A Geometric Approach to Carlitz-Dedekind Sums

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *A Geometric Approach to Carlitz-Dedekind Sums* by Asia R Matthews and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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A Carlitz polynomial is a polynomial generalization of the Dedekind sum, which in turn is an arithmetic sum playing a central role in various mathematical areas, such as theta functions, group actions on manifolds, and integer-point enumeration in polytopes. The most important property of any Dedekind-like sum is reciprocity which Carlitz proved algebraically for his polynomials. In this paper we give a geometric proof of Carlitz reciprocity and derive Dedekind reciprocity from our result. This approach gives rise to alternate geometric pictures from which we get a new version of Carlitz reciprocity and some new theorems. Finally, using Brion's decomposition theorem for lattice points in polyhedra, we discover two new theorems relating Carlitz sums to the generating function of two and three-dimensional simplices. In three dimensions we rederive the Mordell-Pommersheim theorem, which marks the first appearance of Dedekind sums in Ehrhart polynomials.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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My best friend Rueben: thank you for understanding that I had to do it, without really understanding what I was doing. Matthias, my advisor in Mathematics as well as Life Matters: thank you for leading the way. My thesis committee. My professors and teachers along the way who have made Math my passion. My Girlfriends, all. My Baba. My family (including all close, far, extended, half, lawful, unlawful, and undiscovered) for being my family. You are all ingredients in this tasty recipe.

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Introduction

Reciprocity is a very useful mathematical tool that, in many cases, reduces a complex argument to a trivial one. We consider two important reciprocity laws in this paper; Dedekind reciprocity and Carlitz reciprocity.

In the 1880's Richard Dedekind introduced what is now known as a *Dedekind sum*.

Definition 1.1 (Dedekind sum)**.** For positive integers a and b,

$$\mathrm{s}(a,b) := \sum_{k=1}^{b-1} \left(\left(\frac{ka}{b}\right)\right) \left(\left(\frac{k}{b}\right)\right),$$

where

$$((x)) = \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \, , \\ 0 & \text{if } x \in \mathbb{Z} \, . \end{cases} \tag{1.1}$$

Here, the *greatest integer function* $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. The sum $\mathrm{s}(a, b)$ was introduced to study another function that Dedekind was interested in, the *Dedekind eta function* defined on the upper half-plane of the complex numbers as $\eta(\tau) = \mathbf{e}^{\frac{\pi\imath\tau}{12}} \Pi_{n=1}^{\infty} (1 - \mathbf{e}^{2\pi\imath n\tau})$. Dedekind sums are used primarily in the fields of analytic number theory and discrete geometry and are also found in combinatorics, topology, algebraic number theory, algorithmic complexity, and continued fractions.

The Dedekind sum has no closed form and can take a priori quite a long time to compute for large values of b. This is a property of the function that has the potential to make it devastatingly difficult to work with. Fortunately, Dedekind proved the following reciprocity theorem that eliminated this circumstance and which has had profound implications in any field in which Dedekind sums appear.

Theorem 1.1 (Dedekind reciprocity)**.** *Let a and b be relatively prime positive integers. Then*

$$\mathrm{s}\,(a, b) + \mathrm{s}\,(b, a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right).$$

The simple act of switching the parameters in the Dedekind sum and then adding the two sums together gives a very simple and symmetric closed-form result which reduces the computations to a much more practical length of time. In fact, this is one of the reasons that the theorem is so noteworthy. A priori, it takes about b steps to compute a single Dedekind sum, $\mathrm{s}(a,b)$, which can take quite a long time if $b = 1000$, for example. On the other hand, using the reciprocity theorem, it takes about $\log_2 b$ steps: for $b = 1000$, about ten steps! This is due to the property of Dedekind sums that $\mathrm{s}(a,b) = \mathrm{s}(a \bmod b, b)$ where $a \bmod b$ is the remainder of a when divided by b. The process is similar to the Euclidean algorithm and takes the same time to compute. For example, using this property of the modulo we have $\mathrm{s}(381,125) = \mathrm{s}(6,125)$. Then using reciprocity we reduce this to $\mathrm{s}(125,6) = \mathrm{s}(5,6)$.

Leonard Carlitz studied various generalizations of the classical Dedekind sums. He is said to be one of the most prolific mathematicians of all time because he published over 700 research papers. *Carlitz polynomials* are of the following form.

Definition 1.2 (Carlitz polynomial)**.** For indeterminates u and v, and positive integers a and b, define

$$\mathrm{c}(u,v;a,b) := \sum_{k=1}^{a-1} u^{k-1} v^{\left\lfloor \frac{kb}{a} \right\rfloor}.$$

These polynomials arose while Carlitz was studying Dedekind sums through Bernoulli polynomials. Carlitz showed [6] that these polynomials are related by a *reciprocity law* similar to that of Dedekind sums.

