coloring of signed graphs.

3.1 Signed Graphs

A **signed graph** S is a pair $S = (G, \sigma)$ consisting of an (unsigned) graph G and a sign function $\sigma : E \longrightarrow \{+1, -1\}$ that labels each edge and loop with a $+$ or $-$

sign. The sign function σ is defined for all edges except halfedges (not sign-labelled) [17]. Note that an unsigned graph G can be realized as a signed graph where the sign function σ labels each edge and loop with a $+$ sign.

3.1.1 Switching and Balance

Suppose $S = (G, \sigma)$ is a signed graph and $\nu : V \longrightarrow \{\pm 1\}$. **Switching** S by ν means forming the switched graph $S^{\nu} = (G, \sigma^{\nu})$, whose underlying graph is the same but whose sign function is defined on an edge v_1v_2 by

$$\sigma^{\nu}(v_1v_2) = \nu(v_1)\sigma(v_1v_2)\nu(v_2).$$

Observe that we are only reversing the sign of an edge whose endpoints have opposite ν values, i.e., $\sigma^{\nu}(v_1v_2) \neq \sigma(v_1v_2)$ if vertex $v_1 = \nu^{-1}(-)$ and vertex $v_2 = \nu^{-1}(+)$.

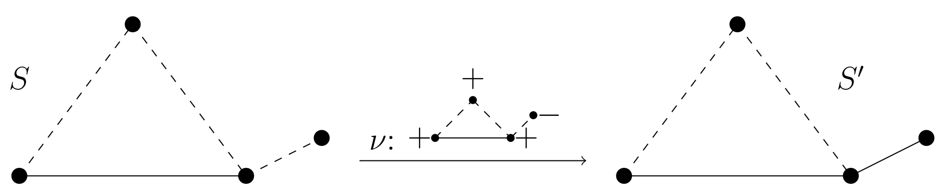

Figure 3.1: Signed graph S switched by ν to signed graph S'

An edge set $F \subseteq E$ is **balanced** if it contains no halfedges and each cycle in it has a positive product of signs.

Proposition 3.1 (Zaslavsky [17]). *Switching S does not alter balance.*

Proof. Since halfedges are not signed-labelled and a signed graph S is balanced if it contains no halfedges and each cycle in it has a positive product of signs, we only need to consider cycles here. Suppose each vertex has a positive sign, then switching at a vertex means to assign that vertex a negative sign. We will do an induction on s, the number of vertex switches. For $s = 1$, the switch affects two edges incident to that vertex because each vertex has degree two. Thus, the switch results in an even number of sign changes (two) and the graph is balanced. For the induction step, suppose that switching s vertices does not alter balance. Since this graph as the result of switching s vertices is balanced, we can again assign each vertex to have a positive sign, and for $s + 1$ switches, we are only adding one more switch to a balanced graph which again is balanced due to the fact that each vertex has degree 2 and each vertex switch affects an even number of edge reversals. $\square$

Corollary 3.2 (Zaslavsky [17]). *A signed graph $S = (V, E, \sigma)$ is balanced if and only if it is switching equivalent to an all-positive graph without halfedges.*

Proof. We only need to consider cycles because a graph S is balanced if it does not contain halfedges and each cycle has a positive product of signs, i.e., each cycle has an even number of negative edges. Let C be a balanced cycle in S. We assume that each vertex has a positive sign and switching at a vertex means to assign that vertex a negative sign. We will do an induction on e, the number of negative edges in C, to show that a balanced cycle C is switching equivalent to an all-positive graph without halfedges. The fewest number of negative edges for a balanced cycle is zero. This is clearly an all-positive cycle. Now, assume that we can switch C with e negative edges such that the switched cycle graph is all-positive. For $e+2$, we have two cases, whether these two negative edges are adjacent or not. If they are adjacent, we simply switch at the vertex incident to both negative edges. If these negative edges are not adjacent, pick one of two all-positive paths between the negative edges and switch at all the vertices in that path. This switch will keep all the edges in the path positive while switching the two negative edges to positive edges. The resulting cycle is all-positive. The converse is true since every switch at a vertex affects an even number of edge reversals. Hence, we can switch any all-positive cycle to have an even number of negative edges. □

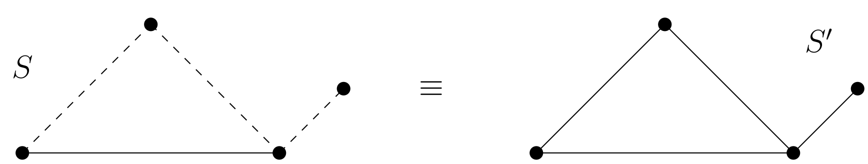

Figure 3.2: Signed graph S is switching equivalent to an all-positive signed graph S'

3.1.2 Contraction

The **restriction** of $S = (V, E, \sigma)$ to an edge set $F \subseteq E$ is the subgraph

$$S|F = (V, F, \sigma|F).$$

The **contraction** of S by $F \subseteq E$ is denoted by S/F. To construct S/F, first S must be switched to S^{ν} such that each balanced components of $S^{\nu}|F$ have positive sign; this is possible by Corollary 3.2. Second, discard all the elements of F and coalesce all the vertices in each balanced component of $S^{\nu}|F$ into a 'new' vertex, i.e., contract all the edges in each balanced component of $S|F$ into a new vertex (the same as the contraction of an unsigned graph). Let V_b be the vertex set that contains all the 'new' vertices after contracting the edges in each balanced component of $S|F$ and

$$W := \{v \in V : v \text{ is a vertex of an unbalanced component of } S|F\}.$$

Now, let $E_b \subseteq E$ such that elements in E_b are links, each connected to a balanced component of $S|F$ and to $S - F$, i.e., for an element $vu \in E_b$, $v \in (S - F)$ and u is in a balanced component of $S|F$. Because we contracted the edges in each balanced components of $S|F$ into a 'new' vertex, we now obtain a new edge set E_c. The set E_c consists of links such that one endpoint of each link is in V and the other is one of the new vertex from V_b, i.e., the vertex $u \in vu \in E_b$ is replaced by a new vertex $w \in V_b$ making vw an element of E_c. The signs of edges of E_c are kept the same as the signs of edges in E_b they replaced, i.e., $\sigma(vw) = \sigma(vu)$. Therefore, the contracted graph S/F is a signed graph (V', E', σ') with vertex set $V' := (V - W) \cup V_b$, edge set $E' := (E - E_b) \cup E_c$, and a sign function σ', which is the same as σ, but restricted to the new edge set E'. By definition, the contraction of an ordinary graph is that described in Section 2.1.4.

