Integer-Point Transforms of Rational Polygons and Rademacher–Carlitz Polynomials

Bachelor-Thesis

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ABSTRACT. In this bachelor's thesis we give a complete description of integer-point transforms

$$\sigma(x,y) := \sum_{(j,k)\in\mathcal{P}\cap\mathbb{Z}^2} x^j y^k$$

of rational polygons $\mathcal{P}$. We show that the main ingredient of these integer-point transforms are Rademacher–Carlitz polynomials

$$R\left(x,y;\beta,\alpha,t;\frac{e}{f}\right) := \sum_{k=\left\lceil \frac{e}{f}\right\rceil}^{\left\lceil \frac{e}{f}\right\rceil+\alpha-1} x^k y^{\left\lfloor \frac{\beta}{\alpha}k+t\right\rfloor}.$$

We prove, using Barvinok's Theorem, that these polynomials can be computed as a sum of short rational functions. Furthermore, we derive a reciprocity statement for Rademacher–Carlitz polynomials, which implies the already known reciprocity statement for Carlitz polynomials

$$c(u,v;a,b) = \sum_{k=1}^{b-1} u^{\left\lfloor \frac{ka}{b}\right\rfloor} v^{k-1}.$$

As a corollary of the reciprocity law for Carlitz polynomials, we get the reciprocity statement for Dedekind sums

$$s(a,b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right).$$

In addition, we prove a new reciprocity theorem for Rademacher–Dedekind sums

$$r_n(a,b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka+n}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right).$$

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CHAPTER 1

Introduction

The goal of this bachelor's thesis is to derive integer-point transforms of rational polygons, i.e., polygons whose vertices only have rational coordinates. Before we start deriving the integer-point transforms of rational polygons, we introduce the topic with a background chapter. In this chapter, we start by defining *polytopes* and give some examples of them. While we do so, we introduce the important construct of a *cone* which turns out to be crucial for this thesis. Next, we define *integer-point transforms* and give some examples of how to compute them explicitly, for instance, by a tiling argument. An integer-point transform is a way of encoding the integer points lying in a polyhedron or, in our case, in a polygon. More explicitly, the integer-point transform $\sigma(x, y)$ of a polygon P is defined as

$$\sigma(x,y) := \sum_{(i,j)\in P\cap\mathbb{Z}^2} x^i y^j.$$

Finally, we mention *Brion's Theorem* which will play an important role in this thesis. Brion says that if we determine the integer-point transforms of the vertex cones and add them, we get the integer-point transform of the polytope. This will be our basic approach to reveal the integer-point transforms of rational polygons. In order to determine the integer-point transform of an arbitrary rational polygon, we triangulate the rational polygon into rational triangles. If we can compute the integer-point transform of a rational triangle, we can put all integer-point transforms of the rational triangles back together and get the desired integer-point transform of the rational polygon. However, we have to be careful with this operations, since we are counting the points on the shared line segments of the triangulation twice. This implies that we have to find the integer-point transform of a rational line segment as well.

Before we derive new results, we devote the third chapter to three mathematicians, namely Richard Dedekind, Hans Rademacher and Leonard Carlitz, whose work we will come across. Here we give definitions of Dedekind sums, Rademacher–Dedekind sums and Carlitz polynomials and present their reciprocity theorems.

In the forth chapter, we treat the case of a rational line segment and derive its integer-point transform. As a warm-up exercise, we start by computing the integer-point transform of a rational line segment going through the origin. Next, we derive the integer-point transform of an arbitrary rational line segment using Brion's Theorem. This representation will be computationally feasible.

In the fifth chapter, we start by reducing the problem to determining the integer-point transform of a rational *right* triangle using a geometric argument. This leads us to:

Theorem 1.1
Let a, b, c, d, e and $f \in \mathbb{N}$. Let Δ denote the triangle with vertices $(\frac{e}{f}, \frac{g}{h})$, $(\frac{a}{b}, \frac{g}{h})$ and $(\frac{e}{f}, \frac{c}{d})$. Moreover, we define $\alpha := dh(be - af)$ and $\beta := bf(ch - dg)$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be the map $f(x) := \frac{\beta}{\alpha}x + \frac{c}{d} - \frac{e\alpha}{f\beta}$ and $f^{-1}(x) = \frac{\alpha(x - \frac{c}{d} + \frac{e\alpha}{f\beta})}{\beta}$ its inverse

map. Then the integer-point transform of Δ *equals*

$$\sigma_\Delta(x,y) = \frac{x^{\lceil \frac{a}{b} \rceil} y^{\lceil \frac{g}{h} \rceil}}{(1-x)(1-y)} + \frac{\sum_{k=\lceil \frac{g}{h} \rceil}^{\beta+\lceil \frac{g}{h} \rceil -1} x^{\lfloor f^{-1}(k) \rfloor} y^k}{(1-x^{-1})(1-x^\alpha y^\beta)} + \frac{\sum_{k=\lceil \frac{e}{f} \rceil}^{\alpha+\lceil \frac{e}{f} \rceil -1} x^k y^{\lfloor f(k) \rfloor}}{(1-y^{-1})(1-x^{-\alpha} y^{-\beta})}.$$

The numerator of this integer-point transform looks similar to Carlitz polynomials. For the Carlitz polynomial there are several results known and they have a deep connection to Dedekind sums, see equation (3.1) below. Actually, the numerators of the integer-point transform are of a more general form than a Carlitz polynomial. They turn out to be important for this thesis and therefore deserve their own definition.

Definition 1.2
Suppose that α, $\beta \in \mathbb{Z}$, $e \in \mathbb{Q}$ *and* $t \in \mathbb{Q}$ *with* $\gcd(\alpha,\beta) = 1$. *We define the Rademacher–Carlitz polynomial* $R(x,y;\beta,\alpha,t;e)$, *or for* $h(x) = \frac{\beta}{\alpha}x + t$ *the shortform* $R(x,y;h;e)$, *as*

$$R(x,y;\beta,\alpha,t;e) = \sum_{k=\lceil e \rceil}^{\lceil e \rceil + \alpha - 1} x^k y^{\lfloor \frac{\beta}{\alpha}k + t \rfloor}.$$

If $e = 0$ *we are writing* $R(x,y;\alpha,\beta,t)$ *instead of* $R(x,y;\alpha,\beta,t;0)$.

Using Barvinok's Theorem 5.1 below, we show that these polynomials can be computed as a sum of short rational functions. This gives us our next theorem.

Theorem 1.3
The Rademacher–Carlitz polynomial $R\left(x,y;\alpha,\beta,t;\frac{g}{h}\right)$ *where* $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{N}$ *with* $\gcd(\alpha,\beta) = \gcd(g,h) = 1$, $t \in \mathbb{Q}$ *can be computed as a sum of short rational functions in* x *and* y *in polynomial time.*

It is known that there is a deep connection between Carlitz polynomials and Dedekind sums. More explicitly, we can derive the reciprocity theorem of Dedekind sums, Theorem 3.1 below, by applying the operators $x\partial x$ twice and $y\partial y$ once to the reciprocity law of Carlitz polynomials, Theorem 3.3. Therefore, our final goal is to find a similar connection between Rademacher–Carlitz polynomials and Rademacher–Dedekind sums, which explains the name of Rademacher–Carlitz polynomials. This connection is derived in Chapter 6.
Taking the reciprocity law of Carlitz polynomials as a motivation, we follow the strategy of [**1**] and derive a novel reciprocity law for Rademacher–Carlitz polynomials. This leads to a reciprocity theorem for Rademacher–Carlitz polynomials which is proved in Chapter 7.

Theorem 1.4
Let $(P,Q) \in \mathbb{Q}^2$ *and let* $f(x) := \frac{\alpha}{\beta}x + \frac{p}{q}$ *be a linear function going through* (P,Q), *where* $p \in \mathbb{N}$, $q \in \mathbb{Z}\backslash\{0\}$ *and* α, $\beta \in \mathbb{N}$ *and* $t \in \mathbb{Q}$. *Without loss of generality we can assume that* $\gcd(p,q) = \gcd(\alpha,\beta) = 1$. *The Rademacher–Carlitz polynomials satisfy the reciprocity law*

$$\begin{aligned} &y(1-x)R\left(x,y;f;P\right) + (1-y)xR\left(y,x;f^{-1};Q\right) + \\ &+ \chi(f)(1-x)(1-y)\left(x^{a+\beta k_{\min}} y^{b+\alpha k_{\min}}\right) \\ &= (1-x^\beta y^\alpha)x^{\lceil P \rceil} y^{\lceil Q \rceil}, \end{aligned}$$

where $\chi(f)$ *equals* 1 *if there are integer points on* f *and* 0 *otherwise, and where*

$C := -p\frac{\beta}{q}$,

$a := \left[C\alpha^{-1}\right]$, *where* $[x]_y$ *denotes the smallest nonnegative integer congruent to* $x \mod y$, $b := \frac{\alpha a - C}{\beta}$,

$k_{\min} = \left\lceil \frac{\lceil P \rceil - a}{\beta} \right\rceil$.

As a corollary, we derive the reciprocity law for Carlitz polynomials which gives us as a corollary of the corollary the reciprocity law for Dedekind sums.

CHAPTER 2

Background

The main purpose of this section is to gather information we need in order to understand the definitions, arguments and theorems that are used and developed in this thesis. Unless otherwise marked, all definitions, arguments and theorems we introduce are from [**2**].
The first part of this section will provide the geometric setup. We first begin by defining a polytope $P \subset \mathbb{R}^d$. For our needs it is fundamental to understand what a polytope is and how we can describe it. After defining it, we will give some examples and we will introduce the term of a rational polytope.

Definition 2.1
Given a finite point set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} \subset \mathbb{R}^d$*, the polytope* P *is defined as*

$$P = \left\{ \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \,:\, \textit{all } \lambda_k \geq 0 \textit{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

In other words, P *is the smallest convex set containing* $v_1, v_2, \dots, v_n$.

This definition is called the vertex description of a polytope P. As it turns out, there is another equivalent description of a polytope P. We can define a polytope as a bounded intersection of finitely many half-spaces and hyperplanes. A *hyperplane* is defined as

$$\{\mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{ax} = b\}$$

for fixed $\mathbf{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, and a *closed half-space* is defined as

$$\{\mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{ax} \leq b\}$$

or

$$\{\mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{ax} \geq b\},$$

again for fixed $\mathbf{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$. An *open half-space* is defined analogously, only the inequalities are turned into *strict* inequalities. The *dimension* of a polytope P is defined as the dimension of the affine space

$$\operatorname{span} P := \{\mathbf{x} + \lambda \mathbf{y} \,:\, \mathbf{x}, \mathbf{y} \in P, \lambda \in \mathbb{R}\}.$$

If P has dimension d, we denote P as a d-polytope.
A hyperplane $H = \{\mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{ax} = b\}$ is called a *supporting hyperplane* of P, if P lies entirely on one side of H, that is

$$P \subset \{\mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{ax} \leq b\}$$

or

$$P \subset \{\mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{ax} \geq b\}.$$

A *face* of P is a set of the form $H \cap P$, where H is a supporting hyperplane of P. This implies that P itself is a face of P, corresponding to the hyperplane $\mathbb{R}^d$ and the empty set is a face of every polytope corresponding to a hyperplane that does not meet P. Some faces have their own names, the most important of them are:

the $(d-1)$-dimensional faces are called *facets*,

the 1-dimensional faces are called *edges*,

the 0-dimensional faces are called *vertices*.

We call a point in $\mathbb{R}^d$ an *integer point* if all coordinates of the points are integral. Enough defined, let's give some examples.

A 0-polytope is a point.

A 1-polytope is a line segment.

A 2-polytope is also called a polygon. We know plenty of them, for example: a triangle, a square, a pentagon, a rectangle, a parallelogram, etc.

Here are some examples for a 3-polytope: cube, pyramid, tetrahedron, octahedron, dodecahedron, icosahedron, etc.

Next, we define an important special case of a polytope, namely a *simplex*.

Definition 2.2
A d-polytope P is called a simplex if it has $d+1$ vertices.

For this thesis, we need to define what a *rational polytope* is.

Definition 2.3
A rational polytope P is a polytope whose vertices all have rational coordinates, i.e., every entry of the coordinates is rational. If all coordinates only have integral entries, we call a P an integral polytope or lattice polytope.

A finite collection of hyperplanes $\mathcal{H}$ is called a *hyperplane arrangement*. If all its hyperplanes are of the form

$$\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}^T\mathbf{x} = b\}$$

for $a \in \mathbb{Z}^d$ and $b \in \mathbb{Z}$, we call this arrangement *rational*. If all its hyperplanes meet in at least one point, we call the hyperplane arrangement $\mathcal{H}$ a *central arrangement*. A *convex cone* is the intersection of finitely many half-spaces for which the corresponding hyperplanes form a central arrangement. If the hyperplanes meet in exactly one point, we call it a *pointed cone*. We call a cone *rational* if all its bounding hyperplanes are rational.
The *tangent cone* of a face $\mathcal{F}$ of a polytope P is defined by

$$\mathcal{K}_{\mathcal{F}} := \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x} \in \mathcal{F}, \mathbf{y} \in P, \lambda \in \mathbb{R}_{\geq 0}\}.$$

Cones and polytopes are special cases of a more general construct, namely the construct of a *polyhedron*. A polyhedron $P \subset \mathbb{R}^d$ is an intersection of finitely many half-spaces, i.e. P is a convex body. We call P rational if all of its vertices only have rational coordinates.

The next part introduces the combinatorial strategies and definitions we need. One of the most important constructions we need is that of a *generating function*.