Theorem 1.2 (Carlitz reciprocity)**.** *If u and v are indeterminates, and a and b are* *relatively prime positive integers, then*

$$(u-1)\,\mathrm{c}(u,v;a,b)+(v-1)\,\mathrm{c}(v,u;b,a)=u^{a-1}v^{b-1}-1. \tag{1.2}$$

The theorem was proved algebraically by Carlitz in 1975, but first appeared in this form in a paper by Berndt and Dieter [4] in 1982. It is interesting to note that the two reciprocity laws that we have mentioned are so closely related that it is possible to algebraically obtain Dedekind reciprocity from Carlitz reciprocity. Before proving this result, we note a few useful identities.

Lemma 1.3. *Let* $\{x\} := x - \lfloor x \rfloor$ *denote the fractional-part function. If* a *and* b *are relatively prime positive integers, then*

$$\sum_{k=1}^{b-1} k = \frac{b(b-1)}{2}, \tag{1.3}$$

$$\sum_{k=1}^{b-1} k^2 = \frac{b(b-1)(2b-1)}{6}, \tag{1.4}$$

$$\sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}, \tag{1.5}$$

$$\sum_{k=1}^{b-1} \left\{ \frac{ka}{b} \right\} = \frac{b-1}{2}, \tag{1.6}$$

$$\sum_{k=1}^{b-1} \left\{ \frac{ka}{b} \right\}^2 = \frac{(b-1)(2b-1)}{6b}. \tag{1.7}$$

Proof. The first two identities are trivial. The sum in equation (1.5) can be interpreted as the number of integer points between the lines $y = 0$ and $y = \frac{a}{b}x$ for $x = 1, 2, \ldots, b-1$. Draw the rectangle with vertices $(0,0), (a,0), (0,b)$, and (a,b). There are exactly $(a-1)(b-1)$ integer points in the interior of this rectangle. If we draw a line between the origin and the point (a,b), we decompose the rectangle into the triangle in which we are interested, and a similar triangle. Because a and b are relatively prime, there are no integer points along the line between the points $(0,0)$ and (a,b). Therefore, there are $\frac{(a-1)(b-1)}{2}$ integer points in each triangle and we have proved equation (1.5). To show (1.6) we substitute the fractional-part function into equation (1.5) as follows:

$$\frac{(a-1)(b-1)}{2} = \sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor = \sum_{k=1}^{b-1} \left(\frac{ka}{b} - \left\{ \frac{ka}{b} \right\} \right) = \frac{a(b-1)}{2} - \sum_{k=1}^{b-1} \left\{ \frac{ka}{b} \right\}.$$

Then, using the identity in (1.3) we have proved (1.6). Note that (1.6) implies that $\sum_{k=1}^{b-1} \left\{ \frac{ka}{b} \right\} = \frac{b(b-1)}{2b} = \sum_{k=1}^{b-1} \frac{k}{b} = \sum_{k=1}^{b-1} \left\{ \frac{k}{b} \right\}$. Because the function $\sum_{k=1}^{b-1} \left\{ \frac{ka}{b} \right\}$ is periodic with period b and $(a,b) = 1$, the sum over a complete period is just a permutation of $\sum_{k=1}^{b-1} \left\{ \frac{k}{b} \right\}$. We use this fact to prove equation (1.7) by noting that $\sum_{k=1}^{b-1} \left\{ \frac{ka}{b} \right\}^2 = \sum_{k=1}^{b-1} \left\{ \frac{k}{b} \right\}^2 = \sum_{k=1}^{b-1} \left(\frac{k}{b} \right)^2 = \frac{1}{b^2} \sum_{k=1}^{b-1} k^2$. This along with equation (1.4) gives the result. □

Lemma 1.4. *If a and b are relatively prime positive integers, then*

$$\mathrm{s}(a,b) = \frac{1}{b} \sum_{k=1}^{b-1} k \left\{ \frac{ka}{b} \right\} - \frac{b-1}{4}, \tag{1.8}$$

and, equivalently,

$$\mathrm{s}(a,b) = \frac{1}{b}\sum_{k=1}^{b-1} k \left\lfloor \frac{ka}{b} \right\rfloor - \frac{a(b-1)(2b-1)}{6b} - \frac{b-1}{4}. \tag{1.9}$$

Proof. Using Equation (1.1), we replace the sawtooth function in the Dedekind sum with the fractional-part function as follows:

$$\begin{aligned}
\mathrm{s}\left(a,b\right) &= \sum_{k=1}^{b-1} \left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right) = \sum_{k=1}^{b-1}\left(\left\{\frac{ka}{b}\right\} - \frac{1}{2}\right)\left(\left\{\frac{k}{b}\right\} - \frac{1}{2}\right) \\
&= \sum_{k=1}^{b-1}\left(\left\{\frac{ka}{b}\right\} - \frac{1}{2}\right)\left(\frac{k}{b} - \frac{1}{2}\right) \\
&= \frac{1}{b}\sum_{k=1}^{b-1} k\left\{\frac{ka}{b}\right\} - \frac{1}{2}\sum_{k=1}^{b-1}\left\{\frac{ka}{b}\right\} - \frac{1}{2b}\sum_{k=1}^{b-1} k + \sum_{k=1}^{b-1}\frac{1}{4} \\
&= \frac{1}{b}\sum_{k=1}^{b-1} k\left\{\frac{ka}{b}\right\} - \frac{b-1}{4}.
\end{aligned}$$