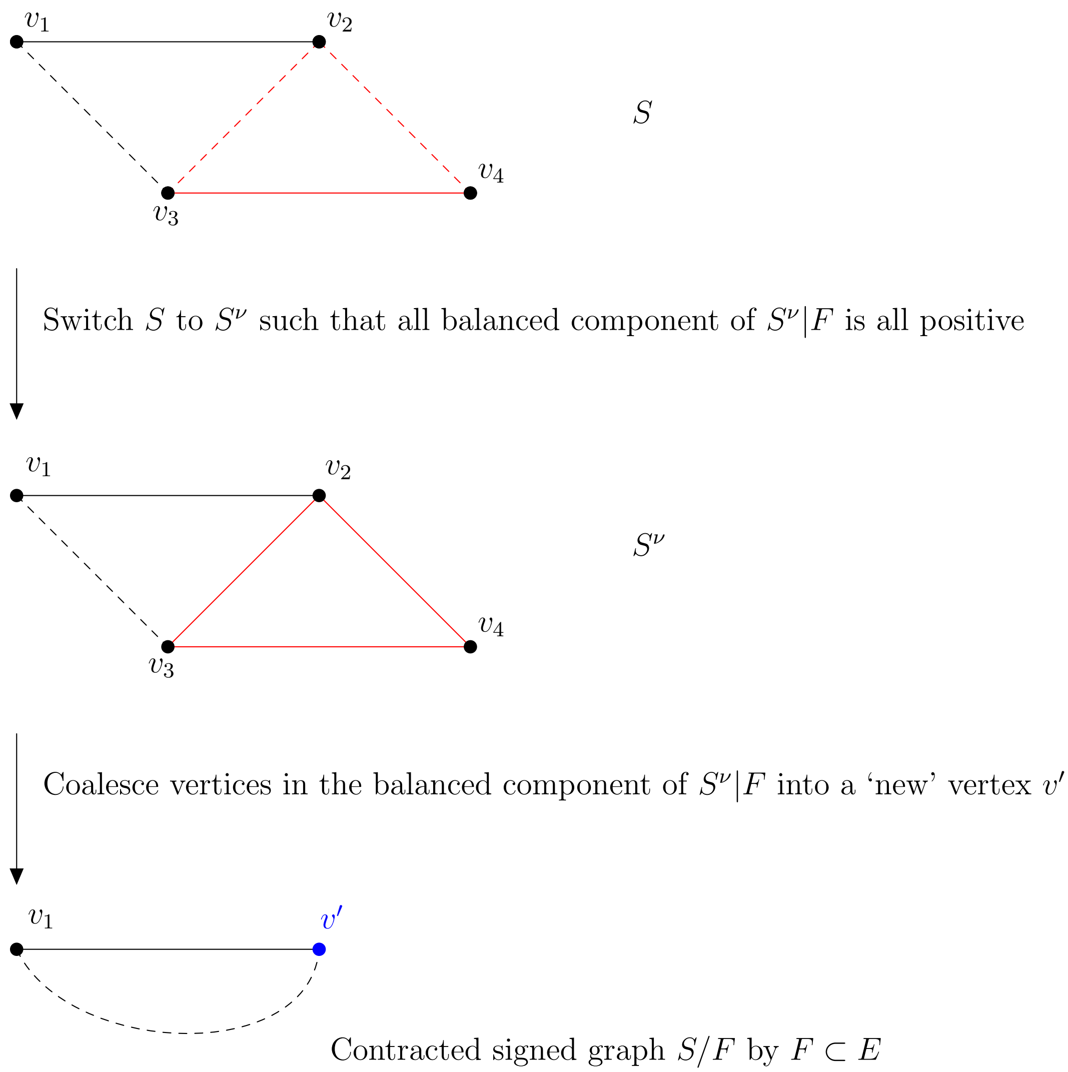

Figure 3.3: A step-by-step example of the contraction of a signed graph S by $F \subset E$. The set F is shown in red and v' is an element of V_b, a vertex set that contains all the 'new' vertices after contracting the edges in each balanced component of $S|F$.

3.2 Coloring and the Chromatic Polynomial

Let $S = (G, \sigma)$ be a signed graph. A (vertex) k-**coloring** of a signed graph with vertex set V is a function

$$\varphi : V \longrightarrow \mathcal{K} := \{0, \pm 1, \ldots, \pm k\}.$$

The coloring φ is **proper** if the endpoints of each positive edge have different signed colors and those of each negative edge have colors that are not negatives of each other, i.e., whenever there is an edge v_1v_2 with sign $+$, then $\varphi(v_1) \neq \varphi(v_2)$ and if the edge v_1v_2 has a $-$ sign, then $\varphi(v_1) \neq -\varphi(v_2)$. In addition, the incident vertex of halfedges and negative loops cannot be colored zero. These conditions imply that a signed graph S that contains a positive loop cannot be properly colored.

The function

$$P(S; x) := \#(\text{proper } k\text{-colorings of } S)$$

is known to be a polynomial, called the **chromatic polynomial** of S defined for odd positive arguments $x = 2k + 1$ [17].

Signed graphs have a second chromatic counting function: the **zero-free chromatic**

polynomial or the **balanced chromatic polynomial**

$$P^*(S;x) := \#(\text{proper zero-free } k\text{-colorings of } S)$$

defined for even positive arguments $x = 2k$. A coloring is **zero-free** if no vertex gets the color value 0. $P(S;x)$ and $P^*(S;x)$ are two constituent polynomials of a single Ehrhart quasipolynomial of an inside-out polytope [3].

Proposition 3.3 (Zaslavsky [17])**.** *Switching does not alter $P(S;x)$ and $P^*(S;x)$.*

Proof. Let $\mathbf{k} := (k_1, k_2, \ldots, k_n)$ be a vector in $\mathbb{Z}^n$ that represents a proper coloring of S. Now, suppose we switch S at vertex w. We define a mapping $(k_1, \ldots, k_w, \ldots, k_n) \longmapsto (k_1, \ldots, -k_w, \ldots, k_n)$ of S. Clearly, this map is a bijection and the coloring is proper. If $k \in (\mathbb{Z} \setminus 0)^n$, then $(k_1, \ldots, -k_w, \ldots, k_n) \in (\mathbb{Z} \setminus 0)^n$. □

We next state a simple rule that enables one to calculate the polynomials $P(S;x)$ and $P^*(S;x)$ recursively via deletion–contraction.

Theorem 3.4 (Zaslavsky [16])**.** *Let $S = (V, E, \sigma)$ be a signed graph and $e \in E$. Then if e is a halfedge or a negative loop,*

$$P(S;x) = P(S - e;x) - P(S/e;x)$$

otherwise,

$$P(S;x) = P(S - e;x) - P(S/e;x)$$

and

$$P^*(S;x) = P^*(S-e;x) - P^*(S/e;x).$$

Proof. Suppose e is a halfedge or a negative loop. Then, $P(S-e;x)$ is precisely the chromatic polynomial of S without the edge e. Let the vertex incident to e be denoted by v. Since v cannot be colored zero (because e is a halfedge or a negative loop), we have to subtract those proper colorings where we allow v to be zero, which means that the vertices adjacent to v cannot be colored zero. We obtain the graph S/e by contracting S at e and the chromatic polynomial of S/e is $P(S/e;x)$. And the result follows.

If e is not a halfedge or a negative loop (i.e., they can be assigned the color zero), then we consider the number of proper k-colorings of $S-e$. The vertices incident with e in S can now be colored distinctly or colored the same color since they are no longer incident with e in $S-e$. This coloring function is given by $P(S-e;x)$. We now have to subtract $P(S/e;x)$, the coloring function where we allow the two vertices incident with e to be colored the same. Similar argument for the $P^*(S;x)$ case. □

On the coloring interpretation of Theorem 3.4. For e, a negative loop or a halfedge, $P^*(S;x)$ is not expected. Let us consider S consisting of two vertices linked by one positive and one negative edges, with a negative loop at each vertex. We have $P(S;x) = (x-1)(x-3)$ because we cannot color the vertices zero, hence, x is defined for even arguments. We see that $P(S;2) = -1$, implying that $P(S;x)$ does not only count colorings for all positive integral x. What $P(S;x)$ can mean for even arguments is not considered here.

Chapter 4

A New Two-variable Chromatic Polynomial for Signed Graphs

In this chapter we extend Dohmen, Pönitz, and Tittmann's two-variable chromatic polynomial in [6] to signed graphs. Our two-variable chromatic polynomial generalizes the chromatic polynomial (Section 4.1), the independence polynomial (Section 4.2) and the antibalance polynomial (Section 4.3) for signed graphs. The latter is unique to signed graphs because antibalance is defined only for signed graphs. Then in Section 4.4, we generalize the Lucas number and other recurrences in the context of signed graph colorings, an extension to Hillar and Windfeldt's work in [11].

4.1 Two-variable Chromatic Polynomials for Signed Graphs

In this section, we extend the results of Dohmen, Pönitz, and Tittmann in [6].