Definition 2.4
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Then its generating function is defined as

$$F(z) = \sum_{k\geq 0} a_k z^k.$$

Generating functions turn out to be very useful. For example, we can use them to develop a formula for the k^{th} term of the Fibonacci sequence. For any further information about how to deduce a formula out of the recursive definition of the Fibonacci sequence using generating functions we refer to [**2**].
This thesis mostly treats questions about what integer points lie in a polygon or a line segment. One way to encode integer points is the integer-point transform $\sigma(x)$. This integer-point transform converts the integer points into a multivariate Laurent polynomial, so we can use analytic and algebraic methods.

Definition 2.5
The integer-point transform $\sigma(\boldsymbol{x})$ of a rational d-polyhedron P is defined as

$$\sigma(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d\cap P} \boldsymbol{x}^{\boldsymbol{k}} := \sum_{\boldsymbol{k}\in\mathbb{Z}^d\cap P} x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}.$$

In dimension 2 we write $\sigma(x, y)$ instead of $\sigma(\boldsymbol{x}) = \sigma(x_1, x_2)$.

After a moment's thought, it becomes clear that if we evaluate the integer-point transform of a polytope at $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$, we get the number of integer points lying in the polytope. Let us show an example in which the integer-point transform is a smart way to encode all information we need.

Example 2.6
Let $[0, \infty)$ be our cone. We already know what integers in this cone are, namely the natural numbers and 0. If we encode this information in the integer-point transform, we get

$$x^0 + x + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

This is only true if the series converges. Since the conditions we need for this series to converge are simple and well known ($|x| < 1$), we don't explicitly take care about this. This rational function contains all information we need and is in a compact form. Furthermore, it is effective to compute.

Let us give another example which is more relevant since it introduces an important strategy. (The strategy is similiar to the one describe in [**2**], but the example is new!) We want to determine the integer-point transform $\sigma(x, y)$ of a cone. More explicitly, we want to compute this transform for the cone

$$\mathcal{K} := \{\lambda_1(2,1) + \lambda_2(1,2) : \lambda_1, \lambda_2 \geq 0\}.$$

The vectors $(2, 1)$ and $(1, 2)$ are called the *generators* of $\mathcal{K}$. The basic technique we need is to tile the cone by its *fundamental parallelogram* which has the form

$$\Pi = \{\lambda_1(2,1) + \lambda_2(1,2) : 0 \leq \lambda_1, \lambda_2 < 1\}.$$

Taking linear combinations of these generators leads to a tiling of the whole cone. Therefore, it is crucial to choose our fundamental parallelogram to be half open, since otherwise the tiling would not be disjoint. Translated into mathematical language these linear combinations can be rewritten as

$$\sum_{j\geq 0,\ k\geq 0} (x, y)^{j(2,1)+k(1,2)} = \frac{1}{(1 - x^2y^1)(1 - x^1y^2)}.$$

The geometric interpretation is that our cone is now tiled by the fundamental parallelogram. If we next take an integer point (m, n) out of Π and add a linear combination of the generators we get a subset of $\mathbb{Z}^2$ which has the form

$$\{(m, n) + j(2, 1) + k(1, 2) : j, k \in \mathbb{Z}_{\geq 0}\}.$$

The cone $\mathcal{K}$ can be described as the disjoint union of all such subsets, if (m, n) ranges over all integer points in Π. Since Π contains the integer points $(0, 0)$, $(1, 1)$ the integer-point transform takes on the form

$$\begin{aligned}\sigma_{\mathcal{K}}(x, y) &= (1 + xy) \sum_{\mathbf{m}=j(2,1)+k(1,2)\ j,\ k\geq 0} (x, y)^{\mathbf{m}} \\ &= \frac{1 + xy}{(1 - x^2y^1)(1 - x^1y^2)}.\end{aligned}$$

The next theorem is very important for this thesis. We are introducing *Brion's theorem* which tells us that the integer-point transform of a rational polytope can be written as the sum of pointed cones in which the apices are the vertices of the polytope. This theorem opens the door to new strategies of determining integer-point transforms of rational polytopes. In this thesis we are using Brion's theorem in dimension 2 as a starting point in order to determine the integer-point transform of both an arbitrary rational line segment and a rational triangle.

Theorem 2.7
Let P be a rational polytope. Then its integer-point transform $\sigma_P(\boldsymbol{z})$ can be written as the sum of its pointed vertex cones $\sigma_{\mathcal{K}_v}(\boldsymbol{z})$:

$$\sigma_P(\boldsymbol{z}) = \sum_{\boldsymbol{v} \text{ vertex of } P} \sigma_{\mathcal{K}_v}(\boldsymbol{z})$$

For a proof of this lovely theorem we refer the reader to [**2**].

CHAPTER 3

Dedekind, Rademacher and Carlitz

This chapter gives a short overview of work by *Richard Dedekind*, *Hans Adolph Rademacher* and *Leonard Carlitz*, since we will meet some of their theorems in the later chapters.
Richard Dedekind was born on October 6^{th} 1831 in Braunschweig, Germany, and died on February 12^{th} 1916 [**6**]. At the age of seven he went to a school in Braunschweig. His main interest was not mathematics, but he instead focused on sciences as physics and chemistry. However, he was bothered by gaps in the logical arguments of the phyisicists. From 1850 on he studied at the University of Göttingen. At this time, Göttingen wasn't the best place to study mathematics, even though they had Gauß who was mostly teaching fundemantal classes. Wilhelm Weber taught the first course that Dedekind was really interested in, in fact, an experimental physics class. In the end of 1850 he was attending his first class taught by Gauß. Gauß became the supervisor of Dedekind's doctoral work and he wrote a PhD thesis on the theory of Eulerian integrals. He was Gauß' last student. In 1852 he received his doctoral degree but he was not satisfied with his own mathematical abilities. Since the University of Göttingen was not the best place to study the latest developments and results of research, Dedekind spent a lot of time studying modern mathematics. Furthermore, he was writing his habilitation thesis which he finished in 1854. After Gauß died in 1855, Dirichlet was offered Gauß' job at the University of Göttingen. For Dedekind this was a lucky event, since he enjoyed working with Dirichlet. In addition they became close friends and Dedekind attended many classes held by Dirichlet, for example, classes on number theory and partial differential equations. In 1858 Dedekind moved to Zürich and got a job at the Polytechnic Zürich. He moved back to Braunschweig in 1862 and retired in 1894. After his retirement, he didn't meet many people and lived somehow isolated from others. Unfortunately, the Jahresbericht of the German Mathematical Society thought he was dead and published an article about the day, month and year of his death. Dedekind was suprised of this publication and wrote a letter: "In your communication, page so and so, of the Jahresbericht, concerning the date, at least the year is wrong."[**5**, p. 172].
Let us now take a closer look at his mathematical discoveries. There are many discoveries named after him, for instance, the *Dedekind η-function* or the *Dedekind sums*. The Dedekind sums first appeared when Dedekind studied the functional equation of the Dedekind η-function [**1**]. Dedekind sums turned out to be very important, not just for this thesis. For any two relatively coprime integers a and $b > 0$ we define

$$s(a,b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right) \,, \tag{3.1}$$

where

$$((x)) := \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

and where $\{x\}$ again denotes the fractional-part function. This sum is called a *Dedekind sum*. To illustrate how important these sums are: Currently, there are 2145 matches on MathSciNet for *"Dedekind AND sum*"* (date:05/01/2013). The Dedekind sums satisfy a reciprocity statement that implies their computational feasibility.

Theorem 3.1
Let $a, b \in \mathbb{N}$ with $\gcd(a,b) = 1$. Then

$$s(a,b) + s(b,a) = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) - \frac{1}{4}.$$

This theorem was originally stated and proved by Richard Dedekind, see [**4**]. The fractional-part function is a periodic function and as a consequence the Dedekind sum is periodic as well. More specifically, the period is the denominator b of $\frac{ak}{b}$. If b is bigger than a, we can choose a to be $1 \leq a < b$. Since we can determine $s(a,b)$ from $s(b,a)$ and vice versa, we can use this periodicity to compute it efficiently (similar to the Euclidean Algorithm or the computation of the Legendre-symbol). We refer the interested reader to [**2**] for a proof of this theorem.
These sums, or better said a slight generalization, appear while computing the Ehrhart polynomials of polyhedra in dimension 2. These generalizations are called *Rademacher–Dedekind* or *Dedekind–Rademacher sums*. They were defined by Hans Adolph Rademacher who was a German mathematician. Rademacher was born in Wandsbeck, Germany, in 1892 and he died in Pennsylvania in 1969 [**8**]. Rademacher first wanted to study philosophy as a main subject, but after having attended classes of Hecke and Weyl he was persuaded to study mathematics with Courant. The first world war started and Rademacher had to serve in the German army. However, he continued doing his research. He received his PhD in 1916 while working on the field of real functions and he completed his habilitation in 1919. His mathematical focus changed when he started working as an extraordinary Professor at the University of Hamburg in 1922 where he worked with Hecke. He then went to Breslau where he became an ordinary Professor. Since Rademacher had a strong political and ethical opinion, the Nazis forced him to give up his chair in Breslau. His pacifistic attitude led him to leave the country: he emigrated to the United States and worked at the University of Pennsylvania.
Even though Hans Rademacher and his wife owned a car, they both didn't know how to drive. Sometimes a friend drove him or his wife. When Ivan Niven offered to teach him how to drive, he declined. However, he let Niven teach his wife how to drive, since he needed a lot of uninterrupted time to do mathematics [**5**, p. 24].
Rademacher's main focus was on number theory in general and on analytic number theory, modular forms, and applications to combinatorics in particular. He defined the Rademacher–Dedekind sums and generalized Theorem 3.1. Let $a, b \in \mathbb{N}$ with $\gcd(a,b) = 1$. Then the *Rademacher–Dedekind sum* is defined as

$$r_n(a,b) := \sum_{k=0}^{b-1} \left(\left(\frac{ka+n}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right) .$$

For $n = 0$ we have $r_0(a,b) = s(a,b)$. These Rademacher–Dedekind sums are important in many branches, the most important for us is their appearance as building blocks of Ehrhart polynomials [**1**] and their appearance later in this thesis. Furthermore, Rademacher–Dedekind sums satisfy a similar reciprocity statement as the Dedekind sums. Again, this ensures the computational feasibitly of the Rademacher–Dedekind sum. The next result was originally published in [**9**].

Theorem 3.2
For a and $b \in \mathbb{N}$ with $\gcd(a,b) = 1$ and for $n = 1, 2, \ldots, a+b$,

$$\begin{aligned} r_n(a,b) + r_n(b,a) &= \frac{n^2}{2ab} - \frac{n}{2}\left(\frac{1}{ab} + \frac{1}{a} + \frac{1}{b}\right) + \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a}\right) \\ &= \frac{1}{2}\left(\left(\left(\frac{a^{-1}n}{b}\right)\right) + \left(\left(\frac{b^{-1}n}{a}\right)\right) + \left(\left(\frac{n}{a}\right)\right) + \left(\left(\frac{n}{b}\right)\right)\right) \\ &= \frac{1}{4}\left(1 + \chi_a(n) + \chi_n(b)\right)\ , \end{aligned}$$

where a^{-1} and b^{-1} denote the inverse of $a \mod b$ and $b \mod a$, respectively. Moreover, we define

$$\chi_a(n) := \begin{cases} 1 & \text{if } a \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

For a proof see [**2**] or [**9**]. Again this reciprocity theorem implies the computational feasibility of the Rademacher–Dedekind sums. Rademacher–Dedekind sums appear while determining the Ehrhart polynomials of rational polygons, so it is likely that they are connected to the integer-point transform of rational polygons as well.

The next important mathematician we like to introduce is *Leonard Carlitz* [**7**].

He was born in 1907 in Philadelphia and died in 1999 in Pittsburgh. Early in his life his outstanding mathematical talent was revealed and he therefore gained a scholarship at the University of Pennsylvania. In 1930 he received his doctorate under Howard Mitchell. He then went to Caltech where he worked with Eric Temple Bell and continued his way to Cambridge. In Cambridge he collaborated with the famous Godfrey Harold Hardy (1931/32) at the University of Cambridge. From 1932 to 1977 he worked as a professor at Duke University. In his working life he mainly published paper in number theory, Galois theory, combinatorics, and some other fields. For us the most important definition is that of a *Carlitz polynomial.* For a, $b \in \mathbb{N}$ and two indeterminates u and v, the Carlitz polynomial is defined as

$$c(u,v;a,b) := \sum_{k=1}^{b-1} u^{\left\lfloor \frac{ka}{b} \right\rfloor} v^{k-1},$$

where we used the definition of [**1**]. This polynomial shows up while computing the integer-point transform of a polyhedral cone. For further details, we refer the reader to [**1**]. A fascinating fact of Carlitz polynomials is that they satisfy a reciprocity statement, which was discovered and proved by Carlitz [**3**].

Theorem 3.3
The Carlitz polynomials $c(u,v;a,b)$ and $c(v,u;b,a)$ satisfy the reciprocity

$$(v-1)c(u,v;a,b) + (u-1)c(v,u;b,a) = u^{a-1}v^{b-1} - 1.$$

This law can be proved by dividing the first quadrant into two polyhedral cones by a linear function starting in the origin and determining the integer-point transform of the first quadrant in two different ways. First, the integer-point transform can be computed straightforward. Second, the integer-point transform can be determined by adding the integer-point transforms of the polyhedral cones. Again, we refer the reader to [**1**]. This reciprocity law has a connection to the reciprocity law of Dedekind sums. More explicitly, the reciprocity statement of Carlitz polynomials implies the reciprocity law of Dedekind sums. This implication can be seen by applying the operators $x\partial x$ twice and $y\partial y$ once to Theorem 3.3 and setting $x = y = 1$

[**1**].
Furthermore, these polynomials are very important to many branches of mathematics. Currently (5/2/2013), there are 1203 matches on MathSciNet ("Carlitz AND polynomial*"). A closer look reveals that these polynomials are not only important in discrete mathematics, geometry and number theory, they are important in applied mathematics as well.