The third equality is obtained by observing that for k ranging from 1 to $b-1$, the fractional-part function $\left\{\frac{k}{b}\right\}$ in the second sum is equal to $\frac{k}{b}$. After expanding, we use a few of the identities from Lemma 1.3 in the final equality. Converting the fractional-part function in (1.8) to the floor function, we have (1.9). $\square$

Proof that Theorem 1.2 implies Theorem 1.1. By applying the operators $u\,\partial u$ twice and then $v\,\partial v$ once to

$$(u-1)\sum_{k=1}^{a-1} u^{k-1} v^{\left\lfloor \frac{kb}{a} \right\rfloor} + (v-1)\sum_{k=1}^{b-1} v^{k-1} u^{\left\lfloor \frac{ka}{b} \right\rfloor} = u^{a-1}v^{b-1} - 1$$

and then setting $u = v = 1$, we obtain

$$\sum_{k=1}^{a-1} k^2 \left\lfloor \frac{kb}{a} \right\rfloor - \sum_{k=1}^{a-1}(k-1)^2 \left\lfloor \frac{kb}{a} \right\rfloor + \sum_{k=1}^{b-1} k \left\lfloor \frac{ka}{b} \right\rfloor - \sum_{k=1}^{b-1}(k-1) \left\lfloor \frac{ka}{b} \right\rfloor^2 = (a-1)^2(b-1).$$

This reduces to

$$2\sum_{k=1}^{a-1} k \left\lfloor \frac{kb}{a} \right\rfloor + \sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor^2 = \frac{(a-1)(b-1)}{2} + (a-1)^2(b-1),$$

which is a relation of greatest integer functions. Now we convert the greatest-integer functions to fractional-part functions using identities from Lemma 1.3:

$$2\sum_{k=1}^{a-1} k \left(\frac{kb}{a} - \left\{ \frac{kb}{a} \right\} \right) + \sum_{k=1}^{b-1} \left(\frac{ka}{b} - \left\{ \frac{ka}{b} \right\} \right)^2 = \frac{(a-1)(b-1)}{2} + (a-1)^2(b-1),$$

which we expand to get

$$2\sum_{k=1}^{a-1} k \left\{ \frac{kb}{a} \right\} + 2\frac{a}{b}\sum_{k=1}^{b-1} k \left\{ \frac{ka}{b} \right\} = \frac{2b}{a}\sum_{k=1}^{a-1} k^2 + \frac{a^2+1}{b^2}\sum_{k=1}^{b-1} k^2 - \frac{1}{2}(a-1)(b-1)\left(2a-1\right).$$

Multiplying the equation by $6b$ and simplify the right-hand side yields

$$\begin{aligned}
&\frac{12b^2}{a}\sum_{k=1}^{a-1} k^2 + \frac{6\left(a^2+1\right)}{b}\sum_{k=1}^{b-1} k^2 - 3b(a-1)(b-1)\left(2a-1\right) \\
&\quad = 2b^2(a-1)(2a-1) + a^2(b-1)(2b-1) + (b-1)(2b-1) \\
&\qquad - 3b(a-1)(b-1)\left(2a-1\right) = 3a^2b + 3ab^2 + a^2 + b^2 - 9ab + 1.
\end{aligned}$$

and thus we have the conversion of Carlitz reciprocity to fractional-part functions, namely,

$$12a\sum_{k=1}^{b-1} k\left\{\frac{ka}{b}\right\} + 12b\sum_{k=1}^{a-1} k\left\{\frac{kb}{a}\right\} = 3a^2b + 3ab^2 + a^2 + b^2 - 9ab + 1. \tag{1.10}$$

Now we turn our attention to Dedekind reciprocity. Using Lemma 1.4, we convert the greatest-integer functions in the left-hand side of Equation (1.1) to fractional-part functions:

$$\begin{aligned}
&\frac{1}{b}\sum_{k=1}^{b-1} k\left\{\frac{ka}{b}\right\} - \frac{b-1}{4} + \frac{1}{a}\sum_{k=1}^{a-1} k\left\{\frac{kb}{a}\right\} - \frac{a-1}{4} = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) \\
&\Rightarrow 12a\sum_{k=1}^{b-1} k\left\{\frac{ka}{b}\right\} + 12b\sum_{k=1}^{a-1} k\left\{\frac{kb}{a}\right\} = 3a^2b + 3ab^2 + a^2 + b^2 - 9ab + 1.
\end{aligned}$$

Multiplying the first equation by $12ab$ and rearranging the terms to isolate the fractional-part functions on the left-hand side of the equation gives the same result as (1.10). $\square$

In this paper we will also consider a more generalized version of the Carlitz polynomial.