Given nonnegative integers k and l, we define the set

$$\{\underbrace{0, \pm 1, \ldots, \pm k}_{\mathcal{K}}, \underbrace{\pm(k+1), \ldots, \pm(k+l)}_{\mathcal{L}}\}$$

to be the set of $2(k+l)+1$ available colors with two disjoint subsets $\mathcal{K}$ and $\mathcal{L}$.

A (k,l)-**coloring** of a signed graph S is a map $\varphi : V \longrightarrow \mathcal{K} \cup \mathcal{L}$. The map φ is called **proper** if whenever vertex v is adjacent to vertex w and

1. the edge vw is positive with $\varphi(v), \varphi(w) \in \mathcal{K}$, then $\varphi(v) \neq \varphi(w)$;

2. the edge vw is negative with $\varphi(v), \varphi(w) \in \mathcal{K}$, then $\varphi(v) \neq -\varphi(w)$.

Otherwise, we say that the coloring is **improper**. In somewhat looser terminology, one can think of the set $\mathcal{L}$ as "wildcard" colors.

The function

$$P(S; x, y) := \#(\text{proper } (k,l)\text{-colorings of } S)$$

is called the **chromatic polynomial** of S defined for odd positive arguments x and y such that $x = 2k+1$ and $y = 2l$.

The **zero-free chromatic polynomial**

$$P^*(S; x, y) := \#(\text{proper zero-free } (k, l)\text{-colorings of } S)$$

is defined for even positive arguments x and y such that $x = 2k$ and $y = 2l$. A coloring is **zero-free** if no vertex gets the color value 0. This extends the definition of [6] for unsigned graphs.

The following theorem combines Theorem 3.4 and Lemma 2.6.

Theorem 4.1. *Let $S = (V, E, \sigma)$ be a signed graph, then the counting functions $P(S; x, y)$ and $P^*(S; x, y)$ are polynomials;*

$$P(S; x, y) = P(S - e; x, y) - P(S/e; x, y) \tag{4.1}$$

if $e \in E$ is a halfedge or a negative loop;

$$P(S; x, y) = P(S - e; x, y) - P(S/e; x, y) + y\, P([S/e] - v; x, y) \tag{4.2}$$

and

$$P^*(S; x, y) = P^*(S - e; x, y) - P^*(S/e; x, y) + y\, P^*([S/e] - v; x, y) \tag{4.3}$$

if $e \in E$ is not a halfedge or a negative loop and we let v be the vertex to which e contracts in S/e.

Proof. The proof to show $P(S; x, y)$ is a polynomial is the same as for Theorem 2.2 and the argument is the same for $P^*(S; x, y)$.

Suppose e is a halfedge or a negative loop. $P(S - e; x, y)$ is the chromatic polynomial of S without the edge e. Let the vertex incident to e be denoted by v. Since v cannot be colored zero (because e is a halfedge or a negative loop), we have to subtract those proper colorings where we allow v to be zero, which means that the vertices adjacent to v cannot be colored zero. We obtain the graph S/e by contracting S at e and the chromatic polynomial of S/e is $P(S/e; x, y)$, and the result follows.

Now, suppose e is not a halfedge or a negative loop, then we consider the number of proper (k, l)-colorings of $S - e$. The vertices incident with e in S can now be colored distinctly or colored the same color since they are no longer incident with e in $S - e$. This coloring function is given by $P(S - e; x, y)$. We now have to subtract $P(S/e; x, y)$, the coloring function where we allow the two vertices incident with e to be colored the same. The remaining proper (k, l)-colorings of S are precisely those for which the vertices sharing the edge e have the same color (chosen from the improper color set $\mathcal{L}$) of which we have $2l$ colors to choose from. The argument for $P^*(S; x, y)$ is the same. $\square$

Lemma 4.2. *Let $S = (V, E, \sigma)$ be a signed graph. Then we have the evaluations*

$$P(S; 0, y) = P^*(S; 0, y) = y^n,$$

where n is the number of vertices.

Proof. We will prove this by induction on n, the number of vertices in S. Since the set $\mathcal{L}$ contains only of improper (wildcard) colors, we will only need to consider equations (4.2) and (4.3) of Theorem 4.1 because halfedges and negative loops do not affect the chromatic polynomial of S when we only have wildcard colors, i.e., ways to color a signed graph S containing halfedges and/or negative loops is the same as ways to color S disregarding those halfedges and/or negative loops. For $n = 0$, $P(S; 0, y) = y^0 = 1 = P^*(S; 0, y)$. Now, suppose the statement is true for n. For $n + 1$ vertices, S/e has n vertices and $[S/e] - v$ has $n - 1$ vertices. Thus, $P(S/e; 0, y) = P^*(S/e; 0, y)$ and $y\, P([S/e] - v; 0, y) = y\, P^*([S/e] - v; 0, y)$ cancel and we are left with $P(S - e; 0, y) = P^*(S - e; 0, y)$ with $n + 1$ vertices. The result follows. $\square$

Now, we generalize Theorem 4.1.

Lemma 4.3. *Let $S = (V, E, \sigma)$ be a signed graph. Then*

$$P(S; x, y) = \sum_{W \subseteq V} y^{|W|} P(S - W; x)$$

and

$$P^*(S; x, y) = \sum_{W \subseteq V} y^{|W|} P^*(S - W; x).$$

Proof. Every generalized proper coloring of S can be obtained by first choosing a subset W of V that is colored with colors of $\mathcal{L}$. There are $y^{|W|}$ different colorings for these vertices since the set $\mathcal{L}$ is the improper set of colors. The remaining subgraph $S - W$ has to be colored properly using the proper colors from the set $\mathcal{K}$, for which there are $P(S - W; x)$ possibilities. Similarly for $P^*(S; x, y)$. $\square$

4.2 Independence Polynomial

Let a_i be the number of independent vertex sets of cardinality i in a signed graph S. The **independence polynomial** of S is then defined by

$$I(S; x) := \sum_{i=0}^{n} a_i x^{n-i},$$

where n denotes the number of vertices in S.

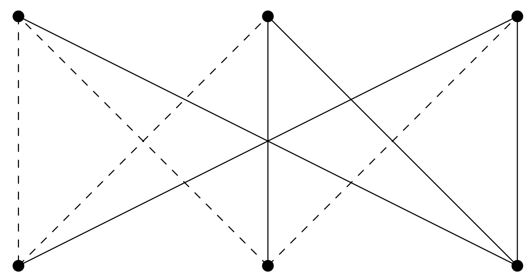

Figure 4.1: Signed $K_{3,3}$ used in Example 4.1

Example 4.1.

$$\begin{aligned} I(S;x) &= \sum_{i=0}^{6} a_i x^{n-i} \\ &= x^6 + 6x^5 + 6x^4 + 2x^3 \end{aligned}$$

We next extend Corollary 2.4 to signed graphs.

Corollary 4.4. *The independence polynomial of S satisfies*

$$I(S;y) = P(S;1,y).$$

Proof. By Lemma 4.3, we have

$$\begin{aligned} P(S;1,y) &= \sum_{W \subseteq V} y^{|W|} P(S-W;1) \\ &= \sum_{W \subseteq V} y^{|W|} Q(S-W), \end{aligned}$$

where $Q(T) := \begin{cases} 0 & \text{if } T \text{ is not independent,} \\ 1 & \text{if } T \text{ is independent.} \end{cases}$

Consequently, the coefficient of y^i in $P(S;1,y)$ counts the independent sets of cardinality $n-i$ in S. $\square$

The following theorem extends Theorem 2.5 to signed graphs.