CHAPTER 4

Integer-point transforms of rational line segments

1. The integer-point transforms of rational line segments going through the origin

In this section we want to determine the integer points on a line segment embedded in the plane. One endpoint of the line segment is assumed to be the origin whereas the other endpoint can be an arbitrary rational point. Due to symmetry we can assume this point to be in the first quadrant excluding endpoints on the y-axis. Let us now fix some notation before we start. In the follow we denote the origin by $\mathbf{0}$ and the other endpoint by $(\frac{a}{b}, \frac{c}{d})$ where a, b, c, $d \in \mathbb{Z}$ and $\gcd(a,b) = \gcd(c,d) = 1$. In order to count the integer points on the line segment, we interpret the line segment as part of the linear graph going through $\mathbf{0}$ and $(\frac{a}{b}, \frac{c}{d})$. This linear graph can be described by the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{bc}{ad}x$.
To count the number of integer points on the line segment, we first derive some conditions for integer points on f.

$$y = f(x) = \frac{bc}{ad}x \overset{y \text{ integer}}{\Longleftrightarrow} \frac{x}{y} = \frac{ad}{bc}$$

This last equality contains all information we need, since it gives us all integer points on the line segment. First we reduce the fraction on the right-hand side as far as possible, e.g., we divide by all common divisors. As a result we get the fraction $\frac{ad}{\gcd(ad,bc)} / \frac{bc}{\gcd(ad,bc)}$. After $\mathbf{0}$ the numerator as x-coordinate and the denominator as the y-coordinate form the next integer point on our line segment, if and only if $\frac{ad}{\gcd(ad,bc)} \leq \lfloor \frac{a}{b} \rfloor$.
In a natural way we get all integer points of the line segment by expanding the fraction as long as the numerator is less or equal $\lfloor \frac{a}{b} \rfloor$. Thus we have a total of

$$\left\lfloor \frac{\lfloor \frac{a}{b} \rfloor \cdot \gcd(ad, bc)}{ad} \right\rfloor + 1$$

integer points on the line segment. Note that if $\frac{ad}{\gcd(ad,bc)} > \lfloor \frac{a}{b} \rfloor$ (which means that there are no integer points on the line segment except for $\mathbf{0}$) our formula gives us the correct value 1.
What does this formula mean for some special cases?

b, $d = 1 \Longleftrightarrow$ the endpoint is an integer:
Our formula reduces to $\lfloor \frac{a \gcd(a,c)}{a} \rfloor + 1 = \gcd(a,c) + 1$.

$c = 0$ or, in other words, the point lies on the x-axis. If we go back to our approach, we realize that it was not valid for $c = 0$. Ignoring this fact for a second, we plug in $c = 0$. Using $\gcd(ad, 0) = ad$, we get $\lfloor \frac{a}{b} \rfloor + 1$. As we know, this is the correct formula! In other words, we have proved that the formula is valid even if the rational endpoint lies on the x-axis.

The goal of this section was to derive an integer-point transform of a line segment going through the origin. Above we have collected enough data to determine $\sigma(x,y)$. We have found an explicit way to reveal all integer points on the line segment and we know how many integer points the line segment contains. In order to get a compact result, we introduce the abbreviation $n_{\max} := \lfloor \frac{\lfloor \frac{a}{b} \rfloor \cdot \gcd(ad,bc)}{ad} \rfloor$. We have now deduced our first lemma.

Lemma 4.1
Let $a,\ b,\ d \in \mathbb{N}$, $c \in \mathbb{Z}_{\geq 0}$ and $\gcd(a,b) = \gcd(c,d) = 1$. Then the integer-point transform $\sigma(x,y)$ of a line segment starting at the origin $\mathbf{0}$ and ending at $(\frac{a}{b}, \frac{c}{d})$ equals

$$\sigma(x,y) = \sum_{k=0}^{n_{\max}} x^{k\frac{ad}{\gcd(ad,bc)}} \cdot y^{k\frac{bc}{\gcd(ad,bc)}} = \frac{1 - \left(x^{\frac{ad}{\gcd(ad,bc)}} \cdot y^{\frac{bc}{\gcd(ad,bc)}}\right)^{n_{\max}+1}}{1 - \left(x^{\frac{ad}{\gcd(ad,bc)}} \cdot y^{\frac{bc}{\gcd(ad,bc)}}\right)}.$$

This expression is computationally feasible.

2. The integer-point transform of an arbitrary rational line segment

The goal of this section is to find a closed and computationally feasible expression for the integer-point transform $\sigma(x,y)$ of such a line segment. Without loss of generality, we can assume the line segment to lie in the first quadrant. This is due to the fact that an integer translation does not change the integer count. More explicitly, we can translate the line segment in such a way that its affine span meets the x-axis in the interval $(0,1)$. If we now want to count the integer-points on the line segment, we can count the integer points in the two line segments that are connecting the x-axis and the endpoints.

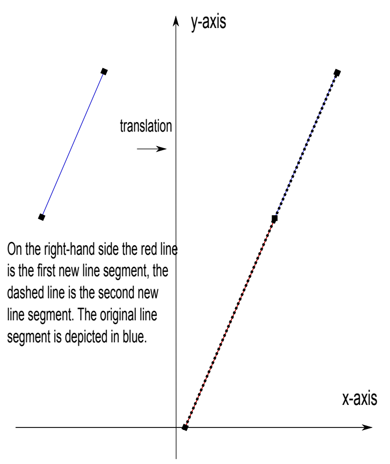

FIGURE 1. Translating a line segment into first quadrant and extending it till it cuts the x-axis in the interval $[0,1)$

Our problem has been reduced to the question how to count integer points on a line segment with rational endpoints in the first quadrant that cuts the x-axis in

the interval $(0,1)$. Let us denote the endpoints of this line segment by $\left(\frac{p}{q},0\right)$ and $\left(\frac{a}{b},\frac{c}{d}\right)$, where we can assume that $\gcd(a,b)=\gcd(c,d)=\gcd(p,q)=1$.
For our next attempt, we go to the underlying algebraic equations and use Brion's Theorem to determine the integer-point transform $\sigma(x,y)$ of the line segment starting in $(\frac{p}{q},0)$ and ending in $(\frac{a}{b},\frac{c}{d})$. In order to use Brion, we need to compute the integer-point transforms of the vertex cones

$$V_1 := \left\{(m,n)\in\mathbb{Z}^2_{\geq 0} : bcqm-(adq-bdp)n = bcp\right\}$$

and

$$V_2 := \left\{(m,n)\in\mathbb{Z}^2 : bcqm-(adq-bdp)n = bcp \text{ and } m\leq \left\lfloor\frac{a}{b}\right\rfloor\right\}$$

of the line segment.
Brion's Theorem tells us that if we compute the integer-point transforms of the vertex cones and add them, we get the integer-point transform for the line segment. For the equation

$$bcqx-(adq-bdp)y = bcp \tag{4.1}$$

this implies:

> $\sigma_1(x,y)=\sum_{(m,n)\in V_1} x^m y^n$ encodes (as exponents of x and y) all integer solutions to (4.1) whose coordinates are both positive;
>
> $\sigma_2(x,y)=\sum_{(m,n)\in V_2} x^m y^n$ encodes all integer solutions to (4.1) that have a negative coordinate or that have a positive x-coordinate $\leq \left\lfloor\frac{a}{b}\right\rfloor$.

Let us next determine the solutions of the linear equation. We do so in general for an equation of the form

$$Ax-By=C, \tag{4.2}$$

where A, B and $C\in\mathbb{N}$.
From Bezout's Lemma we know that this equation is solvable if and only if $\gcd(A,B)\mid C$. We now assume that this condition holds. Without loss of generality we can furthermore assume that $\gcd(A,B)=1$. For our approach, we will need one more fact about linear equations, namely that two solutions differ only in a solution of the homogeneous equation $Ax-By=0$ which we already know how to solve from the previous section. So let's play a bit with (4.2):

$$Ax-By=C \iff x=\frac{C-By}{A} \overset{x \text{ integer}}{\Longrightarrow} x\equiv C\cdot A^{-1} \mod B$$

Note that A^{-1} denotes the inverse of $A \mod B$ (which exists since $\gcd(A,B)=1$). This can be (computationally feasibly) calculated using the extended Euclidean Algorithm. To simplify notation we introduce $\alpha\in\mathbb{N}$ as $\alpha := CA^{-1}$ and $\beta\in\mathbb{N}$ as $\beta := \frac{A\alpha-C}{B}$ (since $y=\frac{Ax-C}{B}$). Then (α,β) is a solution of the inhomogeneous equation (4.2). To get another integer solution to (4.2), we can add an arbitrary integer solution of the homogeneous equation. From the previous section we know that we can write this arbitrary integer solution of the homogeneous equation as k times $(\frac{B}{\gcd(A,B)},\frac{A}{\gcd(A,B)})$. To simplify notation we define $\gamma := \frac{B}{\gcd(A,B)}$ and $\delta := \frac{A}{\gcd(A,B)}$. In order to get all positive integer solutions to $(*)$ (and only those solutions), we need to determine a minimal k called $k_{\min}$ such that $(\alpha+k\gamma,\beta+k\delta)\in\mathbb{Z}^2_{\geq 0}$ for all $k\geq k_{\min}$. We want to have $\alpha+k\gamma\geq 0$, so we have $k\geq\frac{-\alpha}{\gamma}$. Since $k_{\min}$ is an integer, we get $k_{\min}=\left\lceil -CA^{-1}\frac{\gcd(A,B)}{B}\right\rceil$.

Then the integer-point transform $\sigma_1(x,y)$ takes on the form

$$\begin{aligned}\sigma_1(x,y) &= \sum_{k=k_{\min}}^{\infty} x^{\alpha+k\gamma}y^{\beta+k\delta}\\ &= x^{\alpha}y^{\beta}\sum_{k=0}^{\infty} x^{\gamma(k+k_{\min})}y^{\delta(k+k_{\min})}\\ &= \frac{x^{\alpha+\gamma k_{\min}}y^{\beta+\delta k_{\min}}}{1-x^{\gamma}y^{\delta}}.\end{aligned}$$

Next we need to determine the second integer-point transform $\sigma_2(x,y)$, which counts all integer points with x-coordinate $\le \lfloor \frac{a}{b} \rfloor$. We want to find a $k_{\max}$ such that $\alpha + k\gamma \le \lfloor \frac{a}{b} \rfloor$ which is equivalent to $k \le \frac{\lfloor \frac{a}{b} \rfloor - \alpha}{\gamma}$. Solving for $k_{\max}$ leads us to $k_{\max} = \left\lfloor \frac{\lfloor \frac{a}{b} \rfloor - \alpha}{\gamma} \right\rfloor$.
We are now able to determine $\sigma_2(x,y)$ using the geometric series and shifting the indices:

$$\begin{aligned}\sigma_2(x,y) &= \sum_{k=-\infty}^{k_{\max}} x^{\alpha+k\gamma}y^{\beta+k\delta}\\ &= x^{\alpha}y^{\beta}\sum_{k=-\infty}^{k_{\max}} (x^{-\gamma})^{-k}(y^{-\delta})^{-k}\\ &= x^{\alpha}y^{\beta}\sum_{k=-k_{\max}}^{\infty} (x^{-\gamma})^{k}(y^{-\delta})^{k}\\ &= x^{\alpha}y^{\beta}\sum_{k=0}^{\infty}(x^{-\gamma(k-k_{\max})}y^{-\delta(k-k_{\max})})\\ &= x^{\alpha+k_{\max}\gamma}y^{\beta+k_{\max}\delta}\sum_{k=0}^{\infty}(x^{-\gamma k}y^{-\delta k})\\ &= \frac{x^{\alpha+k_{\max}\gamma}y^{\beta+k_{\max}\delta}}{1-x^{-\gamma}y^{-\delta}}.\end{aligned}$$

The underlying linear equation that led to this expression contained the coefficients A, B and C. Now we have to adapt the results to the case of our line segment that led to the linear equation

$$bcqx - (adq - bdp)y = bcp.$$

As above, let us assume that $\gcd(bcq, adq-bdp) \mid bcp$. Next, we divide this equation by $\gcd(bcq, adq-bdp)$ in order to make bcq and $adq-bdp$ co-prime. Our new parameters now look like

$A := \frac{bcq}{\gcd(bcq,adq-bdp)}$

$B := \frac{adp-bdq}{\gcd(bcq,adq-bdp)}$

$C := \frac{bcp}{\gcd(bcq,adq-bdp)}$

Let us next take care of the condition $\gcd(bcq, adq-bdp) \mid bcp$: We introduce a trick factor, which takes on the value 1 in case the condition holds and 0 otherwise, and place it in front of the sum. This factor can for instance look like

$$1-\left\lceil\left\{\frac{bcp}{\gcd(bcq,adq)}\right\}\right\rceil ,$$

where $\{x\}$ again denotes the fractional part function.
Finally, we have all ingredients we need to apply Brion. Adding the two integer-point transforms $\sigma_1(x,y)$ and $\sigma_2(x,y)$ gives us

$$\sigma(x,y)=\left(1-\left\lceil\left\{\frac{bcp}{\gcd(bcq,adq)}\right\}\right\rceil\right)\left(\frac{x^{\alpha+k_{\max}\gamma}y^{\beta+k_{\max}\delta}}{1-x^{-\gamma}y^{-\delta}}+\frac{x^{\alpha+\gamma k_{\min}}y^{\beta+\delta k_{\min}}}{1-x^{\gamma}y^{\delta}}\right).$$

Again, we have found a formula for the integer-point transform, but is this formula computationally feasible? To answer this question, we examine each parameter and the operations used.