Definition 1.3. The generalization of the Carlitz polynomial, where $u_1, u_2, \ldots, u_n$ are indeterminates and $a_1, a_2, \ldots, a_n$ are positive integers, is defined as the polynomial

$$\mathrm{c}\left(u_1, u_2, \ldots, u_n; a_1, a_2, \ldots, a_n\right) := \sum_{k=1}^{a_1-1} u_1^{k-1} u_2^{\left\lfloor \frac{ka_2}{a_1} \right\rfloor} u_3^{\left\lfloor \frac{ka_3}{a_1} \right\rfloor} \cdots u_n^{\left\lfloor \frac{ka_n}{a_1} \right\rfloor}.$$

Berndt and Dieter proved a general polynomial reciprocity theorem [4, Theorem 5.1] for Carlitz polynomials in n indeterminates. We use the slightly more specific form given in [1, Theorem 1.2] which states the following.

Theorem 1.5 (Berndt–Dieter)**.** *If $a_1, a_2, \ldots, a_n$ are pairwise relatively prime positive integers, then*

$$\begin{aligned}
&(u_1 - 1)\,\mathrm{c}\left(u_1, u_2, \ldots, u_n; a_1, a_2, \ldots, a_n\right) \\
&\qquad + (u_2 - 1)\,\mathrm{c}\left(u_2, u_3, \ldots, u_n, u_1; a_2, a_3, \ldots, a_n, a_1\right) \\
&\qquad + \cdots + (u_n - 1)\,\mathrm{c}\left(u_n, u_1, \ldots, u_{n-1}; a_n, a_1, \ldots, a_{n-1}\right) \\
&\qquad = u_1^{a_1-1} u_2^{a_2-1} \cdots u_n^{a_n-1}.
\end{aligned}$$

In the following chapters we will give a novel proof of Theorem 1.2 and of its higher-dimensional analog, Theorem 1.5. Our approach will lead us to some new reciprocity theorems and relations for Carlitz polynomials. Finally, we derive a relation between Carlitz polynomials and certain familiar geometric figures by realizing Carlitz sums as generating functions of polyhedra.

Chapter 2

Generating Functions and Geometry

2.1 A Two-Dimensional Example of Generating Functions

In this paper we take a geometric approach to Carlitz reciprocity. It was recently observed that Carlitz polynomials show up in the generating functions of certain geometric objects. A geometric proof of Carlitz reciprocity was introduced by Beck in [1] in 2006. Here we present a novel way of realizing Carlitz polynomials in a geometric setting.

Looking at the elements that make up a Carlitz polynomial $\mathrm{c}(u, v; a, b)$, it is natural to associate the variables u and v with the x and y axes, and the parameters a and b with the point (a, b) in the first quadrant. From there we draw a ray from the origin through the point (a, b) and we have decomposed the first quadrant into two cones.

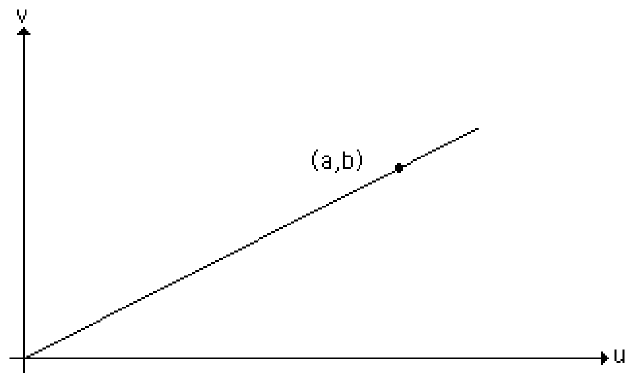

Figure 2.1: A ray in the first quadrant.

These *pointed cones* are defined as the intersection of finitely many half-spaces that intersect in exactly one point, the vertex. The pointed cones defined above are

$$K_1 = \{\lambda_1(0,1) + \lambda_2(a,b) : \lambda_1, \lambda_2 \geq 0\} \subset \mathbb{R}^2,$$
$$K_2 = \{\lambda_1(1,0) + \lambda_2(a,b) : \lambda_1 \geq 0, \lambda_2 > 0\} \subset \mathbb{R}^2,$$

such that K_1 is closed and K_2 is half-open.

Proposition 2.1. *Define* $Q_1^{(2)} := \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$*, the first quadrant in the plane. Then* $K_1 \cup K_2 = Q_1^{(2)}$ *and* $K_1 \cap K_2 = \emptyset$*.*

Proof. It is clear that $K_1 \cup K_2 = Q_1^{(2)}$. Suppose by contradiction that $(u,v) \in K_1 \cap K_2$. Note that, $(u,v) \in K_1$ implies that $(u,v) = (\lambda a, \lambda_2 + \lambda b)$ for $\lambda, \lambda_2 \geq 0$ and

$(u, v) \in K_2$ implies that $(u, v) = (\lambda_1 + \lambda a, \lambda b)$ for $\lambda \geq 0, \lambda_1 > 0$. Then $\lambda a = \lambda_1 + \lambda a$ and hence $\lambda_1 = 0$ which is a contradiction. $\square$

The process now requires that we list the integer points in each of these cones. The method that we employ uses *generating functions* to encode integer points in a set $S \subset \mathbb{R}^d$. Instead of vectors, each coordinate point is listed as the multidegree of a monomial. For example, the point (a, b) is encoded in the monomial $u^a v^b$ in the indeterminates u and v. This clever method is helpful for it raises the discrete set of points to a continuous function, thereby making it easier to work with.