Theorem 4.5. *Let $S = (V, E, \sigma)$ be a signed graph. Then*

$$\frac{\partial}{\partial y} P(S;x,y) = \sum_{v \in V} P(S - v; x, y)$$

and

$$\frac{\partial}{\partial y} P^*(S;x,y) = \sum_{v \in V} P^*(S - v; x, y).$$

Proof. From Lemma 4.3, it follows that

$$\sum_{v \in V} P(S - v; x, y) = \sum_{v \in V} \sum_{U \subseteq [V - v]} y^{|U|} P(S - [U \cup v]; x).$$

Setting $W = U \cup v$, we get

$$\sum_{v \in V} \sum_{U \subseteq [V - v]} y^{|U|} P(S - [U \cup v]; x) = \sum_{W \subseteq V} |W| y^{|W|-1} P(S - W; x),$$

since $|W|$ is the number of ways of getting the set W by first choosing a vertex

$v \in W$ then the rest of the vertices in W, the set U. Now,

$$\begin{aligned}\sum_{W \subseteq V} |W| y^{|W|-1} P(S-W; x) &= \frac{\partial}{\partial y} \sum_{W \subseteq V} y^{|W|} P(S-W; x) \\ &= \frac{\partial}{\partial y} P(S; x, y).\end{aligned}$$

The proof for the zero-free chromatic polynomial $P^*(S; x, y)$ is the same. $\square$

4.3 Antibalance Polynomials

An **antibalanced** signed graph is the negative (or negation) of a balanced signed graph, i.e., a signed graph $S = (V, E, \sigma)$ is antibalanced if every cycle in S has an even number of positive edges.

Figure 4.2: Examples of antibalanced signed graphs.

Let b_i be 2^m times the number of antibalanced subgraphs of a signed graph S with i vertices and m components. The **antibalance polynomial** of S is then defined by

$$A(S;x) := \sum_{i=0}^{n} b_i x^{n-i},$$

where n denotes the number of vertices in S.

Proposition 4.6. *The chromatic polynomial $P^*(S;2)$ is an indicator function for antibalanced signed graphs, in the sense that*

$$P^*(S;2) := \begin{cases} 0 & \text{if } S \text{ is not antibalanced,} \\ 2^m & \text{if } S \text{ is antibalanced with } m \text{ components.} \end{cases}$$

Proof. It is enough to consider one component of a signed graph S. Suppose $S' \subset S$ is a component of S such that S' is not antibalanced, i.e., S' contains a cycle C with an odd number of positive edges. An induction on the number of positive edges in C starting with the base case of one positive edge shows that S' cannot be properly 2-colored. Now, consider an antibalanced component S'' of S. Pick a vertex v in S'' and properly color S''. The color of v determines the colors of all the other vertices. $\square$

Theorem 4.7. *The antibalance polynomial of S satisfies*

$$A(S; y) = P^*(S; 2, y).$$

Proof. From Lemma 4.3, it follows that

$$P^*(S; 2, y) = \sum_{W \subseteq V} y^{|W|} P^*(S - W; 2).$$

The result follows from Proposition 4.6 and the proof is analogous to that of Lemma 4.3. □

4.4 Recurrences and Graph Colorings

In this section we generalize the Lucas number and other recurrences in the context of signed graph colorings, an extension to Hillar and Windfeldt's work in [11]. We show how these recurrences can be generated from the different choices for decomposing a signed graph into smaller pieces.

With such recurrences as in Theorem 4.1, initial conditions for $P(S; x, y)$ need to be specified. When S consists of one vertex and has no edges, we have

$$P(S; x, y) = x + y$$

and when S is the empty graph,

$$P(S; x, y) = 1.$$

A **link of a vertex** v is defined to be the subgraph $\operatorname{link} v$ consisting of v, the edges incident to v, and the vertices incident to the edges incident to v. If uv is an edge denoted by e, then we define

$$\operatorname{link}(e) := \operatorname{link}(u) \cup \operatorname{link}(v)$$

and

$$\deg(e) := \deg(u) + \deg(v) - 2.$$

Lemma 4.8. *Let v be a vertex not incident to a halfedge or a negative loop of S, then*

$$P(S; 1, y) = y\, P(S - v; 1, y) + y^{\deg(v)}\, P(S - \operatorname{link}(v); 1, y). \tag{4.4}$$

If v is incident to a halfedge or a negative loop, then

$$P(S; 1, y) = y\, P(S - v; 1, y).$$

Proof. The number of proper$(1, l)$-colorings of S with vertex v colored with a wildcard is $y\, P(S - v; 1, y)$. If v is not incident to a halfedge or a negative loop and v is colored zero, then each vertex adjacent to v must be colored with a wildcard color chosen from $\mathcal{L}$, hence the second term. However, if v is incident to a halfedge or a negative loop, then it cannot be colored zero and must be colored with a wildcard, then the second term vanishes and we have $P(S; 1, y) = y\, P(S - v; 1, y)$. □

Lemma 4.9. *Let e be an edge that is not a halfedge or a negative loop, then*

$$P(S; 1, y) = P(S - e; 1, y) - y^{\deg(e)}\, P(S - \mathrm{link}(e); 1, y) \tag{4.5}$$

Proof. Let v be the vertex to which e contracts in S/e. From (4.4), substituting S/e for S, we get

$$P(S/e; 1, y) = y\, P([S/e] - v; 1, y) + y^{\deg(v)}\, P([S/e] - \mathrm{link}(v); 1, y) \tag{4.6}$$

Subtracting (4.6) from (4.2) with $x = 1$, the result follows. □

Let Q_n be a signed path graph, H_n be a signed path graph with a terminal halfedge, B_n be a signed balanced cycle graph, and C_n be a signed unbalanced cycle graph, all on n vertices. Fixing nonnegative integers k and l not both zero, we define the following sequences of numbers ($n \geq 1$):

$$\begin{aligned} q_n &= P(Q_n; 2k+1, 2l), \\ h_n &= P(H_n; 2k+1, 2l), \\ b_n &= P(B_n; 2k+1, 2l), \\ c_n &= P(C_n; 2k+1, 2l). \end{aligned} \tag{4.7}$$

Figure 4.3: Signed graphs with four vertices

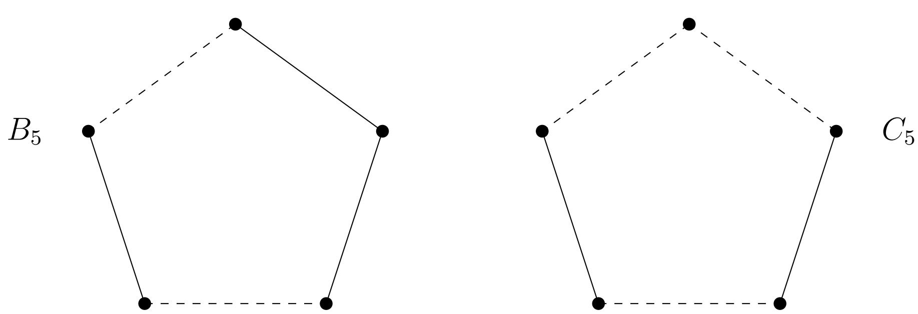

Figure 4.4: A balanced cycle B_5 and an unbalanced cycle C_5 with five vertices

The following lemmas use signed graph decomposition to give simple recurrences for these sequences.

Lemma 4.10.

$$q_1 = 2(k+l)+1, \quad q_2 = 4(k+l)^2 + 2k + 4l, \quad q_n = 2[(k+l)q_{n-1} + lq_{n-2}]. \quad (4.8)$$

In particular, if $k = 1$ and $l = 0$, we have

$$q_n = 2\, q_{n-1} \implies P(Q_n; 3, 1) = 3(2^{n-1}).$$

Proof. The result follows from deleting an outer edge of the signed path graph Q_n and using Theorem 4.1 (see example of graph Q_4 in Figure 4.4). □

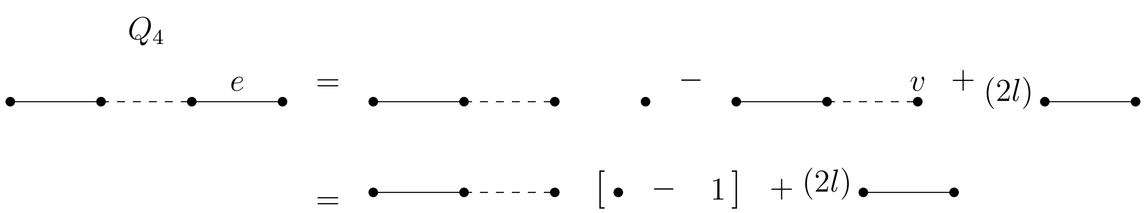

Figure 4.5: Deletion–contraction of Q_4 on an outer edge e. The vertex v is the vertex to which e contracts in Q_4/e.