> α: C is easy to compute, since it is just multiplication and the computation of the $\gcd(bcq, adq-bdp)$. Both operations can be effectively computed. Moreover, we need to determine the inverse of $A \mod B$. This can be done via the extended Euclidean Algorithm, which is quickly computed.
>
> β, γ, and δ: All of them are fast to compute, since they need at most one multiplication, one subtraction, one division and the computation of the gcd.
>
> $k_{\min}$, $k_{\max}$ and the trick factor: Rounding up and down and taking the fractional-part function is fastly done, furthermore the other computations of the parameters do not require us to do a lot of multiplication, subtraction, division and determining the gcd. As a consequence, they are computationally feasible as well.
>
> $\sigma(x,y)$: All parameters used are effectively computed and since the computation of σ consists of a few easy-to-do computations like addition, multiplication, subtraction, division, powers, etc., the integer-point transform is computationally feasible.

Summing up what we have shown so far gives us:

Theorem 4.2
Let a, c, $p \in \mathbb{Z}_{\geq 0}$ and b, d, $q \in \mathbb{N}$, where without loss of generality $\gcd(a,b) = \gcd(c,d) = \gcd(p,q) = 1$. *Then the integer-point transform $\sigma(x,y)$ of the line segment with endpoints $(\frac{p}{q},0)$, $(\frac{a}{b},\frac{c}{d})$ equals*

$$\sigma(x,y)=\begin{cases}\left(\frac{x^{\alpha+k_{\max}\gamma}y^{\beta+k_{\max}\delta}}{1-x^{-\gamma}y^{-\delta}}+\frac{x^{\alpha+\gamma k_{\min}}y^{\beta+\delta k_{\min}}}{1-x^{\gamma}y^{\delta}}\right) & \textit{if } \frac{bcp}{\gcd(bcq,adq)}\in\mathbb{Z},\\ 0 & \textit{otherwise,}\end{cases}$$

where

> $A := \frac{bcq}{\gcd(bcq,adq-bdp)}$, $B := \frac{adp-bdq}{\gcd(bcq,adq-bdp)}$ *and* $C := \frac{bcp}{\gcd(bcq,adq-bdp)}$;
>
> $\alpha := CA^{-1} \in \mathbb{N}$ *(here A^{-1} denotes the inverse of $A \mod B$),* $\beta := \frac{A\alpha - C}{B} \in \mathbb{N}$, $\gamma := \frac{B}{\gcd(A,B)}$ *and* $\delta := \frac{A}{\gcd(A,B)}$;
>
> $k_{\min} := \left\lceil -CA^{-1}\frac{\gcd(A,B)}{B}\right\rceil$ *and* $k_{\max} := \left\lfloor \frac{\lfloor \frac{a}{b}\rfloor-\alpha}{\gamma}\right\rfloor$.

This expression is computationally feasible.

CHAPTER 5

The integer-point transform of an arbitrary rational triangle

In the previous sections, we examined arbitrary line segments with rational endpoints. As a result, we got the integer-point transforms $\sigma(x, y)$ and it turned out that there is an explicit formula for them that is computationally feasible. The main reason for this endeavor comes from the triangulation of a rational polygon. If we want to find the integer-point transform of an arbitrary rational polygon P, we can triangulate it and determine the integer-point transforms of the triangles. However, we have to be careful with this operation, since we will count points on the shared line segments twice. So the explicit formula of the integer-point transform $\sigma(x, y)$ of a line segment in combination with the integer-point transform of a rational triangle enables us to do inclusion-exclusion.
Next we need to examine the integer-point transforms of triangles. We can embed an arbitrary triangle in a rectangle in such a way that we only need rectangles and right triangles to fill the blanks. The coordinates of the endpoints of the new rectangles and right triangles are still rational.

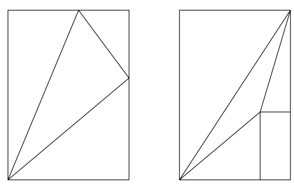

FIGURE 2. Rational triangles embedded in a rational rectangulars and rational right triangles

Moreover, from the same reasoning we can deduce that two sides of the triangle are parallel to the x- and y-axis (see (5)). We can translate the right triangles into a suitable quadrant, i.e., a quadrant where the right angle is at the vertex that is closest to the origin. Without loss of generality, we assume that this is the first quadrant $\mathbb{R}^2_{\geq 0}$.
Since we know how the integer-point transform of a rectangle looks, we need to find out what the integer-point transform of a right triangle with rational vertices looks like in order to determine the integer-point transform of an arbitrary rational

polygon.

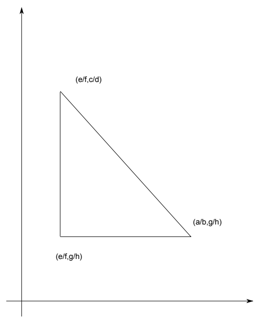

FIGURE 3. The shifted rational right triangle placed in the first quadrant

Taking this train of thought as a motivation and preparation, we can begin to examine the integer-point transform of a right triangle Δ with vertices $v_1 = (\frac{e}{f}, \frac{g}{h})$, $v_2 := (\frac{a}{b}, \frac{g}{h})$ and $v_3 := (\frac{e}{f}, \frac{c}{d})$. As a starting point, we again use Brion's Theorem (2.7) and compute the integer-point transforms of the vertex cones of Δ:

$$V_1 = \left\{\lambda_1(1,0) + \lambda_2(0,1) + (\tfrac{e}{f}, \tfrac{g}{h}) \,:\, \lambda_1,\, \lambda_2 \geq 0\right\}$$
$$V_2 = \left\{\lambda_1(-1,0) + \lambda_2(dh(be-af), bf(ch-dg)) + (\tfrac{a}{b}, \tfrac{g}{h}) \,:\, \lambda_1,\, \lambda_2 \geq 0\right\}$$
$$V_3 = \left\{\lambda_1(0,-1) + \lambda_2(-dh(be-af), -bf(ch-dg)) + (\tfrac{e}{f}, \tfrac{c}{d}) \,:\, \lambda_1,\, \lambda_2 \geq 0\right\}$$

To shorten notation, we define $\alpha := dh(be-af)$ and $\beta := bf(ch-dg)$.
To determine the integer-point transform of the first vertex cone V_1, we observe that the cone is just a shifted quadrant. So we can list all integer points in the integer-point transform

$$\sigma_{V_1}(x,y) = \sum_{k \geq \lceil \frac{e}{f} \rceil,\ j \geq \lceil \frac{g}{h} \rceil} x^k y^j = \frac{x^{\lceil \frac{e}{f} \rceil} y^{\lceil \frac{g}{h} \rceil}}{(1-x)(1-y)}$$

The other two vertex cones are not as easy to compute, but by tiling the cones with fundamental parallelepipeds of height 1, we can determine their integer-point transforms. Since we want the fundamental parallelepipeds to be disjoint, we choose them to be half-open. Here is our definition:

$$\Pi_2 := \left\{\lambda_1(-1,0) + \lambda_2(\alpha, \beta) + (\tfrac{a}{b}, \tfrac{g}{h}) \,:\, 0 \leq \lambda_1,\, \lambda_2 < 1\right\}$$
$$\Pi_3 := \left\{\lambda_1(0,-1) + \lambda_2(-\alpha, -\beta) + (\tfrac{e}{f}, \tfrac{c}{d}) \,:\, 0 \leq \lambda_1,\, \lambda_2 < 1\right\}$$

Which integer points lie in Π_2? To answer this question, we determine the linear graph that runs through $v_2 = (\frac{a}{b}, \frac{g}{h})$ and $v_3 = (\frac{e}{f}, \frac{c}{d})$. This graph can be described by the function $f(x) = \frac{\beta}{\alpha}x + \frac{c}{d} - \frac{e\alpha}{f\beta}$. To make things easier we define $m := \frac{\beta}{\alpha}$ and $t := \frac{c}{d} - \frac{e\alpha}{f\beta}$. For our goal, it turns out that the inverse function $f^{-1}(x) = \frac{x-t}{m}$ can make our computational life easier. Our strategy is now to walk along the integer points on the y-axis. Since our half-open parallelepiped has height 1, we know that for every integral y-coordinate between $\lceil \frac{g}{h} \rceil$ and $\beta + \lceil \frac{g}{h} \rceil - 1$ (-1 since the parallelepiped is half-open) there is exactly one point with integral x-coordinate lying in the parallelepiped, namely $\left\lfloor f^{-1}(y)\right\rfloor$. Thus

$$\sigma_{\Pi_2}(x,y) = \sum_{k=\lceil \frac{g}{h}\rceil}^{\beta+\lceil \frac{g}{h}\rceil-1} x^{\left\lfloor f^{-1}(k)\right\rfloor} y^k.$$

To tile the vertex cone completely, all positive integral combinations of the two generating vectors are needed. For the integer-point transform this implies (compare [**2**], page $60f$)

$$\sigma_{V_2}(x,y) = \frac{\sigma_{\Pi_2}}{(1-x^{-1})(1-x^{\alpha}y^{\beta})} = \frac{\sum_{k=\lceil \frac{g}{h}\rceil}^{\beta+\lceil \frac{g}{h}\rceil-1} x^{\left\lfloor f^{-1}(k)\right\rfloor} y^k}{(1-x^{-1})(1-x^{\alpha}y^{\beta})}$$

For the third integer-point transform of the vertex cone V_3, the argument repeats almost verbatim. We only have to adapt the generators and walk along the x-axis which implies that we use f instead of f^{-1}. Let's explain it more explicitly. The fundamental parallelepiped projected on the x-axis contains the integer points from $\left\lceil \frac{e}{f}\right\rceil$ to $\lfloor\alpha\rfloor+\left\lceil \frac{e}{f}\right\rceil-1$. The -1 is due to the fact that the parallelepiped is half-open. Like before, the height 1 of the parallelepiped implies that there is exactly one point with integral y-coordinate for every x in the given range, namely $\lfloor f(x)\rfloor$. Therefore

$$\sigma_{V_3}(x,y) = \frac{\sum_{k=\lceil \frac{e}{f}\rceil}^{\alpha+\lceil \frac{e}{f}\rceil-1} x^k y^{\lfloor f(k)\rfloor}}{(1-y^{-1})(1-x^{-\alpha}y^{-\beta})}$$

Now we know what the integer-point transforms of the vertex cones look like and we can hence compute the integer-point transform $\sigma_{\Delta}(x,y)$ of the right triangle. Brion tells us that

$$\begin{aligned}\sigma_{\Delta}(x,y) &= \sigma_{V_1}+\sigma_{V_2}+\sigma_{V_3}\\ &= \frac{x^{\lceil \frac{a}{b}\rceil}y^{\lceil \frac{g}{h}\rceil}}{(1-x)(1-y)} + \frac{\sum_{k=\lceil \frac{g}{h}\rceil}^{\beta+\lceil \frac{g}{h}\rceil-1} x^{\left\lfloor f^{-1}(k)\right\rfloor} y^k}{(1-x^{-1})(1-x^{\alpha}y^{\beta})} + \frac{\sum_{k=\lceil \frac{e}{f}\rceil}^{\alpha+\lceil \frac{e}{f}\rceil-1} x^k y^{\lfloor f(k)\rfloor}}{(1-y^{-1})(1-x^{-\alpha}y^{-\beta})}\end{aligned}$$

and this proves Theorem 1.1. As in [**1**], it makes sense to keep the following Theorem in mind.

Theorem 5.1
[**1**, Section 4, Thm. 4]
In fixed dimensions, the integer-point transform $\sigma_{\mathcal{P}}(z_1,\ldots,z_d)$ of a rational polyhedron $\mathcal{P}$ can be computed as a sum of short rational functions in z_1, z_2, $\ldots$, z_d in time polynomial in the input size of $\mathcal{P}$.

Since vertex cones are polyhedra, we can safely apply this nice theorem to the Rademacher–Carlitz polynomials and get their computational feasibility which proves Theorem 1.3.