We define a *polyhedron* as the intersection of finitely many half-spaces, and call it a *rational polyhedron* if the half-spaces defining the polyhedron can be written in terms of rational parameters.

Definition 2.1. If S is a rational polyhedron, $\sigma_S \in \mathbb{R}^d$ is called the **integer-point transform** of S, where

$$\sigma_S(\mathbf{z}) = \sigma_S(z_1, z_2, \cdots z_d) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}.$$

We use the notation $\mathbf{m} := (m_1, \ldots, m_d)$ and $\mathbf{z}^{\mathbf{m}} := z_1^{m_1} \cdots z_d^{m_d}$.

The two cones that we created in the first quadrant are rational cones because the two rays that bound each cone go through rational points. We tile the cone with copies of the *fundamental parallelogram*. By tiling, we are simplifying the infinite list of integer points in each cone to just translates of the lattice points in the fundamental parallelogram of each cone. Consistent with our definition of the cones K_1 and K_2, we define the fundamental parallelograms

$$\Pi_1 = \{\lambda_2 \mathbf{e}_2 + \lambda(a,b) : 0 \le \lambda_2, \lambda < 1\},$$

$$\Pi_2 = \{\lambda_1 \mathbf{e}_1 + \lambda(a,b) : 0 < \lambda_1 \le 1, 0 \le \lambda < 1\}.$$

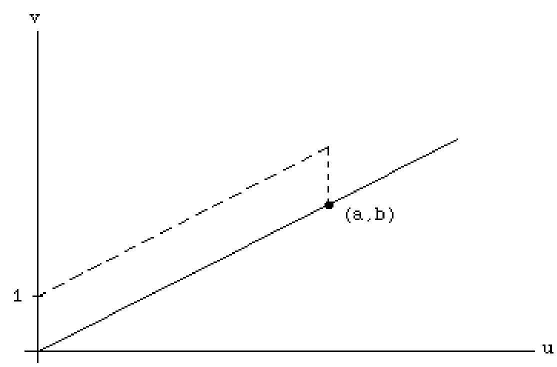

Figure 2.2: The fundamental parallelogram Π_1.

In order to illustrate the process of writing the integer-point transform of each cone, consider the fundamental parallelogram Π_1. We begin by listing the translates of the vertices of Π_1. These are the non-negative integer combinations of the generators $(0,1)$ and (a,b) and so we list them using the generating function:

$$\sum_{\substack{(m_1,m_2)=j(0,1)+k(a,b)\\ j,k\geq 0}} u^{m_1}v^{m_2} = \sum_{j\geq 0}(u^0v^1)^j \sum_{k\geq 0}(u^a v^b)^k = \frac{1}{(1-v)(1-u^a v^b)}\,. \tag{2.1}$$

Now we list the integer points in the interior of Π_1. Because the parallelogram is half-open, it contains the point $(0,0)$. Recall that the lattice point (a,b) is the first integer point from the origin along the ray (a,b). Because this property remains fixed for any integer translation of the ray in the plane, the point $(a,b+1)$ is the first integer point along the ray extending from $(0,1)$ through $(a,b+1)$. This means that there are no integer points on the boundary lines of the parallelogram, except at the vertices. Now, for each point along the horizontal axis between 0 and a, the vertical height of the fundamental parallelogram is one unit. Therefore, at each of these values along the horizontal axis, there will be exactly one interior lattice point in the parallelogram. We use the greatest integer function to denote this point. The vertical integer point will lie between the lines $y = \frac{b}{a}x$ and $y = \frac{b}{a}x + 1$, for these lines describe two sides of the parallelogram. Therefore, the vertical integer point will have the value $y = \left\lfloor \frac{b}{a}x + 1 \right\rfloor = \left\lfloor \frac{b}{a}x \right\rfloor + 1$ for each x and we have that the set of interior lattice points in Π_1 is the set $\left\{(0,0), \left(x, \left\lfloor \frac{b}{a}x \right\rfloor + 1\right) : 1 \leq x \leq a-1, x \in \mathbb{Z}\right\}$. Putting this information together with (2.1) we have

$$\sigma_{K_1}(u,v) = \left(u^0v^0 + \sum_{k=1}^{a-1} u^k v^{\left\lfloor \frac{kb}{a} \right\rfloor + 1}\right) \sum_{\substack{(m_1,m_2)=j(0,1)+k(a,b)\\ j,k\geq 0}} u^{m_1}v^{m_2}$$

$$= \frac{1+\sum_{k=1}^{a-1} u^k v^{\left\lfloor \frac{kb}{a} \right\rfloor + 1}}{(1-v)(1-u^a v^b)} = \frac{1+uv\,\mathrm{c}\,(u,v;a,b)}{(v-1)\,(u^a v^b - 1)}\,.$$