Lemma 4.11.

$$h_1 = 2(k+l), \quad h_2 = 4(k+l)^2 + 2l, \quad h_n = 2(k+l)q_{n-1} - h_{n-1} + 2l\, q_{n-2}. \quad (4.9)$$

In particular, if $k = 1$ and $l = 0$, we have

$$h_n = 2q_{n-1} - h_{n-1} \implies P(H_n; 3, 1) = 2^n.$$

Proof. The result follows from deleting and contracting an edge adjacent to the terminal halfedge of H_n and using Theorem 4.1 (see example of graph H_3 in Figure 4.5). □

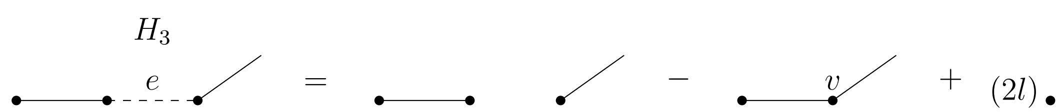

Figure 4.6: Deletion–contraction of H_3 on an edge e that is adjacent to the terminal halfedge. The vertex v is the vertex to which e contracts in H_3/e.

Lemma 4.12.

$$b_1 = 2l, \quad b_2 = q_2 = 4(k+l)^2 + 2(k+l) + 2l, b_3 = q_3 - b_2 + 2l\, q_1\,; \tag{4.10}$$

$$b_n = [2(k+l) - 1]b_{n-1} + 2(k+2l)b_{n-2} + 2l\, b_{n-3}. \tag{4.11}$$

In particular, if $k = 0$ and $l = 1$, we have

$$P(B_n; 1, 1) = L_{n+2}, \tag{4.12}$$

where L_n are the Lucas numbers.

If $\mathcal{K} = \emptyset$, we get a shorter recurrence $b_n = 2l\, b_{n-1}$.

Proof. The first recurrence follows by using Theorem 4.1 by deleting and contracting any edge in B_n to give

$$b_n = q_n - b_{n-1} + 2l\, q_{n-2}. \tag{4.13}$$

Now, let $d_n = b_n + b_{n-1}$.

Since $b_n = q_n - b_{n-1} + 2l\, q_{n-2}$, we have

$$\begin{aligned}
d_n &= q_n + 2l\, q_{n-2} \\
&= 2(k+l)q_{n-1} + 2l\, q_{n-2} + 2l[2(k+l)q_{n-3} + 2l\, q_{n-4}] \\
&= 2(k+l)q_{n-1} + 2l\, q_{n-2} + 4l(k+l)q_{n-3} + 4l^2\, q_{n-4} \\
&= 2(k+l)(q_{n-1} + 2l\, q_{n-3}) + 2l\,(q_{n-2} + 2l\, q_{n-4})
\end{aligned}$$

$$= \quad 2(k+l)d_{n-1} + 2l\, d_{n-2}.$$

[Notice that d_n satisfies the same recurrence as q_n.]

Thus, we have

$$\begin{aligned} b_n &= 2(k+l)[b_{n-1} + b_{n-2}] + 2l(b_{n-2} + b_{n-3}) - b_{n-1} \\ &= [2(k+l) - 1]b_{n-1} + 2(k+2l)b_{n-2} + 2l\, b_{n-3}. \end{aligned}$$

□

Lemma 4.13.

$$c_1 = 2(k+l), \quad c_2 = 4(k+l)^2 + 4l, \quad c_n = q_n + 2l\, q_{n-2} - c_{n-1}\,; \tag{4.14}$$

$$c_1 = 2(k+l), \quad c_2 = q_2 = 4(k+l)^2 + 2k + 4l, \quad c_n = q_n + 2l\, q_{n-2} - b_{n-1}\,; \tag{4.15}$$

Proof. The first recurrence follows by using Theorem 4.1 by deleting and contracting at a positive edge in C_n. The second recurrence also follows by using Theorem 4.1, but here we delete and contract at a negative edge in C_n. □

Chapter 5

Graph Coloring Reciprocity

In this chapter, we extend the results of [3] on Ehrhart–Macdonald reciprocity and its connections to graph coloring to the two-variable chromatic polynomials for unsigned and signed graphs. By introducing improper colors, Ehrhart–Macdonald reciprocity gives us both the acyclic orientations of a graph as described in [14] and the number of ways to color a graph using at least one of the wildcards. In Section 5.2 we begin with an introduction of Ehrhart-Macdonald reciprocity as applied to signed graphs in one variable. By introducing improper colors, Ehrhart–Macdonald reciprocity gives us both the compatible acyclic orientations of a signed graph as described in [16] and a coloring of S using colors from the set $\{\pm(k+2), \pm(k+3), \ldots, \pm(k+l+1)\}$.

5.1 (Unsigned) Graphs and Ehrhart–Macdonald Reciprocity

We begin by first discussing the basics of lattice polytopes and Ehrhart theory as well as connections with graph coloring.

A **convex polytope** $\mathcal{P}$ is the convex hull of finitely many points in $\mathbb{R}^d$, i.e., given any finite set $\{v_1, \ldots, v_n\} \subset \mathbb{R}^d$, $\mathcal{P}$ is the smallest convex set containing those points [2]. For an integer $k \in \mathbb{Z}_{>0}$, $k\mathcal{P}$ denotes the **dilated** polytope

$$\{(kx_1, kx_2, \ldots, kx_d) : (x_1, x_2, \ldots, x_d) \in \mathcal{P}\},$$

for any polytope $\mathcal{P}$. If $\mathcal{P} = \text{conv}\{V\}$ where $V \subset \mathbb{Z}^d$, then $\mathcal{P}$ is called a **lattice polytope**. We denote the interior of $\mathcal{P}$ by $\mathcal{P}^\circ$. The **unit** d**-cube** is defined as $\Box := [0,1]^d$. A **hyperplane** of a d-dimensional Euclidean or affine space is an affine subspace of dimension $d-1$. A **hyperplane arrangement** $\mathcal{H}$ is a set of finitely many hyperplanes in $\mathbb{R}^d$. This hyperplane arrangement divides up the space into regions: an **open region** is a connected component of $\mathbb{R}^d \backslash \bigcup \mathcal{H}$ and a **closed region** is the topological closure of an open region. A **face** of $\mathcal{P}$ is the intersection of a d-dimensional polytope with a tangent hyperplane. The $(d-1)$-dimensional faces are called **facets** and the 0-dimensional faces are its **vertices**.

We are interested in lattice–point counting in polytopes with boundary on the inside, i.e., we consider a convex polytope $\mathcal{P}$ with a hyperplane arrangement $\mathcal{H}$ and count the lattice–points that lie interior to $\mathcal{P}$ but not in any of the hyperplanes, because of a geometrical interpretation of coloring of graphs and signed graphs. This pair is denoted by $(\mathcal{P}, \mathcal{H})$, called an **inside–out polytope** [3]. $(\mathcal{P}, \mathcal{H})$ has dimension equal to the dimension of $\mathcal{P}$ and its vertices are all the intersection points in $\mathcal{P}$ formed by the hyperplanes of $\mathcal{H}$ and the facets of $\mathcal{P}$, including the vertices of $\mathcal{P}$.