CHAPTER 6

Connections to Rademacher–Dedekind sums

While computing the integer-point transform of a rational triangle, Rademacher–Carlitz polynomials appeared naturally. A special case of those polynomials is the Carlitz polynomial. The Carlitz polynomials satisfy a reciprocity theorem from which we can deduce the reciprocity of the Dedekind sums by applying the operator $x\partial x$ twice and the operator $y\partial y$ once and setting $x = y = 1$. The goal of this section is to derive a similar connection between the Rademacher–Carlitz polynomials and the Rademacher–Dedekind sums. For this endeavor, we are following the same train of thought, with the only difference that we apply the operator $x\partial x$ once instead of twice. In what follows, we assume that $\gcd(\alpha, \beta) = 1$.
In the previous section, we came across the Rademacher–Carlitz polynomials

$$R\left(y, x; \alpha, \beta, -\frac{\alpha t}{\beta}; \frac{g}{h}\right) = \sum_{k=\lfloor \frac{g}{h} \rfloor}^{\lfloor \frac{g}{h} \rfloor + \beta - 1} x^{\left\lfloor \frac{\alpha(k-t)}{\beta} \right\rfloor} y^k.$$

As mentioned above, we apply the operator $x\partial x$ and $y\partial y$ once, set $x = 1 = y$ and rewrite the floor function as $\lfloor x \rfloor = x - \{x\}$ using the definition of the fractional-part function. Furthermore, we have (from the previous section) the explicit form of $f^{-1}(k) = \frac{k-t}{m} = \frac{\alpha(k-t)}{\beta}$. Thus we have, using the abbreviation $A := \left\lceil \frac{g}{h} \right\rceil$,

$$y\,(\partial y)\,x\,(\partial x)\,R\left(y, x; \alpha, \beta, -\frac{\alpha t}{\beta}; \frac{g}{h}\right)|_{x=y=1} = \sum_{k=A}^{\beta-1+A} \left(\frac{\alpha(k-t)}{\beta} - \left\{\frac{\alpha(k-t)}{\beta}\right\}\right) k. \tag{6.1}$$

Since we want to find a connection to the Rademacher–Dedekind sums, we order the terms so that the aperiodic terms are put together. Hence (6.1) equals

$$\frac{\alpha}{\beta} \sum_{k=A}^{\beta-1+A} (k^2 - tk) - \beta \sum_{k=A}^{\beta-1+A} \frac{k}{\beta} \left\{\frac{\alpha(k-t)}{\beta}\right\}.$$

For the next steps we pay closer attention to the second sum and introduce $\Lambda := \frac{\alpha}{\beta} \sum_{k=A}^{\beta-1+A} k^2 - kt$. Moreover, we are shifting the indices of the second sum such that the summation range is from $k = 0$ to $\beta - 1$. Therefore we get an extra term and we have to slightly change the fractional-part function. This extra term comes from the change $k \mapsto k + A$. The argument of the fractional-part function is adjusted (this is done by setting $\widetilde{t} := t - A$.) and the first fraction is expanded in two summands. We can now rewrite (6.1) as

$$\Lambda - A \sum_{k=0}^{\beta-1} \left\{\frac{\alpha(k - \widetilde{t})}{\beta}\right\} - \beta \sum_{k=0}^{\beta-1} \left(\frac{k}{\beta} \left\{\frac{\alpha(k - \widetilde{t})}{\beta}\right\}\right).$$

In order to get a form similar to the Rademacher–Dedekind sum we are adding a 0. This means (6.1) is equal to

$$\Lambda - \beta \sum_{k=0}^{\beta-1} \left(\left\{ \frac{k}{\beta} \right\} \left\{ \frac{\alpha(k-\widetilde{t})}{\beta} \right\} - \frac{1}{2} \left\{ \frac{\alpha(k-\widetilde{t})}{\beta} \right\} - \frac{1}{2} \left\{ \frac{k}{\beta} \right\} + \frac{1}{4} \right)$$
$$+ \quad \beta \sum_{k=0}^{\beta-1} \left(\frac{1}{2} \left\{ \frac{\alpha(k-\widetilde{t})}{\beta} \right\} - \frac{1}{2} \left\{ \frac{k}{\beta} \right\} + \frac{1}{4} \right) - A \sum_{k=0}^{\beta-1} \left\{ \frac{\alpha(k-\widetilde{t})}{\beta} \right\}.$$

We pause for a second to properly rewrite this expression using Rademacher–Dedekind sums. For this encounter, we have to take care of what happens if $\frac{\alpha k}{\beta} \in \mathbb{Z}$ and $\frac{\alpha(k-\widetilde{t})}{\beta} \in \mathbb{Z}$. The second summand is exactly once an integer, namely for $k = 0$. By the definiton of $((x))$ there are the two cases $x \in \mathbb{Z}$ (then $((x)) = 0$) and $x \notin \mathbb{Z}$ (then $((x)) = \{x\} - \frac{1}{2}$). In our expression above, there is no such distinction of cases. So we have added a

$$\frac{1}{2} \left\{ \frac{\alpha(-\widetilde{t})}{\beta} \right\} + \frac{1}{4}$$

too much. The other part is an integer if and only if $\widetilde{t} \in \mathbb{Z}$. This is due to the fact that the difference between $\frac{\alpha(k-\widetilde{t})}{\beta}$ and an integer has to be a multiple of $\frac{1}{\beta}$, otherwise the fractional-part function cannot give 0. If $\widetilde{t} \in \mathbb{Z}$ then we have added a

$$\frac{1}{2} \left\{ \frac{\alpha k_*}{\beta} \right\} + \frac{1}{4}$$

too much, where k_* is the $k \in 0, \ldots, \beta - 1$ for which the expression $\frac{\alpha(k-\widetilde{t})}{\beta}$ is integral. So we have two options for this additional summand now; we could either distinguish the two cases $\widetilde{t} \in \mathbb{Z}$ and $\widetilde{t} \notin \mathbb{Z}$ explicitly, or introduce a trick factor that equals 1 in case of an integral t and 0 otherwise. This factor might look like $1 - \lceil \{\widetilde{t}\} \rceil$. Finally, we can correctly embed the Rademacher–Dedekind sums into our equation. As a result we get that $(y\partial y)(x\partial x)1, 1; \alpha, \beta, \frac{-\alpha t}{\beta}; \frac{g}{h})$ equals

$$\Lambda - \beta \sum_{k=0}^{\beta-1} \left(\left(\frac{\alpha(k-t)}{\beta} \right) \right) \left(\left(\frac{k}{\beta} \right) \right) + \beta \sum_{k=0}^{\beta-1} \left(\frac{1}{2} \left\{ \frac{\alpha(k-t)}{\beta} \right\} - \frac{1}{2} \left\{ \frac{k}{\beta} \right\} + \frac{1}{4} \right)$$
$$-\beta \left((1 - \lceil \{\widetilde{t}\} \rceil) \left(\frac{1}{2} \left\{ \frac{\alpha k_*}{\beta} \right\} + \frac{1}{4} \right) + \left(\frac{1}{2} \left\{ \frac{\alpha(-\widetilde{t})}{\beta} \right\} + \frac{1}{4} \right) \right) +$$
$$-A \sum_{k=0}^{\beta-1} \left\{ \frac{\alpha(k-\widetilde{t})}{\beta} \right\} \quad (6.2)$$

As a consequence, we have shown that there is a natural connection between Rademacher–Carlitz polynomials and Rademacher–Dedekind sums. Besides the Rademacher–Dedekind sums, we have some additional terms in (6.2), but it turns out that all of them are fairly easy to compute. There are closed expressions for $\sum_k k^2$ and $\sum_{k=0}^{\beta-1} \frac{k}{\beta} = \frac{\beta-1}{2}$ (we can use the periodicity of the fractional-part function to translate our summation range to 0 to $\beta - 1$).

It remains to find a compact form of sums that looks like

$$\sum_{k=0}^{\beta-1} \left\{ \frac{\alpha(k-\widetilde{t})}{\beta} \right\}.$$

Let us rewrite

$$\sum_{k=0}^{\beta-1}\left\{\frac{\alpha(k-\widetilde{t})}{\beta}\right\}=\sum_{k=0}^{\beta-1}\left\{\frac{\alpha k}{\beta}-\frac{-\alpha\widetilde{t}}{\beta}\right\}. \tag{6.3}$$

Since the fractional-part function has period β, we are summing over the entire period. The fact that $\gcd(\alpha,\beta)=1$ implies that it does not matter if we are summing over $\frac{\alpha k}{\beta}$ or $\frac{k}{\beta}$. Moreover, by definition of the fractional-part function (6.3) equals

$$\sum_{k=0}^{\beta-1}\left(\frac{k}{\beta}-\frac{-\alpha\widetilde{t}}{\beta}-\left\lfloor\frac{k}{\beta}-\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor\right).$$

For the first term, we know the result:

$$\sum_{k=0}^{\beta-1}\frac{k}{\beta}-\frac{-\alpha\widetilde{t}}{\beta}=\frac{\beta-1}{2}+\alpha\widetilde{t}.$$

How to compute the second term? We first make the observation that if $\frac{-\alpha\widetilde{t}}{\beta}\in\mathbb{Z}$, then as k runs through $\{0,\ldots,\beta-1\}$ the floored number stays the same. So we are adding β times $\frac{-\alpha\widetilde{t}}{\beta}$, thus

$$\sum_{k=0}^{\beta-1}\frac{k}{\beta}-\frac{-\alpha\widetilde{t}}{\beta}-\left\lfloor\frac{k}{\beta}-\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor=\frac{\beta-1}{2}-\alpha\widetilde{t}+\beta\frac{\alpha\widetilde{t}}{\beta}.$$

What happens if $\frac{-\alpha\widetilde{t}}{\beta}\notin\mathbb{Z}$? Then the floor function makes exactly one leap. This means that if we know how often we add $\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor$ and how often $\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor+1$, we have found a compact form. Our strategy is to find a $k_{\min}$ such that for all $k_{\min}\le k\le \beta-1$ the equation $\left\lfloor\frac{k}{\beta}-\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor=\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor+1$ holds, in other words, $\frac{k}{\beta}-\frac{\alpha\widetilde{t}}{\beta}\ge\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor+1$. After a few transformations we have the condition $k\ge\beta\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor+1+\alpha\widetilde{t}$. If k is running from

$$k_{\min}:=\left\lceil\beta\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor+1+\alpha\widetilde{t}\right\rceil$$

to $\beta-1$ the floor function equals $\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor+1$. If $k\in\{0,\ldots,k_{\min}-1\}$, then the floor function equals $\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor$. Putting it all together gives

$$\begin{aligned}k_{\min}\quad&\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor+(\beta-1-(k_{\min}-1))\left(\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor+1\right)=\\ =\quad&\beta-k_{\min}+\beta\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor.\end{aligned}$$

Thus, in the case $\frac{-\alpha\widetilde{t}}{\beta}\notin\mathbb{Z}$ (6.3) equals

$$\sum_{k=0}^{\beta-1}\left\{\frac{\alpha(k-\widetilde{t})}{\beta}\right\}=\beta-k_{\min}+\beta\left\lfloor\frac{-\alpha\widetilde{t}}{\beta}\right\rfloor+\frac{\beta-1}{2}.$$

Let us sum up what we have shown in this chapter.

By applying the operators $x\partial x$ and $y\partial y$ to the Rademacher–Carlitz polynomial $R\left(y,x;f^{-1}(x);\frac{g}{h}\right)$ and by setting $x=y=1$ a connection to Rademacher–Dedekind *sums is revealed. This connection is of the form*

$$
\begin{aligned}
&(y\partial y)(x\partial x)R\left(y,x;f^{-1}(x);\frac{g}{h}\right)\Big|_{x=y=1} = \\
= \quad & \Lambda-\beta\sum_{k=0}^{\beta-1}\left(\left(\frac{\alpha(k-t)}{\beta}\right)\right)\left(\left(\frac{k}{\beta}\right)\right)+\beta\sum_{k=0}^{\beta-1}\left(\frac{1}{2}\left\{\frac{\alpha(k-t)}{\beta}\right\}-\frac{1}{2}\left\{\frac{k}{\beta}\right\}+\frac{1}{4}\right) \\
- \quad & \beta\left((1-\lceil\{\widetilde{t}\}\rceil)\left(\frac{1}{2}\left\{\frac{\alpha k_*}{\beta}\right\}+\frac{1}{4}\right)++\left(\frac{1}{2}\left\{\frac{\alpha(-\widetilde{t})}{\beta}\right\}+\frac{1}{4}\right)\right)-A\sum_{k=0}^{\beta-1}\left\{\frac{\alpha(k-\widetilde{t})}{\beta}\right\},
\end{aligned}
\tag{6.4}
$$

where $\Lambda := \frac{\alpha}{\beta}\sum_{k=A}^{\beta-1+A} k^2 - kt$.

CHAPTER 7

Reciprocity

1. Preparations

This section is technical but important for our later journey. We are presenting compact forms of certain sums that appear while dealing with the reciprocity statement of Rademacher–Carlitz polynomials, so this section is basically a cornucopia of lemmata.

Lemma 7.1
Suppose that α, $\beta \in \mathbb{Z}$, and $c \in \mathbb{Q}$, with $\gcd(\alpha, \beta) = 1$. *Then*

$$\sum_{k=0}^{\beta-1} \left\{ \frac{\alpha k}{\beta} + c \right\} = \frac{\beta - 1}{2} + c\beta - \beta \lfloor c \rfloor - \beta + k_{\min} \tag{7.1}$$

and, consequently,

$$\sum_{k=0}^{\beta-1} \left\lfloor \frac{k}{\beta} + c \right\rfloor = k_{\min} \lfloor c \rfloor + (\beta - k_{\min})(\lfloor c \rfloor + 1), \tag{7.2}$$

where $k_{\min} = \lceil \beta(\lfloor c \rfloor - c + 1) \rceil$.

Proof. Since α and β are co-prime we can equivalently take a look at the sum

$$\sum_{k=0}^{\beta-1} \left\{ \frac{k}{\beta} + c \right\} = \frac{1}{\beta} \sum_{k=0}^{\beta-1} k + c \sum_{k=0}^{\beta-1} 1 - \sum_{k=0}^{\beta-1} \left\lfloor \frac{k}{\beta} + c \right\rfloor, \tag{7.3}$$

by definiton of the fractional-part function. The first part of the sum is easy to compute. For the second part, we are making the observation that the floor-function changes its value exactly one time. The first part is $\lfloor c \rfloor$ and the last part is $\lfloor c \rfloor + 1$. The next step is to define $k_{\min}$ in such a way that $k_{\min} \leq k \leq \beta - 1$ implies $\left\lfloor \frac{k}{\beta} + c \right\rfloor = \lfloor c \rfloor + 1$. As in Section 2, we get $\frac{k}{\beta} + c \geq \lfloor c \rfloor + 1$ and therefore $k_{\min} = \lceil \beta(\lfloor c \rfloor - c + 1) \rceil$. This means that we can split

$$\sum_{k=0}^{\beta-1} \left\lfloor \frac{k}{\beta} + c \right\rfloor = \lfloor c \rfloor \sum_{k=0}^{k_{\min}-1} 1 + (\lfloor c \rfloor + 1) \sum_{k=k_{\min}}^{\beta-1} 1,$$

which proves (7.2). Now we can finally rewrite (7.3) in a compact form:

$$(7.3) = \frac{\beta - 1}{2} + c\beta - \beta \lfloor c \rfloor - \beta - k_{\min}.$$

□

The next lemma is well known and nearly every student has proven the following results in a calculus class. They are the only equations we don't prove and find on our own.