We obtain the integer-point transform for K_2 in the same way, careful to adjust our sums for the half-open cone: Π_2 does not contain the point $(0,0)$. Instead, the point that will tile the vertices of the paralellogram is the point $(1,0)$ which is on the closed side of the cone. Thus,

$$\sigma_{K_2}(u,v) = \frac{u+\sum_{k=1}^{b-1} v^k u^{\left\lfloor \frac{ka}{b} \right\rfloor + 1}}{(1-u)(1-u^b v^a)} = \frac{u+uv\,\mathrm{c}\,(v,u;b,a)}{(u-1)\,(u^b v^a - 1)}\,. \tag{2.2}$$

2.2 An Example in 3 Dimensions

This construction is more complex for dimensions > 2, and is also much more difficult to depict, so we illustrate it in three dimensions before generalizing. Let $Q_1^{(3)}$ denote the first (non-negative) orthant in 3-space. In three dimensions, for coprime positive integers a, b, c, draw an infinite ray from the origin through the point (a, b, c). Now we decompose $Q_1^{(3)}$ into three (half-open) vertex cones,

$$K_1 = \{\lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 + \lambda(a,b,c) : \lambda_2, \lambda_3, \lambda \geq 0\},$$

$$K_2 = \{\lambda_1 \mathbf{e}_1 + \lambda_3 \mathbf{e}_3 + \lambda(a,b,c) : \lambda_1, \lambda_3, \lambda \geq 0\},$$

$$K_3 = \{\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda(a,b,c) : \lambda_1, \lambda_2, \lambda \geq 0\}.$$

The fundamental parallelepiped

$$\Pi_3 = \{\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda\,(a,b,c) : 0 \leq \lambda_1, \lambda_2 < 1, 0 \leq \lambda < 1\}$$

corresponding to the cone K_3 is bounded by the three rays that extend from the origin through the points $(1,0,0)$, $(0,1,0)$, and (a,b,c).

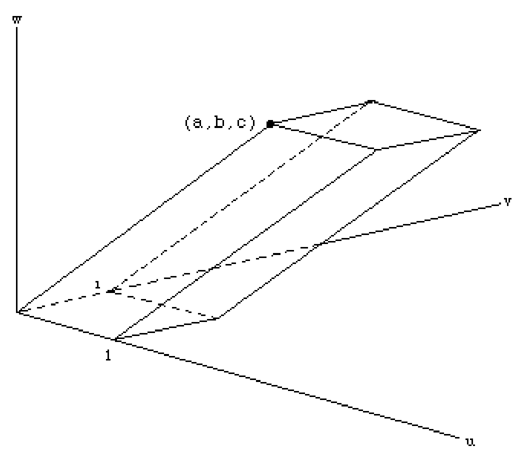

Figure 2.3: The fundamental parallelepiped Π_3.

As in the two-dimensional case, the integer translates of the vertices of Π_3 gives the denominator for the integer-point transform of K_3:

$$\sum_{\substack{(m_1,m_2,m_3)=i(1,0,0)+j(0,1,0)+k(a,b,c)\\ i,j,k\geq 0}} u^{m_1}v^{m_2}w^{m_3} = \frac{1}{(1-u)\,(1-v)\,(1-u^a v^b w^c)}\,.$$

We multiply this by the embedded list of interior points of Π_3 in order to generate the infinite list of lattice points of K_3. To get the interior points of Π_3, consider a "slice" of the parallelepiped at $w = k$ for the integer $0 < k < c$ as in Figure 2.3.

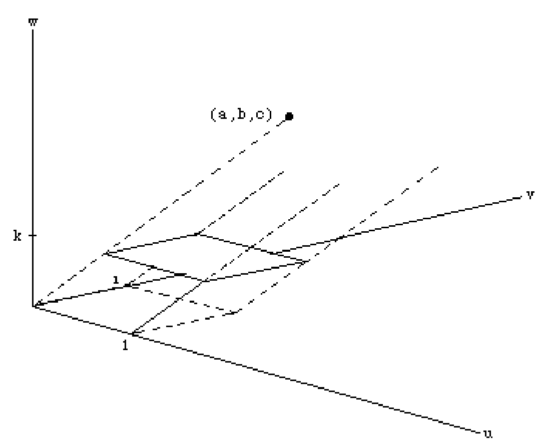

Figure 2.4: A slice of the fundamental parallelepiped.