Recall that a coloring of a graph G is proper if $\varphi(i) \neq \varphi(j)$ whenever there is an edge ij. We regard φ as a point in the real affine space $\mathbb{R}^n$ where $n = |V|$; it is proper if it lies in none of the hyperplanes $h_{ij} : \varphi(i) = \varphi(j)$ for $ij \in E$, the edge set of G. We write

$$\mathcal{H}[G] := \{h_{ij} : ij \in E\},$$

which is the **graphical arrangement** of G, then counting proper k-colorings of G means counting the integral points in $\{1, 2, \ldots, k\}^n \setminus \bigcup \mathcal{H}[G]$.

An **orientation** of a graph is an assignment of a direction to each edge ij, denoted by $i \to j$ or $j \to i$. An orientation of a graph is said to be **acyclic** if it has no directed cycles. Given an orientation α of G and a k-coloring $\varphi : V \to \{1, 2, \ldots, k\}$, Stanley calls them **compatible** if $\varphi(i) \leq \varphi(j)$ whenever there is an edge oriented from $i \to j$, and **proper** if $\varphi(i) < \varphi(j)$ under the same condition [14]. Our goal in this chapter is to extend Stanley's theorem below to the two-variable chromatic polynomial for ordinary and signed graphs.

Theorem 5.1 (Stanley [14])**.** *Let $G = (V, E)$ be a graph with n vertices. Then $(-1)^n P(G; -k)$ equals the number of pairs (α, φ) consisting of an acyclic orientation α of G and a compatible k-coloring φ. In particular, $(-1)^n P(G; -1)$ equals the number of acyclic orientations of G.*

The idea of Ehrhart theory is to study convex lattice polytopes by studying the number of lattice points in integral dilates of $\mathcal{P}$. For a d-dimension lattice polytope $\mathcal{P}$ in $\mathbb{R}^d$, we denote the **lattice–point enumerator** for the k^{th} dilate by

$$E_{\mathcal{P}}(k) := \#(k\mathcal{P} \cap \mathbb{Z}^d)$$

for $k \in \mathbb{Z}_{>0}$. $E_{\mathcal{P}}(k)$ is a polynomial function in k of degree d with rational coefficients called the **Ehrhart polynomial** [7]. In 1971, Macdonald proved the reciprocicty law called **Ehrhart–Macdonald reciprocity**.

Theorem 5.2 (Ehrhart–Macdonald Reciprocity [12])**.** *For any d-dimension lattice polytope $\mathcal{P}$,*

$$E_{\mathcal{P}^\circ}(k) = (-1)^d E_{\mathcal{P}}(-k).$$

For a polytope $\mathcal{P}$, a **region** of $(\mathcal{P}, \mathcal{H})$ is one of the components of $\mathcal{P} \setminus \bigcup \mathcal{H}$, or the closure of such a component. A **vertex** of $(\mathcal{P}, \mathcal{H})$ is a vertex of any of its regions.

The **multiplicity** of $\varphi \in \mathbb{R}^n$ *with respect to* $(\mathcal{P}, \mathcal{H})$ is

$$m_{\mathcal{H}}(\varphi) := \begin{cases} \text{the number of closed regions of } (\mathcal{P}, \mathcal{H}) \text{ that contain } \varphi, & \text{if } \varphi \in \mathcal{P}, \\ 0, & \text{if } \varphi \notin \mathcal{P}. \end{cases}$$

The Ehrhart polynomials of $(\mathcal{P}, \mathcal{H})$ are the **closed** and **open Ehrhart polynomials**

$$E_{\mathcal{P},\mathcal{H}}(k) := \sum_{\varphi \in k^{-1}\mathbb{Z}^n} m_{\mathcal{P},\mathcal{H}}(\varphi)$$

and

$$E^{\circ}_{\mathcal{P},\mathcal{H}}(k) := \# \left(k^{-1}\mathbb{Z}^n \cap \left[\mathcal{P} \setminus \bigcup \mathcal{H} \right] \right),$$

respectively.

We define $E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}[G]}(k)$ to be the **inside-out Ehrhart polynomial** that counts the points in $\mathcal{P}^{\circ}$ that do not lie in any of the graphical arrangement $\mathcal{H}[G]$.

Theorem 5.3 (Beck & Zaslavsky [3])**.** *If $(\mathcal{P}, \mathcal{H})$ is a closed inside-out polytope in $\mathbb{R}^d$ such that $\mathcal{H}$ does not contain the degenerate hyperplane, then $E_{\mathcal{P},\mathcal{H}}(k)$ and $E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}}(k)$ are polynomials in k, with leading term $c_d k^d$ where $c_d = \mathrm{vol}\, \mathcal{P}$, and with the constant term $E_{\mathcal{P},\mathcal{H}}(0)$ equal to the number of regions of $(\mathcal{P}, \mathcal{H})$. Furthermore,*

$$E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}}(k) = (-1)^d E_{\mathcal{P},\mathcal{H}}(-k).$$

Lemma 5.4. *Let*

$$\Phi(k+1,l) := E_{\square}(k+l+1) - E_{\square}(k+1)$$

the difference of the Ehrhart polynomial of the $(k+l+1)^{st}$ *dilate of the unit cube and the* $(k+1)^{st}$ *dilate of the unit cube, and let*

$$\Phi^{\circ}(k+1,l) := E_{\square^{\circ}}(k+l+1) - E_{\square^{\circ}}(k+1).$$

Then

$$\Phi^{\circ}(-k+1,-l) = (-1)^n \Phi(k-1,l),$$

for $k, l \in \mathbb{Z}_{\geq 0}$.

Proof. By Theorem 5.2,

$$\begin{aligned} \Phi^{\circ}(-k+1,-l) &= E_{\square^{\circ}}(-k-l+1) - E_{\square^{\circ}}(-k+1) \\ &= (-1)^n [E_{\square}(k+l-1) - E_{\square}(k-1)] \\ &= (-1)^n \Phi(k-1,l). \end{aligned}$$

□

In the following, we express the two-variable chromatic polynomial $P(G;k,l)$ geometrically.

Proposition 5.5. *Let G be an ordinary graph with n vertices and let $\Box = [0,1]^n$. The inside-out Ehrhart polynomial of $(\Box, \mathcal{H}[G])$ and Φ satisfy*

$$E^{\circ}_{\Box^{\circ},\mathcal{H}[G]}(k+1) + \Phi^{\circ}(k+1,l) = P(G;k,l).$$

Proof. In $\frac{1}{k+1}\mathbb{Z}^n$ the points that are counted by $E^{\circ}_{\Box^{\circ},\mathcal{H}[G]}(k+1)$ are those of $\left(\frac{1}{k+1}\{1,2,\ldots,k\}\right)^n$ that do not lie in any of the graphical arrangement $\mathcal{H}[G]$. The number of such points is the number of proper k-colorings of G. The second term $\Phi^{\circ}(k+1,l)$ comes from Lemma 5.4, which is the difference of interior points of $\left(\frac{1}{k+l+1}\{1,2,\ldots,k+l\}\right)^n$ and $\left(\frac{1}{k+1}\{1,2,\ldots,k\}\right)^n$, i.e., those colorings of G where we use at least one of the wildcard colors from the set $\mathcal{L}$. The sum of both terms gives us the two–variable chromatic polynomial for G (see example Figure 5.1 for $k=6$ and $l=4$). $\Box$

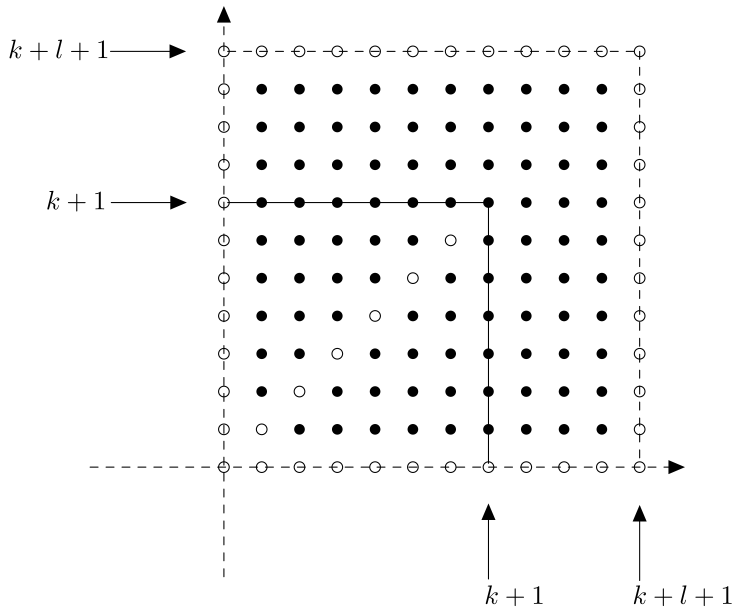

Figure 5.1: The lattice points corresponding to (k,l)-colorings of graph K_2 with $k=6$ and $l=4$.