Lemma 7.2
Suppose $A \in \mathbb{Q}$ and $\beta \in \mathbb{Z}$. Then

$$\sum_{k=\lceil A\rceil}^{\lceil A\rceil+\beta-1} k^2 = \frac{(\lceil A\rceil+\beta-1)(\lceil A\rceil+\beta)(2\lceil A\rceil+2\beta-1)}{6} - \frac{(\lceil A\rceil-1)(\lceil A\rceil)(2\lceil A\rceil-1)}{6}, \tag{7.4}$$

and

$$\sum_{k=\lceil A\rceil}^{\lceil A\rceil+\beta-1} k = \frac{(\lceil A\rceil+\beta-1)(\lceil A\rceil+\beta)}{2} - \frac{(\lceil A\rceil-1)(\lceil A\rceil)}{2}. \tag{7.5}$$

The next result slightly generalizes (7.2). To increase diversity, we use some different Greek letters.

Lemma 7.3
For m, γ, δ, $n \in \mathbb{Z}$, where $m - n = \delta - 1$, $\epsilon \in \mathbb{Q}$ and $\gcd(\delta, \gamma) = 1$,

$$\sum_{k=n}^{m} \left\lfloor \frac{\gamma}{\delta}k + \epsilon \right\rfloor = \frac{\gamma}{\delta}\left(\frac{m(m+1)}{2} - \frac{(n-1)(n)}{2}\right) - \frac{\delta-1}{2} + \delta\lfloor\epsilon\rfloor + \delta - \lceil \delta\left(\lfloor\epsilon\rfloor - \epsilon + 1\right)\rceil . \tag{7.6}$$

Proof. We can use the periodicity of the fractional-part function to eliminate γ. The next step is to convert the fractional-part function back into a floor function. These transformations may seem arbitrary, but the term is fairly easy to compute if there is no γ in the numerator:

$$\begin{aligned}\sum_{k=n}^{m} \left\lfloor \frac{\gamma}{\delta}k + \epsilon \right\rfloor &= \sum_{k=n}^{m}\left(\frac{\gamma k}{\delta} + \epsilon\right) - \sum_{k=0}^{\delta-1}\left\{\frac{k}{\delta} + \epsilon\right\} \\ &= \sum_{k=n}^{m}\left(\frac{\gamma k}{\delta} + \epsilon\right) - \sum_{k=0}^{\delta-1}\left(\frac{k}{\delta} + \epsilon\right) + \sum_{k=0}^{\delta-1}\left\lfloor\frac{k}{\delta} + \epsilon\right\rfloor\end{aligned}$$

The important observation here is that the floor function changes its value only once. As in the first sum we can determine $k_{\min} = \lceil \delta\left(\lfloor\epsilon\rfloor - \epsilon + 1\right)\rceil$. So the sum reduces to $\sum_{k=0}^{\delta-1}\left\lfloor\frac{k}{\delta}+\epsilon\right\rfloor = \delta\lfloor\epsilon\rfloor + \delta - k_{\min}$. Plugging in shows (7.6). □

Lemma 7.4
Let α, $\beta \in \mathbb{Z}$ such that $\gcd(\alpha, \beta) = 1$ and let t, and $P \in \mathbb{Q}$. Then

$$\begin{aligned}&\sum_{k=\lfloor P\rfloor}^{\lfloor P\rfloor+\beta-1} \frac{k}{\beta}\left\lfloor\frac{\alpha k + n}{\beta}\right\rfloor = \\ &= \frac{\beta\left(4\alpha\beta^2 - 6\alpha\beta + 2\alpha - 3\beta + 6(\beta-1)c + 12\right) + 6\lceil P\rceil\left(\alpha\beta^2 + \beta(-\alpha + 2c - 1) + 1\right)}{12\beta} \\ &- r_c(\alpha, \beta) - \frac{\left[-c\alpha^{-1}\right]_\beta}{2\beta} - \frac{[c]_\beta}{2\beta},\end{aligned} \tag{7.7}$$

where

$[x]_y$ *denotes the smallest nonnegative number congruent to* $x \mod y$,

$c := \frac{\alpha\lfloor P\rfloor + n}{\beta}$.

PROOF. By shifting the indices and by using the periodicity of the fractional-part function we can reveal the Rademacher–Dedekind sums. Involving the fractional-part function gives us the following sums which we already know how to compute by Lemma 7.2:

$$\frac{\alpha}{\beta^2} \sum_{k=\lfloor P \rfloor}^{\lfloor P \rfloor + \beta - 1} k^2 =$$

$$= \frac{\alpha}{\beta^2} \left(\frac{(\lceil P \rceil + \beta - 1)(\lceil P \rceil + \beta)(2\lceil P \rceil + 2\beta - 1)}{6} - \frac{(\lceil P \rceil - 1)(\lceil P \rceil)(2\lceil P \rceil - 1)}{6} \right)$$

and

$$\frac{n}{\beta^2} \sum_{k=\lfloor P \rfloor}^{\lfloor P \rfloor + \beta - 1} k = \frac{n}{\beta^2} \left(\frac{(\lceil P \rceil + \beta - 1)(\lceil P \rceil + \beta)}{2} - \frac{(\lceil P \rceil - 1)(\lceil P \rceil)}{2} \right).$$

Using Lemma 7.1 repeatedly, we get the equations

$$\frac{\lfloor P \rfloor}{\beta} \sum_{k=0}^{\beta-1} \left\{ \frac{\alpha k + c}{\beta} \right\} = \frac{\lfloor P \rfloor}{\beta} \frac{\beta - 1}{2}$$

and

$$\frac{1}{2} \sum_{k=0}^{\beta-1} \left\{ \frac{\alpha k + c}{\beta} \right\} = \frac{1}{2} \frac{\beta - 1}{2},$$

and

$$\frac{1}{2} \sum_{k=0}^{\beta-1} \left\{ \frac{\alpha k}{\beta} \right\} = \frac{1}{2} \frac{\beta - 1}{2}.$$

The last summand can now be represented using a Rademacher–Dedekind sum. As in the proof of Theorem 6.4, we get extra summands. □

2. Reciprocity of Rademacher–Carlitz polynomials

We know from Chapter 6 that there is a connection between Rademacher–Carlitz polynomials and Rademacher–Dedekind sums. This and the fact that there is a reciprocity theorem for Carlitz polynomials (Theorem 3.3) should motivate us to try to find such a reciprocity theorem for the Rademacher–Carlitz polynomials (which implies the reciprocity of Rademacher–Dedekind sums). In order to find such a reciprocity statement, we follow the strategy of [**1**]. In their paper, the authors take the first quadrant and compute its integer-point transform in two different ways. The first way is the natural way, namely by computing it straightforward. The second way of determining the integer-point transform is by taking a linear graph starting in the origin and cutting the first quadrant into two halves. Then the integer-point transforms of these two cones are determined by tiling the cones with their fundamental parallelepipeds. We take this approach and change it slightly, namely by shifting the first quadrant by the vector (P, Q) where $(P, Q) \in \mathbb{Q}^2$. The function that cuts the shifted quadrant will be denoted by $f(x) := \frac{\alpha}{\beta}x + t$ with $\gcd(\alpha, \beta) = 1$ and $\alpha,\ \beta \in \mathbb{Z}$ and $t \in \mathbb{Q}$. We choose the cones to be half-open in such a way that f is not contained in either one of them. This gives us a symmetric geometric picture. Our two cones take on the form

$$V_1 = \{\lambda_1(0, 1) + \lambda_2(\beta, \alpha) + (P, Q) : 0 < \lambda_1,\ 0 \le \lambda_2\}$$
$$V_2 = \{\lambda_1(1, 0) + \lambda_2(\beta, \alpha) + (P, Q) : 0 \le \lambda_1,\ 0 < \lambda_2\}.$$

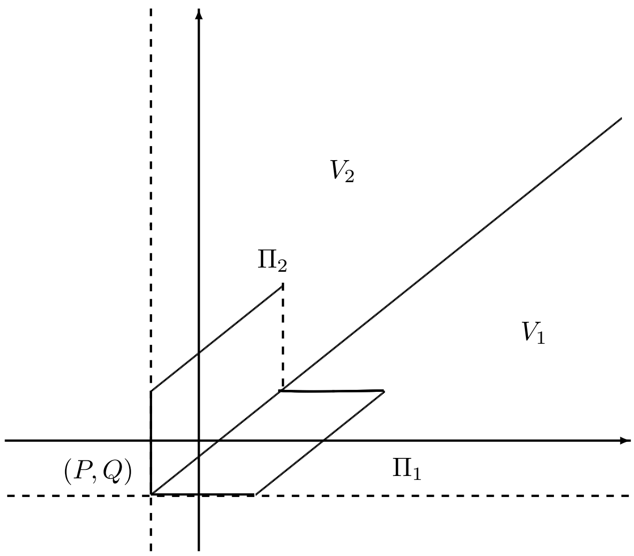

Figure 4. Shifted first quadrant split into two pointed cones

Since we want to use a tiling argument, we need to define the fundamental parallelepipeds. We choose the fundamental parallelepipeds to be half-open so that there are no two parallelepipeds overlapping:

$$\Pi_1 = \{\lambda_1(0, 1) + \lambda_2(\beta, \alpha) + (P, Q) : 0 < \lambda_1 \le 1,\ 0 \le \lambda_2 < 1\}$$
$$\Pi_2 = \{\lambda_1(1, 0) + \lambda_2(\beta, \alpha) + (P, Q) : 0 \le \lambda_1 < 1,\ 0 < \lambda_2 \le 1\}.$$

For the integer-point transform $\sigma_{\Pi_1}(x, y)$ of Π_1, we again imagine ourselves walking along the x-axis. Since our parallelogram has height 1, there is exactly one integer point in Π_1 for each x running from $k = \lceil P \rceil$ to $\lceil P \rceil + \beta - 1$. The y-coordinate of those integer points has the form $\lfloor f(k) \rfloor + 1$. With a similar argument, changing the roles of the axes, we get our second integer-point transform:

$$\sigma_{\Pi_1}(x,y) = \sum_{k=\lceil P \rceil}^{\lceil P \rceil + \beta - 1} x^k y^{\lfloor f(k) \rfloor + 1} = y\ R(x,y;f;P)$$

$$\sigma_{\Pi_2}(x,y) = \sum_{k=\lceil Q \rceil}^{\lceil Q \rceil + \alpha - 1} x^{\lfloor f^{-1}(k) \rfloor + 1} y^k = x\ R\left(y,x;f^{-1};Q\right)$$

Using the same tiling argument as in Chapter 2, we get the integer-point transform $\sigma(x,y)$ of all integer points lying in V_1 and V_2:

$$\sigma_{V_1 \cup V_2}(x,y) = \frac{yR(x,y;f;P)}{(1-y)(1-x^\beta y^\alpha)} + \frac{xR\left(y,x;f^{-1};Q\right)}{(1-x)(1-x^\beta y^\alpha)}$$

It remains to find the integer-point transform of f. Luckily, we know how to compute it from Section 2. There we have treated the case of integer points of the diophantic equation $Ax - By = C$. We have to do two things to properly apply the result to our case. First, we have to transform $f(x) = y = \frac{\alpha}{\beta}x + \frac{p}{q}$ where $\gcd(\alpha,\beta) = \gcd(p,q) = 1$ and $\frac{p}{q} = t$ to a diophantic equation. This is no problem at all. Second, we have to be determine another $k_{\min}$, since we are starting walking along the x-(respectively y-)axis at $\lceil P \rceil$ (respectively $\lceil Q \rceil$). Using the notation of Section 2 (except we are using a instead of α and b instead of β to avoid any ambiguity) this means

$A := \alpha q$, $B := \beta q$, $C := p\beta$;

$a := CA^{-1}$, where A^{-1} denotes the inverse of $A \mod B$, $b := \frac{Aa-C}{B}$, $\gamma := \frac{B}{\gcd(A,B)}$ and $\delta := \frac{A}{\gcd(A,B)}$;

$k_{\min} = \left\lceil \frac{\lceil P \rceil - a}{\gamma} \right\rceil$.

Thus the integer-point transform of the line segment starting in (P,Q) equals

$$\frac{x^{a+\gamma k_{\min}} y^{b+\delta k_{\min}}}{1 - x^\gamma y^\delta}, \tag{7.8}$$

where (a,b) is a solution to the associated inhomogeneous diophantic equation and (γ,δ) a solution to the associated homogeneous equation. Adjusting all parameters gives us (using $\gamma = \beta$ and $\delta = \alpha$)

$C := -p\frac{\beta}{q}$ (which is an integer if there are integer points on f);

$a := C\alpha^{-1}$, $b := \frac{\alpha a - C}{\beta}$;

$k_{\min} = \left\lceil \frac{\lceil P \rceil - a}{\beta} \right\rceil$.

Recall our goal from the beginning of this section: we wanted to compute the integer-point transform of the shifted first quadrant in two different ways. Therefore, we need to find another representation of this integer-point transform. This time we are determining it straightforward. We know that all x-coordinates have to be bigger than $\lceil P \rceil$ and all y-coordinates have to be bigger than $\lceil Q \rceil$. Hence

$$\sigma(x,y) = \frac{x^{\lceil P \rceil} y^{\lceil Q \rceil}}{(1-x)(1-y)}.$$

Finally, we can compare both representations and obtain a reciprocity law for Rademacher–Carlitz polynomials:

$$\frac{x^{\lceil P \rceil} y^{\lceil Q \rceil}}{(1-x)(1-y)} = \frac{yR(x,y;f;P)}{(1-y)(1-x^\beta y^\alpha)} + \frac{xR\left(y,x;f^{-1};Q\right)}{(1-x)(1-x^\beta y^\alpha)} + \chi(f)\frac{x^{a+\beta k_{\min}} y^{b+\alpha k_{\min}}}{1-x^\beta y^\alpha},$$

where χ equals 1 if there are integer points on f and 0 otherwise. By clearing denominators we get a nicer form:

$$\begin{aligned}&y(1-x)R\left(x,y;f;P,1\right)+(1-y)xR\left(y,x;f^{-1};Q,1\right)+\\&+\chi(f)(1-x)(1-y)\left(x^{a+\beta k_{\min}}y^{b+\alpha k_{\min}}\right)\\&=(1-x^{\beta}y^{\alpha})x^{\lceil P\rceil}y^{\lceil Q\rceil}.\end{aligned}$$

This proves Theorem 1.4.