Note that the parallelogram at this "slice" of Π_3 has area 1. Also note that it contains no lattice points on its edges or vertices. Π_3 therefore contains exactly one lattice point at $w = k$. It is immediately clear that this point has coordinates

$\left(\left\lfloor\frac{ka}{c}\right\rfloor+1,\left\lfloor\frac{kb}{c}\right\rfloor+1,k\right)$. Hence, the interior lattice points of Π_3 can be listed as the set of these points for each integer $1 \leq k \leq c-1$.

The final step of this process is to identify the edges of the parallelepiped that are open and those that are closed in order to list the one vertex point that will tile the vertices of Π_3. For the cone K_3, the closed face is the one along the u_1 - u_2 plane, though the edges along each of those axes are open. Therefore, the point $(1,1,0)$ is contained in Π_3 and will tile the vertices of the parallelepipeds. Thus,

$$\sigma_{K_3}(u,v,w)=\frac{uv+\sum_{k-1}^{w-1}u^{\left\lfloor\frac{ka}{c}\right\rfloor+1}v^{\left\lfloor\frac{kb}{c}\right\rfloor+1}c^{k}}{(1-u)\,(1-v)\,(1-u^{a}v^{b}w^{c})}=\frac{uv\,\mathrm{c}\,(w,u,v;c,a,b)}{(1-u)\,(1-v)\,(1-u^{a}v^{b}w^{c})}\,.$$

Chapter 3

Carlitz Reciprocity

3.1 2 Dimensions

We now have an expression that lists the lattice points in each of our cones, so that in fact we have a list of the lattice points in the first quadrant $Q_2^{(1)}$ of the plane. Note that we can also write the integer-point transform of the first quadrant directly for it is also a rational cone. The integer-point transform of $Q_1^{(2)}$ is

$$\sigma_{Q_1^{(2)}}(u,v) = \frac{1}{(1-u)(1-v)} \, .$$

Proof of Theorem 1.2. By Proposition 2.1 we have that $K_1 \cup K_2 = Q_1^{(2)}$ and $K_1 \cap K_2 = \emptyset$. This means that we have created two different expressions $\sigma_{K_1} + \sigma_{K_2}$ and $\sigma_{Q_1^{(2)}}$ that list the integer points in the first quadrant. Naturally we set these two expressions equal to each other to obtain

$$\frac{1 + uv\,\mathrm{c}\,(u, v; a, b)}{(v-1)\,(u^a v^b - 1)} + \frac{u + uv\,\mathrm{c}\,(v, u; b, a)}{(u-1)\,(u^a v^b - 1)} = \frac{1}{(u-1)(v-1)}.$$

To simplify, multiply the equation by $(u-1)(v-1)(u^a v^b - 1)$ and we have:

$$(u-1) + uv(u-1)\,\mathrm{c}\,(u, v; a, b) + u(v-1) + uv(v-1)\,\mathrm{c}\,(v, u; b, a) = u^a v^b - 1$$

$$\Rightarrow uv\left[(u-1)\,\mathrm{c}(u, v; a, b) + (v-1)\,\mathrm{c}(v, u; b, a)\right] = u^a v^b - 1 - (u-1) - u(v-1)$$

$$\Rightarrow (u-1)\,\mathrm{c}(u, v; a, b) + (v-1)\,\mathrm{c}(v, u; b, a) = u^{a-1} v^{b-1} - 1.$$

$\square$

Thus we have constructed a novel proof of Theorem 1.2, the polynomial reciprocity theorem in two dimensions due to Carlitz, using a geometric picture.

3.2 n Dimensions

It is now possible to generalize this geometric approach to Carlitz reciprocity. We will prove Theorem 1.5 in a geometric manner similar to the two-dimensional case. Namely, we construct a single ray in n-dimensional space, and then we decompose the non-negative orthant into n cones, $K_1, K_2, \ldots, K_n$. Again we avoid the issue of over-counting along the common faces of the cones by defining our cones to be half-open. Let $\mathbf{a} := (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ such that $(a_i, a_j) = 1$ whenever $i \neq j$.

Then we define

$$K_1 = \{\lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 + \cdots + \lambda_n \mathbf{e}_n + \lambda \mathbf{a} : \lambda_i \geq 0 \ \forall\ i = 2, 3, \ldots, n, \lambda \geq 0\},$$

$$K_2 = \{\lambda_1 \mathbf{e}_1 + \lambda_3 \mathbf{e}_3 + \cdots + \lambda_n \mathbf{e}_n + \lambda \mathbf{a} : \lambda_i \geq 0 \ \forall\ i > 2, \lambda_1 > 0, \lambda \geq 0\},$$

$$\vdots$$

$$K_j = \left\{ \begin{array}{l} \lambda_1 \mathbf{e}_1 + \cdots + \lambda_{j-1} \mathbf{e}_{j-1} + \lambda_{j+1} \mathbf{e}_{j+1} + \cdots + \lambda_n \mathbf{e}_n + \lambda \mathbf{a} : \\ \lambda_i \geq 0 \ \forall\ i > j, \lambda_1 > 0 \ \forall\ i < j, \lambda \geq 0 \end{array} \right\},$$

$$\vdots$$

$$K_n = \{\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_{n-1} \mathbf{e}_{n-1} + \lambda \mathbf{a} : \lambda_i > 0 \ \forall\ i = 1, 2, \ldots, n, \lambda \geq 0\}.$$

Proposition 3.1. *Let* $Q_1^{(n)} := \{\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n : z_i \geq 0 \ \forall\, i = 1, 2, \ldots, n\}$, *the first orthant in n-space. Then* $\bigcup_{i=1}^n K_i = Q_1^{(n)}$ *and* $\bigcap_{i=1}^n K_n = \emptyset$.