Theorem 5.6. *Let $G=(V,E)$ be a graph with n vertices. Then $(-1)^n\,P(G;-k,-l)$ counts the $(k+l)$-colorings of G with multiplicity: a k-coloring has multiplicity equal to the number of compatible acyclic orientations of G; and a coloring of G with at least one wildcard, i.e., a color $> k$, has multiplicity of one.*

Proof. From Proposition 5.5,

$$P(G;-k,-l) = E^{\circ}_{\square^{\circ},\mathcal{H}[G]}(-k+1) + \Phi^{\circ}(-k+1,-l).$$

Lemma 5.4 and Theorems 5.2 and 5.3 imply

$$(-1)^n P(G; -k, -l) = E_{\square,\mathcal{H}[G]}(k-1) + \Phi(k-1, l).$$

$E_{\square,\mathcal{H}[G]}(k-1)$ counts the number of pairs (φ, R) where φ is a coloring with color set $\{0, 1, \ldots, k-1\}$, R is a closed region of $\mathcal{H}[G]$, and $\varphi \in R$. Greene observed that regions R correspond with acyclic orientation α in the following way: R° is determined by converting each equation $\varphi(i) = \varphi(j)$ corresponding to an edge of G into an inequality $\varphi(i) < \varphi(j)$; then in α the edge ij is directed from i to j [9]. The orientation is acyclic because $R^\circ \neq \emptyset$. Thus φ is compatible with α if and only if $\varphi \in R$. The second term $\Phi(k-1, l)$ equals

$$E_{\square}(k+l-1) - E_{\square}(k-1),$$

which counts the number of ways to (k, l)-color G using at least one wildcard color.

$\square$

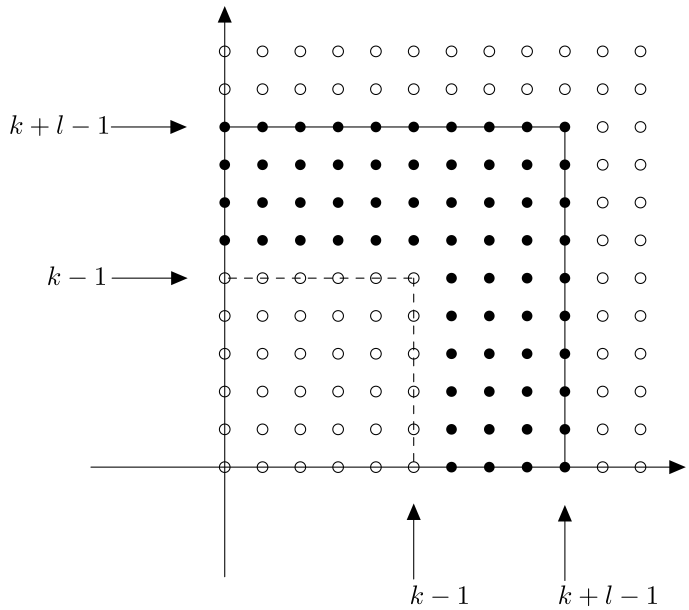

Figure 5.2: The lattice points of (k,l)-coloring of the graph K_2 using at least one wildcard color after applying the Ehrhart–Macdonald reciprocity law to $(-1)^n\Phi^\circ(-k+1,-l)$.

5.2 Signed Graphs and Ehrhart–Macdonald Reciprocity

We begin this section with some definitions necessary for our main theorem.

A **bidirection** of an unsigned graph G is a mapping

$$\tau : I(G) \to \{\pm\},$$

where $I(G)$ is the set of **incidences** between an edge e and its vertex v denoted as (v, e). The interpretation of τ:

1. if $\tau(v, e) = +$, then (v, e) points into the vertex v;
2. if $\tau(v, e) = -$, then (v, e) points away from v.

From the definition of τ, we see that

$$\sigma(e) = -\tau(v, e)\,\tau(w, e), \tag{5.1}$$

for the link $e = vw$. This determines a signed graph $S(\tau)$. Conversely, given a signed graph S we call any bidirection τ satisfying Equation (5.1) an **orientation of** S. We call (S, τ) an **oriented signed graph**. An orientation of S then is an assignment to each endpoint of an edge, i.e., "bidirecting" the edge, such that the two arrows on a positive edge agree (as in an unsigned graph G) but those on a negative edge are opposed (see Figure 5.3).

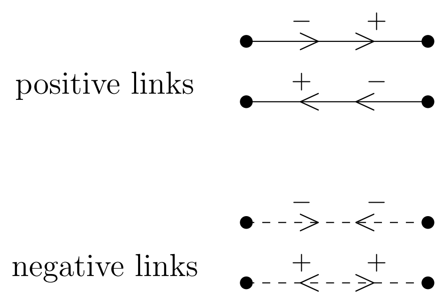

Figure 5.3: Possible orientations on positive and negative links.

Recall a **proper** k-coloring φ of a signed graph in Section 3.2. Geometrically,

$$\varphi \in \{0, \pm 1, \ldots, \pm k\} \setminus \bigcup \mathcal{H}[S],$$

where

$$\begin{aligned} \mathcal{H}[S] \quad := \quad & \{\varphi_j = \zeta\varphi_i : S \text{ has an edge } ij \text{ with sign } \zeta\} \\ & \cup \{\varphi_i = 0 : S \text{ has a halfedge at vertex } v_i\}. \end{aligned}$$

A coloring φ and an orientation τ are **compatible** if

$$\tau(v, e)\varphi(v) + \tau(w, e)\varphi(w) \geq 0, \tag{5.2}$$

whenever $e = vw$ is a link and

$$\tau(v, e)\varphi(v) \geq 0, \tag{5.3}$$

whenever $e = \{v\}$ is a halfedge or a negative loop [16].

A **cycle in an oriented signed graph** S has neither a source vertex nor a sink vertex. (see examples in Figure 5.4). A **source** vertex is one where no edges are directed towards it and a **sink** vertex is one where no edges are directed away from it. An orientation τ of an oriented signed graph S is **acyclic** if there are no cycles in S.

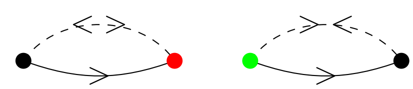

Figure 5.4: Acyclic orientation τ of oriented signed graphs $\pm K_2$. The red vertex is a sink vertex, while the green is a source vertex.

Now, let $\mathcal{P} = [-1,1]^n$. Beck and Zaslavsky observed that $P(S;x) = E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}[S]}(k+1)$, the number of lattice points that lie in $(k+1)\mathcal{P}^{\circ}$, but not in any of the graphical arrangement $\mathcal{H}[S]$ [3]. The regions of $\mathcal{H}[S]$ correspond to the acyclic orientations of S [18] and the regions that contain a coloring φ correspond to the acyclic orientations that are compatible with φ [16].