3. The reciprocity statement for Carlitz polynomials

This reciprocity law of Rademacher–Carlitz polynomials implies the reciprocity statement of Carlitz polynomials. In order to see that, we are treating the special case for $(P,Q)=\mathbf{0}$ and p, $q=0$. This implies that all parameters equal 0 except for γ and δ. So we arrive at

$$\begin{aligned}&y(1-x)R\left(x,y;\alpha,\beta;0,1\right)+(1-y)xR\left(y,x;\beta,\alpha;0,1\right)+\\&+(1-x)(1-y)\\&=(1-x^{\beta}y^{\alpha}).\end{aligned}$$

We can rewrite this as

$$\begin{aligned}&y(x-1)R\left(x,y;\alpha,\beta;0,1\right)+(y-1)xR\left(y,x;\beta,\alpha;0,1\right)+\\&=x^{\beta}y^{\alpha}-y-x+xy.\end{aligned}\tag{7.9}$$

As it turns out, from that representation we can derive the reciprocity law of Carlitz polynomials. Let us be more explicit: (7.9) equals

$$x(y-1)R(y,x;\beta,\alpha)+y(x-1)R(x,y;\alpha,\beta)=x^{\beta}y^{\alpha}-xy+xy+xy-x-y.$$

It is easy to see that this can be transformed into

$$x(y-1)R(y,x;\beta,\alpha)-x(y-1)+y(x-1)R(x,y;\alpha,\beta)-y(x-1)=x^{\beta}y^{\alpha}-xy.$$

Since we want to introduce the notation for Carlitz polynomials, we should make a short remark on the differences between the notation for Rademacher–Carlitz polynomials and the notation used for Carlitz polynomials. By definition of $R(x,y;\alpha,\beta)$ the sum is starting at $k=0$; the notation of $c(x,y;a,b)$, however, starts at $k=1$. Thus we are subtracting a zero in order to switch from one notation to another. Hence

$$(7.9)=x(y-1)\left(\sum_{k=0}^{\alpha-1}\left(x^{\left\lfloor\frac{k\beta}{\alpha}\right\rfloor}y^{k}\right)-1\right)+x(y-1)\left(\sum_{k=0}^{\beta-1}\left(y^{\left\lfloor\frac{k\alpha}{\beta}\right\rfloor}x^{k}\right)-1\right)=(x^{\beta}y^{\alpha}-xy).$$

Dividing the equation by xy gives us

$$(y-1)c(x,y,\beta,\alpha)+(x-1)c(y,x,\alpha,\beta)=x^{\beta-1}y^{\alpha-1}-1,$$

which is indeed equivalent to Theorem 3.3 by substituting $a=\beta$, $b=\alpha$ and $u=x$, $v=y$. This proves Theorem 3.3.

4. A reciprocity theorem for Rademacher–Dedekind sums

We start this section with a definition.

Definition 7.5
Let x and $y \in \mathbb{N}$. We define

$$[x]_y := \min\left\{z \in \mathbb{N} : z \equiv x \mod y\right\},$$

as the smallest nonnegative integer congruent to $x \mod y$.

Recall how we proved the reciprocity statement for Rademacher–Carlitz polynomials. The shifted origin (P, Q) is closely related to the linear function $f(x) = \frac{\alpha x + n}{\beta}$ going through (P, Q). However, we can choose $P = 0$ without getting additional constraints, since we can set $Q = \frac{n}{\beta}$. We take this geometric train of thought as a starting point in order to derive a reciprocity statement for Rademacher–Dedekind sums.
Applying $x\partial x$ twice and $y\partial y$ once to the reciprocity law for Rademacher–Carlitz polynomials gives us

$$\begin{aligned} &- \underbrace{\sum_{k=0}^{\beta-1} \left\lfloor \frac{k\alpha + n}{\beta} \right\rfloor}_{=:\sigma_1} -2 \underbrace{\sum_{k=0}^{\beta-1} k \left\lfloor \frac{k\alpha + n}{\beta} \right\rfloor}_{=:\sigma_2} -2 \underbrace{\sum_{k=\lfloor \frac{n}{\beta} \rfloor}^{\alpha+\lfloor \frac{n}{\beta} \rfloor -1} \left\lfloor \frac{k\beta - n}{\alpha} \right\rfloor}_{=:\sigma_3} + \\ &- \underbrace{\sum_{k=\lfloor \frac{n}{\beta} \rfloor}^{\alpha+\lfloor \frac{n}{\beta} \rfloor -1} \left\lfloor \frac{k\beta - n}{\alpha} \right\rfloor^2}_{=:\sigma_4} + 2a - \alpha + 2\beta k_{\min}^{\text{linesegment}} + 1 - \beta^2 \\ &= \beta(\beta-1)\left(\alpha + \left\lceil \frac{n}{\beta} \right\rceil\right) - \beta\left(\alpha + \left\lceil \frac{n}{\beta} \right\rceil\right) = \left(\alpha + \left\lceil \frac{n}{\beta} \right\rceil\right)(\beta^2 - \beta), \quad (7.10) \end{aligned}$$

where a and $k_{\min}^{\text{linesegment}}$ are defined as in Theorem 1.4. In order to derive a reciprocity theorem for Rademacher–Dedekind sums, we have to find a compact form of all sums appearing. Luckily, Section 1 provides us with all we need. In order to keep an overview, we present most intermediate steps as lemmata.

Lemma 7.6
Let α, $\beta \in \mathbb{Z}\backslash\{0\}$ and $n \in \mathbb{N}$ with $\gcd(\alpha, \beta) = 1$. Then σ_1 can be rewritten in compact form

$$\sigma_1 = \frac{(\beta-1)\alpha}{2} - \frac{\beta-1}{2} + n,$$

and

$$\sigma_3 = \frac{\beta}{\alpha}\left(\frac{(\alpha - 1 + \left\lfloor \frac{n}{\beta} \right\rfloor)(\alpha + \left\lfloor \frac{n}{\beta} \right\rfloor) - \left\lfloor \frac{n}{\beta} \right\rfloor (\left\lfloor \frac{n}{\beta} \right\rfloor - 1)}{2}\right) - \frac{\alpha - 1}{2} - n.$$

Proof. The formula for σ_1 from Lemma 7.3. In a similar fashion, we get a compact form for σ_3. Lemma 7.3 gives

$$\sigma_3 = \frac{\beta}{\alpha}\left(\frac{(\alpha - 1 + \left\lfloor \frac{n}{\beta} \right\rfloor)(\alpha + \left\lfloor \frac{n}{\beta} \right\rfloor) - \left\lfloor \frac{n}{\beta} \right\rfloor (\left\lfloor \frac{n}{\beta} \right\rfloor - 1)}{2}\right) - \frac{\alpha - 1}{2} - n.$$

□

Rewritting σ_2 reveals the Rademacher–Dedekind sum involving f, which may not come as a surprise.

Lemma 7.7
Let $\alpha,\ \beta \in \mathbb{Z}\backslash\{0\}$ *and* $n \in \mathbb{N}$ *with* $\gcd(\alpha,\beta)=1$. *Then* σ_2 *equals*

$$\beta\sum_{k=0}^{\beta-1}\frac{k}{\beta}\left\lfloor\frac{\alpha k+n}{\beta}\right\rfloor = \frac{1}{6}\alpha(2\beta-1)(\beta-1)+\frac{1}{2}(\beta-1)n-\frac{\beta}{2}\left(\frac{[n]_\beta}{\beta}-\frac{1}{2}\right) - \beta\left(\frac{1}{2}\left(\frac{\left[-n\alpha^{-1}\right]_\beta}{\beta}-\frac{1}{2}\right)+\frac{\beta}{4}+r_n(\alpha,\beta)-\frac{1}{2}\right).$$

Proof. The result follows from Lemma 7.4. □

Let us next derive an expression for σ_4. The squared floor function can be expressed using a Rademacher–Dedekind sum. We replace the floor function using the fractional-part function and we then have to treat three sums:

$A_1 := \sum_{k=\lfloor\frac{n}{\beta}\rfloor}^{\alpha+\lfloor\frac{n}{\beta}\rfloor-1}\left(\frac{\beta k-n}{\alpha}\right)^2,$

$A_2 := \sum_{k=\lfloor\frac{n}{\beta}\rfloor}^{\alpha+\lfloor\frac{n}{\beta}\rfloor-1}\left(\frac{\beta k-n}{\alpha}\right)\left\{\frac{\beta k-n}{\alpha}\right\}$, and

$A_3 := \sum_{k=\lfloor\frac{n}{\beta}\rfloor}^{\alpha+\lfloor\frac{n}{\beta}\rfloor-1}\left\{\frac{\beta k-n}{\alpha}\right\}^2.$

A_2 is easy to get by. We just multiply everything out and we shift the summation range using $k \mapsto k+\lfloor\frac{n}{\beta}\rfloor$. Therefore, the argument of the fractional-part function changes from $\frac{\beta k-n}{\alpha}$ to $\frac{\beta k-n+\beta\lfloor\frac{n}{\beta}\rfloor}{\alpha}$. To shorten notification, we define $d := \beta\left\lfloor\frac{n}{\beta}\right\rfloor-n$. Using Lemma 7.1 immediately gives

$$\frac{d}{\alpha}\sum_{k=0}^{\alpha-1}\left\{\frac{\beta k+d}{\alpha}\right\}=\frac{d(\alpha-1)}{2\alpha}.$$

A similar reasoning as in the proof of Lemma 7.4 provides us with the compact form

$$\beta\sum_{k=0}^{\alpha-1}\left\{\frac{k}{\alpha}\right\}\left\{\frac{\beta k+d}{\alpha}\right\}=\beta\left(r_d(\beta,\alpha)+\frac{\alpha}{4}-1+\frac{\left[\beta^{-1}n\right]_\alpha}{2\alpha}+\frac{[d]_\alpha}{2\alpha}\right).$$

Putting everything together shows

$$A_2=\beta\left(r_d(\beta,\alpha)+\frac{\alpha}{4}-1+\frac{\left[\beta^{-1}n\right]_\alpha}{2\alpha}+\frac{[d]_\alpha}{2\alpha}\right)+\frac{d(\alpha-1)}{2\alpha}.$$

It remains to find a compact expression for A_3. We will compute A_3 in two different ways, so we can compare both representations and, as a small bonus, we get a compact form for $r_{-n}(1,\alpha)$. First, the more complicated way. We start by using the periodicity of the fractional-part function and $\gcd(\alpha,\beta)=1$. Hence

$$A_3=\sum_{k=\lfloor\frac{n}{\beta}\rfloor}^{\alpha+\lfloor\frac{n}{\beta}\rfloor-1}\left\{\frac{\beta k-n}{\alpha}\right\}^2=\sum_{k=0}^{\alpha-1}\left\{\frac{k-n}{\alpha}\right\}^2.$$

Now, we can plug in the definition of the fractional-part function and immediately get

$$\sum_{k=0}^{\alpha-1}\left\{\frac{k}{\alpha}-\frac{n}{\alpha}\right\}^2=\sum_{k=0}^{\alpha-1}\left(\left(\frac{k}{\alpha}-\frac{n}{\alpha}\right)-\left\lfloor\frac{k}{\alpha}-\frac{n}{\alpha}\right\rfloor\right)^2,$$

which is equal to

$$\underbrace{\sum_{k=0}^{\alpha-1}\left(\frac{k}{\alpha}-\frac{n}{\alpha}\right)^2}_{=:S_1}+\underbrace{2\frac{n}{\alpha}\sum_{k=0}^{\alpha-1}\left\lfloor\frac{k}{\alpha}-\frac{n}{\alpha}\right\rfloor}_{=:S_2}\underbrace{-2\sum_{k=0}^{\alpha-1}\frac{k}{\alpha}\left\lfloor\frac{k}{\alpha}-\frac{n}{\alpha}\right\rfloor}_{=:S_3}+\underbrace{\sum_{k=0}^{\alpha-1}\left\lfloor\frac{k}{\alpha}-\frac{n}{\alpha}\right\rfloor^2}_{=:S_4}.$$

Basic calculus or Lemma 7.2 proves

$$S_1=\frac{2\alpha^2-3\alpha+6n^2-6\alpha n+6n+1}{6\alpha},$$

and for S_2 we can use Lemma 7.3, which implies

$$S_2=2\frac{n}{\alpha}\sum_{k=0}^{\alpha-1}\left\lfloor\frac{k-n}{\alpha}\right\rfloor=\frac{-2n^2}{\alpha}.$$

In order to compute S_2+S_3, we make use of the compact forms

$$\sum_{k=0}^{\alpha-1}\left\lfloor\frac{k-n}{\alpha}\right\rfloor=-n,$$

and

$$\sum_{k=0}^{\alpha-1}\frac{k}{\alpha}\left\lfloor\frac{k-n}{\alpha}\right\rfloor=\frac{-n(\alpha-1)}{2\alpha}+\frac{(\alpha-1)(2\alpha-1)}{6}+$$
$$-\left(r_{-n}(1,\alpha)+\frac{\alpha}{4}-1+\frac{[-n]_\alpha+[n]_\alpha}{2\alpha}\right).$$

As a result we get

$$S_2+S_3=$$
$$=-2\left(\frac{-n(\alpha-1)}{2\alpha}+\frac{(\alpha-1)(2\alpha-1)}{6}-\left(r_{-n}(1,\alpha)+\frac{\alpha}{4}-1+\frac{[-n]_\alpha+[n]_\alpha}{2\alpha}\right)\right)-\frac{2n^2}{\alpha}.$$

Next, we need to find a compact form for S_4. For this endeavor, we have to keep in mind that the summand changes its value only once. Therefore, we can determine its compact form similar to the proof of Lemma 7.1. As a consequence, we arrive at

$$S_4=\left\lfloor\frac{-n}{\alpha}\right\rfloor^2-2\alpha\left\lfloor\frac{-n}{\alpha}\right\rfloor-2\alpha\left\lfloor\frac{-n}{\alpha}\right\rfloor^2-2n\left\lfloor\frac{-n}{\alpha}\right\rfloor-n.$$

Let us next find a compact form of A_3. We start with the representation

$$A_3=\sum_{k=\lfloor\frac{n}{\beta}\rfloor}^{\alpha+\lfloor\frac{n}{\beta}\rfloor-1}\left\{\frac{\beta k-n}{\alpha}\right\}^2=\sum_{k=0}^{\alpha-1}\left\{\frac{k-n}{\alpha}\right\}^2.$$

Since n is an integer and k is running through a whole period α, $k-n$ is running through a whole period of α as well. This implies that we can neglect n. Thus

$$A_3=\sum_{k=0}^{\alpha-1}\left\{\frac{k}{\alpha}\right\}^2\overset{k\le\alpha}{=}\frac{1}{\alpha^2}\sum_{k=0}^{\alpha-1}k^2=\frac{(\alpha-1)(2\alpha-1)}{6\alpha}.$$

Comparing both sides gives us:

Corollary 7.8
Let n, $\alpha \in \mathbb{Z}$ $\beta \in \mathbb{N}$ with $\gcd(\alpha, \beta) = 1$. Then

$$r_{-n}(1,\alpha) = \\ = \frac{4\alpha^2 - 9\alpha + 14 + 12\alpha \left\lceil \frac{n}{\alpha} \right\rceil^2 - 12\alpha \left\lceil \frac{n}{\alpha} \right\rceil}{12} + \\ + \frac{-6\alpha \left\lceil \frac{n}{\alpha} \right\rceil^2 - 12\alpha n \left\lceil \frac{n}{\alpha} \right\rceil + 6n^2 + 6\alpha n - 6\left[-n\right]_\alpha - 6\left[n\right]_\alpha}{12\alpha}.$$

We not only have found a compact expression for $r_{-n}(1, \alpha)$, we have, furthermore, proved our next lemma.

Lemma 7.9
Let α, $\beta \in \mathbb{Z}\backslash\{0\}$ and $n \in \mathbb{N}$ with $\gcd(\alpha, \beta) = 1$. Then σ_4 equals

$$\sigma_4 = \frac{1}{6\alpha}\Bigg(\alpha^2(\beta(2\beta-3)+2) + \beta^2 - 6\beta(\left[n\beta^{-1}\right]_\alpha + [d]_\alpha) + \\ + 6\beta \left\lfloor \frac{n}{\beta} \right\rfloor \left((\alpha-1)(\beta-1) + \beta \left\lfloor \frac{n}{\beta} \right\rfloor - 2n\right) + 6n^2 + \\ + 6n(-\alpha\beta + \alpha + \beta - 1) - 3\alpha(\beta(\beta + 4r_d(\beta,\alpha) - 4) + 1) + 1\Bigg).$$

Actually, we are done here. However, the parameter d that is appearing in A_2 due to the shift of the indices, seems to be artificial. After a moment's thought, it becomes clear that we can equivalently write $[n]_\beta$ instead. According to Theorem 1.4, we set $a = \left[-n\alpha^{-1}\right]$ and $k_{\min}^{\text{linesegment}} = \left\lceil \frac{-\left[-n\alpha^{-1}\right]_\beta}{\beta} \right\rceil$ Finally, we can put everything together:

Theorem 7.10
Let α, $\beta \in \mathbb{Z}\backslash\{0\}$ and $n \in \mathbb{Z}$ with $\gcd(\alpha, \beta) = 1$. The Rademacher–Dedekind sums $r_n(\alpha, \beta)$ and $r_{[n]_\beta}(\alpha, \beta)$ satisfy

$$r_n(\alpha,\beta)\,(1-\beta) - \beta r_{[n]_\beta}(\beta,\alpha) = \frac{1}{12\alpha\beta}\Bigg(6\beta\left(\left(\left(\beta\left\lfloor\frac{n}{\beta}\right\rfloor - n\right)(\alpha-1) + \\ - \alpha(\beta-1)\beta\left\lceil\frac{n}{\beta}\right\rceil + \beta\left\lfloor\frac{n}{\beta}\right\rfloor\left(-\alpha\beta + 2\alpha + \beta - \beta\left\lfloor\frac{n}{\beta}\right\rfloor + 2n\right)\right)\right) + \\ + \alpha^2\left(-(6\left[-n\alpha^{-1}\right]_\beta + \beta(12(\beta-1)\beta + 7) + 2)\right) + \\ \alpha\left(3\left(\beta\left(2\left[-n\alpha^{-1}\right]_\beta + \beta^2 + \beta + 3\right) + 2\right) - 4\alpha\beta^2 + 6\alpha\beta + (6-18\beta)n\right) + \\ - \beta\left(\beta^2 - 6\beta\left(\left[n\beta^{-1}\right]_\alpha + \left[-[n]_\beta\right]_\alpha\right) + 6n^2 + 6\beta n + 1\right)\Bigg) + \\ + \left[-n\alpha^{-1}\right]_\beta + \beta\left\lceil\frac{-\left[-n\alpha^{-1}\right]_\beta}{\beta}\right\rceil$$

It is not clear whether or not Theorem 7.10 is equivalent to Theorem 3.2. This question will be subject of further investigations.

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Deutsche Zusammenfassung

Das Ziel dieser Bachelor-Thesis ist es, Gitterpunkttransformationen rationaler Polygone vollständig zu beschreiben. Das erste Kapitel ist eine Übersicht der wichtigsten Sätze und Definitionen. Außerdem wird die Bachelor-Thesis in Grundzügen beschrieben. Kapitel 2 stellt eine Einleitung in das Thema dar, in der die nötigen Grundlagen vorgestellt und an Beispielen erklärt werden. Es wird der, für diese Arbeit sehr wichtige, Begriff der rationalen Polygone definiert. Der Begriff der Gitterpunkttransformation (engl. integer-point transforms) $\sigma(x, y)$ rationaler Polygone $\mathcal{P}$

$$\sigma(x, y) := \sum_{(j,k)\in\mathbb{Z}^d\cap\mathcal{P}}$$

wird eingeführt; hierbei werden Kegel vorgestellt und es wird gezeigt, wie man diese möglichst geschickt mit Parallelogrammen disjunkt überdecken kann. Desweiteren wird der Brion'sche Satz erläutert, der in Kapitel 4 und in Kapitel 5 der Ausgangspunkt für die explizite Bestimmung der Gitterpunkttransformationen ist. Das dritte Kapitel beschreibt das Leben der Mathematiker Richard Dedekind, Hans Rademacher und Leonard Carlitz, deren theoretische Vorarbeit und Resultate in dieser Arbeit eine große Rolle spielen. Hierbei werden die Dedekind–Summen, vergleiche Definition 3.1 und deren Reziprozitätsgesetz vorgestellt, siehe Theorem 3.1. Desweiteren werden Rademacher–Dedekind–Summen, sowie deren Reziprozitätsgesetz eingeführt, vergleiche hierzu Theorem 3.2. Carlitz–Polynome werden definiert, vergleiche Definition 3. Das Reziprozitätsgesetz der Carlitz–Polynome, Theorem 3.3, wird erwähnt und später in Kapitel 7 bewiesen.
In Kapitel 4 werden Gitterpunkttransformationen von rationalen Strecken untersucht. Als eine Einführung in die Vorgehensweise und die grundlegenden Ideen wird der Spezialfall einer rationalen Strecke, die im Ursprung beginnt, betrachtet. Danach wird der allgemeine Fall rationaler Strecken untersucht, siehe Theorem 4.2. Ausgangspunkt der Überlegung im allgemeinen Fall ist der Brion'sche Satz.
In Kapitel 5 wird das Problem rationaler Polygone untersucht. Trianguliert man ein beliebiges, rationales Polygon, kann man das Problem auf die Bestimmung von Gitterpunkttransformationen von Dreiecken reduzieren, sofern man weiß, wie die Gitterpunkttransformationen rationaler Strecken aussehen (dies tritt auf, da man die Siebformel beim Zusammensetzen mehrerer Dreiecke anwenden muss, da man sonst Gitterpunkte auf der gemeinsamen Strecke doppelt zählt). Da man ein rationales Dreieck in ein rationales Rechteck dergestalt einbetten kann, dass nur rechtwinklige Dreiecke und Rechtecke zum großen Rechteck "fehlen", vergleiche Figure 5, reduziert sich das Problem Gitterpunkttransformationen rationaler Polygone zu bestimmen, darauf, dass man rechtwinklige, rationale Dreiecke betrachtet. Eine vollständige Beschreibung der Gitterpunkttransformation rechtwinkliger, rationaler Dreiecke (und somit eine vollständige Beschreibung beliebiger rationaler Polygone), wird in Kapitel 5 mit Theorem 1.1 gegeben.
Hierbei stellt sich heraus, dass Rademacher–Carlitz–Polynome ein Hauptbestandteil solcher Gitterpunkttransformationen sind. Die Beschreibung der Gitterpunkttransformationen in rationalen, rechtwinkligen Dreiecken enthalten nämlich eine verallgemeinerte Form der Carlitz–Polynome, die in dieser Arbeit definierten Rademacher–Carlitz–Polynome

$$R(x, y; \beta, \alpha, t; e) = \sum_{k=\lceil e\rceil}^{\lceil e\rceil+\alpha-1} x^k y^{\left\lfloor \frac{\beta}{\alpha}k+t\right\rfloor},$$

vergleiche Definition 1.2. Als eine Folgerung kann man den Satz von Barvinok, vergleiche Theorem 5.1, auf Rademacher–Carlitz–Polynome anwenden und man hat

somit bewiesen, dass Rademacher–Carlitz–Polynome als Summe kurzer, rationaler Funktionen effektiv berechnet werden können. Da Carlitz–Polynome einen engen Zusammenhang mit Dedekind-Summen haben, könnte man vermuten, dass die etwas allgemeineren Rademacher–Dedekind–Summen einen Zusammenhang mit den Rademacher–Carlitz–Polynomen besitzen. Dieser Frage wird in Kapitel 6 nachgegangen. Dabei stellt sich heraus, dass durchaus ein Zusammenhang vorhanden ist. Die Art und Gestalt des Zusammenhanges wird ausführlich hergeleitet.
Carlitz–Polynome erfüllen ein sehr schönes Reziprozitätsgesetz. Dieses Gesetz kann mittels eines geometrischen Argumentes, vergleiche [**1**], bewiesen werden. Die grundlegende Idee ist es, die Gitterpunkttransformation des ersten Quadranten auf zwei verschiedene Arten zu berechen. Erst wird eine lineare Funktion f genommen, die durch den Ursprung geht. Diese teilt den ersten Quadranten in zwei Kegel auf. Die Gitterpunkttransformationen dieser Kegel kann berechnet werden. Die zweite Art, die Gitterpunkttransformation des ersten Quadranten zu berechnen, ist eine direkte Berechnung. Beide Darstellungen können verglichen werden; dieser Vergleich führt auf das Reziprozitätsgesetz von Carlitz–Polynomen. Kapitel 7 nimmt die grundlegende Idee auf, verallgemeinert diese aber. Hierbei wird der Ursprung verschoben und f ist eine beliebige lineare Funktion, die im verschobenen Ursprung startet. Berechnet man nun die Gitterpunkttransformation des verschobenen ersten Quadranten auf zwei verschiedene Weisen, bekommt man ein Reziprozitätsgesetz für Rademacher–Carlitz–Polynome, vergleiche Theorem 1.4. Der dritte Abschnitt von Kapitel 7 zeigt, dass das Reziprozitätsgesetz von Carlitz–Polynomen direkt aus Theorem 1.4 folgt. Hieraus ergibt sich als Korollar des Korollars das Reziprozitätsgesetz für Dedekind–Summen. Im vierten Abschnitt wird der in Kapitel 6 gezeigte Zusammenhang zwischen Rademacher–Carlitz–Polynomen und Rademacher–Dedekind–Summen ausgenutzt, um aus dem Theorem 1.4 ein Reziprozitätsgesetz für Rademacher–Dedekind–Summen herzuleiten, vergleiche Theorem 7.10. Hierbei werden mehrmals auf den ersten Abschnitt des siebten Kapitels verwiesen, der sich damit beschäftig, kompakte Darstellungen für gewissen Summen zu finden. Als kleiner Bonus ergibt sich eine kompakte Formel für Rademacher–Dedekind–Summen der Form $r_{-n}(1, \alpha)$.

Erklärung

Hiermit erkläre ich, dass ich die Arbeit selbstständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die diesen Quellen und Hilfsmitteln wörtlich oder sinngemäß entnommenen Ausführungen als solche kenntlich gemacht habe. Die Arbeit wurde keiner anderen Prüfungsbehörde vorgelegt.

Würzburg, den 21.06.2013 ....