Theorem 5.7 (Zaslavsky [16])**.** *The number of compatible pairs (τ, φ) consisting of an acyclic orientation τ and a k-coloring of a signed graph S is equal to $(-1)^n\, P(S;-x)$ where $x = 2k+1$. In particular, $(-1)^n\, P(S;-1) =$ the number of acyclic orientations of S.*

In the following, we set up some definitions and express the two-variable chromatic polynomial $P(S;k,l)$ geometrically.

Lemma 5.8. *Let*

$$\Psi(k+1,l) := E_{\mathcal{P}}(k+l+1) - E_{\mathcal{P}}(k+1)$$

the difference of the Ehrhart polynomial of the $(k+l+1)^{st}$ *dilate of* $\mathcal{P}$ *and the* $(k+1)^{st}$ *dilate of* $\mathcal{P}$ *where* $\mathcal{P} = [-1,1]^n$*, and let*

$$\Psi^{\circ}(k+1,l) := E_{\mathcal{P}^{\circ}}(k+l+1) - E_{\mathcal{P}^{\circ}}(k+1).$$

Then

$$\Psi^{\circ}(-k-1,-l) = (-1)^n \Psi(k+1,l),$$

for $k,l \in \mathbb{Z}_{\geq 0}$.

Proof. By Theorem 5.2,

$$\begin{aligned}
\Psi^{\circ}(-k-1,-l) &= E_{\mathcal{P}^{\circ}}(-k-l-1) - E_{\mathcal{P}^{\circ}}(-k-1) \\
&= (-1)^n[E_{\mathcal{P}}(k+l+1) - E_{\mathcal{P}}(k+1)] \\
&= (-1)^n \Psi(k+1,l).
\end{aligned}$$

□

Proposition 5.9. *Let S be a signed graph with n vertices and let $\mathcal{P} = [-1,1]^n$. The inside-out Ehrhart polynomial of $(\mathcal{P}, \mathcal{H}[S])$ and Ψ satisfy*

$$E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}[S]}(k+1) + \Psi^{\circ}(k+1,l) = P(S; 2k+1, 2l) = P(S; x, y).$$

Proof. In $\frac{1}{k+1}\mathbb{Z}^n$ the points that are counted by $E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}[S]}(k+1)$ are those of $\left(\frac{1}{k+1}\{0, \pm1, \pm2, \ldots, \pm k\}\right)^n$ that do not lie in any of the graphical arrangement $\mathcal{H}[S]$. The number of such points is the number of proper k-colorings of S. The second term $\Psi^{\circ}(k+1, l)$ counts the difference of interior points of $\left(\frac{1}{k+l+1}\{0, \pm1, \pm2, \ldots, \pm(k+l)\}\right)^n$ and

$\left(\frac{1}{k+1}\{0, \pm1, \pm2, \ldots, \pm k\}\right)^n$, i.e., those colorings of S where we use at least one of the wildcard colors from the set $\mathcal{L}$. The sum of both terms gives us the two–variable chromatic polynomial for S (see example Figure 5.5 for $k = 6$ and $l = 4$). $\square$

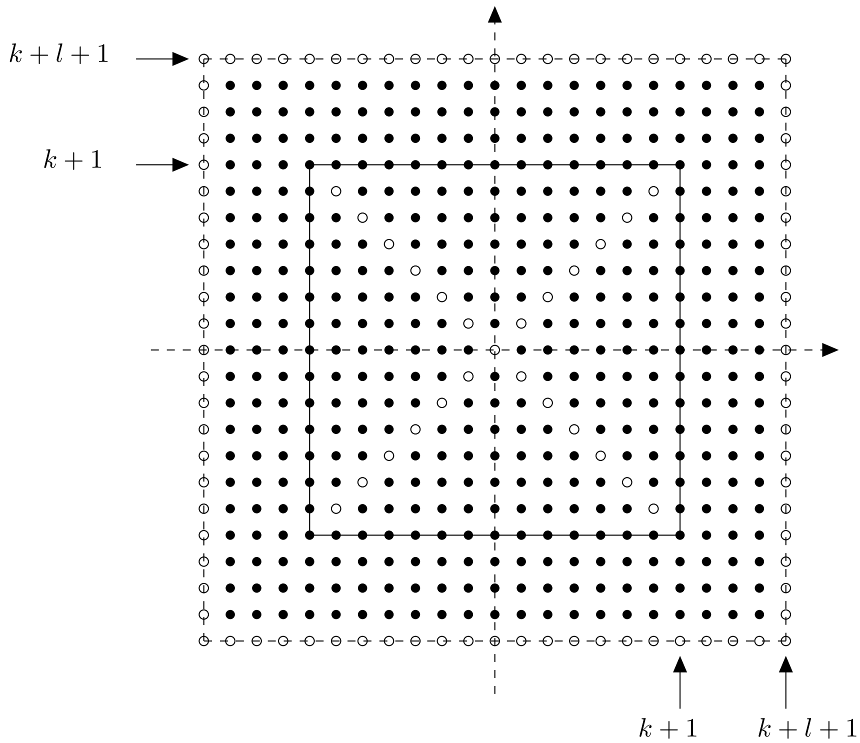

Figure 5.5: The lattice points corresponding to (k,l)-colorings of graph $\pm K_2$ with $k=6$ and $l=4$.

Theorem 5.10. *Let $S=(V,E,\sigma)$ be a signed graph with n vertices and let $\mathcal{P}=[-1,1]^n$. Then $(-1)^n\,P(S;-x,-y)$ counts the $(k+l+1)$-colorings of S with multiplicity: a $(k+1)$-coloring has a multiplicity equal to the number of compatible pairs (τ,φ) consisting of an acyclic orientation τ and a $(k+1)$-coloring of S; and a coloring of S using at least one color from the set $\{\pm(k+2),\pm(k+3),\ldots,\pm(k+l+1)\}$ has multiplicity of one.*

Proof. From Proposition 5.9,

$$P(S;-x,-y) = P(S;-2k-1,-2l) = E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}[S]}(-k-1) + \Psi^{\circ}(-k-1,-l).$$

Lemma 5.8 and Theorems 5.2 and 5.3 imply

$$(-1)^n\, P(S;-x,-y) = E_{\mathcal{P},\mathcal{H}[S]}(k+1) + \Psi(k+1,l).$$

$E_{\mathcal{P},\mathcal{H}[S]}(k+1)$ counts the number of compatible pairs (φ, R) where φ is a coloring with color set $\{0, \pm 1, \ldots, \pm(k+1)\}$, R is a closed region of $\mathcal{H}[S]$, and $\varphi \in R$. Regions R correspond with acyclic orientation τ in the following way: for a positive edge, the result is clear from Stanley's theorem [14], for a negative edge ij, we need to follow the rule (Equation (5.2)) for compatible pairs (τ, φ):

1. if $\varphi(i), \varphi(j) > 0$, (τ, φ) are compatible if both arrows point towards vertices i and j since we are adding $\tau(i,e)\varphi(i) + \tau(j,e)\varphi(j)$ and the sum must be ≥ 0;

2. if $\varphi(i), \varphi(j) < 0$, (τ, φ) are compatible if both arrows point away from vertices i and j for the same reason as previous;

3. if $|\varphi(i)| > |-\varphi(j)|$, then the arrows would point towards both vertices i, j;

4. if $|\varphi(i)| < |-\varphi(j)|$, then the arrows would point away from vertices i, j;

5. if $\varphi(i) = -\varphi(j)$ or $\varphi(i) = \varphi(j) = 0$, then there are two orientations, both arrows pointing away from the vertices i, j or towards them.

The orientation is acyclic because $R^\circ \neq \emptyset$. Thus φ is compatible with τ if and only if $\varphi \in R$. The second term $\Psi(k+1, l)$ equals

$$E_{\mathcal{P}}(k+l+1) - E_{\mathcal{P}}(k+1),$$

which counts the number of ways to color S using at least one of the colors $\{\pm(k+2), \pm(k+3), \ldots, \pm(k+l+1)\}$. $\square$

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