Proof. It is clear that $\bigcup_{i=1}^n K_i = Q_1^{(n)}$. Suppose by contradiction that $\mathbf{z} \in \bigcap_{i=1}^n K_n$. Note that for $j \neq k$, where $j, k = 1, 2, \ldots, n$, if $\mathbf{z} \in K_j$ then

$$\mathbf{z} = (\lambda_1 + \lambda a_1, \lambda_2 + \lambda a_2, \ldots \lambda_{j-1} + \lambda a_{j-1}, \lambda a_j, \lambda_{j+1} + \lambda a_{j+1}, \ldots, \lambda_n + \lambda a_n),$$

where $\lambda_i \geq 0 \,\forall\, i > j$ and $\lambda_1 > 0 \,\forall\, i < j$. Also, $\mathbf{z} \in K_k$ implies that

$$\mathbf{z} = (\gamma_1 + \gamma a_1, \gamma_2 + \gamma a_2, \ldots \gamma_{k-1} + \gamma a_{k-1}, \gamma a_k, \gamma_{k+1} + \gamma a_{k+1}, \ldots, \gamma_n + \gamma a_n),$$

where $\gamma_i \geq 0 \,\forall\, i > k$ and $\gamma_i > 0 \,\forall\, i < k$. Suppose, without loss of generality, that $j < k$. Then $\lambda a_j = \gamma_j + \gamma a_j$ and $\gamma a_k = \lambda_k + \lambda a_k$. Note that $\lambda \neq \gamma$ since otherwise $\gamma a_j = \gamma_j + \gamma a_j$ and $\lambda a_k = \lambda_k + \lambda a_k$ imply that $\gamma_j = 0$ and $\lambda_k = 0$. Therefore, $a_j = \frac{\gamma_j}{\lambda - \gamma}$ and $a_k = \frac{-\lambda_k}{\lambda - \gamma}$. If $\lambda > \gamma$,then $a_k < 0$ which is a contradiction. Now suppose that $\lambda < \gamma$. Then $a_j < 0$. Thus K_j and K_k are disjoint for all $j, k = 1, 2, \ldots, n$ and we have shown that the set $\{K_i : i = 1, 2, \ldots, n\}$ is a disjoint decomposition of the first orthant. □

We want to construct the integer-point transform for each cone $K_1, K_2, \ldots, K_n$, as well as for $Q_1^{(n)}$. Because we are now in n dimensions, we identify the *fundamental parallelepiped* for each cone by the n rays that describe it. The construction of this generating function proceeds as in the two-dimensional case where the process is broken down to three steps:

1. List the vertices of the translates of the fundamental parallelepiped Π_j and encode these in the generating function.

2. Encode the list of the interior points in Π_j.

3. Determine the vertex of the half-open parallelepiped that will tile all of the vertices and add this point to the list of interior points.

Proof of Theorem 1.5. For each parallelepiped $\Pi_j, j = 1, \ldots, n$, we first compute the integer translates of the vertices: the non-negative integer combination of all unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, and the vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and *not* the unit vector $\mathbf{e}_j$. If we let $\mathbf{m} \in K_j \cap \mathbb{Z}^n$ be defined through

$$\mathbf{m} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \cdots + k_{j-1}\mathbf{e}_{j-1} + k_{j+1}\mathbf{e}_{j+1} + \cdots + k_n\mathbf{e}_n + k\mathbf{a},\ k_i \geq 0,$$

and let the monomial $\mathbf{u}^{\mathbf{x}} := u_1^{x_1} u_2^{x_2} \cdots u_n^{x_n}$ for all $\mathbf{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$. Then

$$\begin{aligned}\sum_{\mathbf{m}\in\mathbf{K_j}\cap\mathbb{Z}^\mathbf{n}} \mathbf{u}^{\mathbf{m}} &= \frac{1}{(1-u_1)\cdots(1-u_{j-1})\,(1-u_{j+1})\cdots(1-u_n)\,(1-u_1^{a_1}u_2^{a_2}\cdots u_n^{a_n})} \\ &= \frac{1}{(1-\mathbf{u}^{\mathbf{a}})\prod_{\substack{1\leq k\leq n\\ k\neq j}}(1-u_k)}\,.\end{aligned}$$

Note that the interior points at each "slice" of Π_j where $u_j = k$, $k \neq j$, have coordinates

$$\left(\left\lfloor \frac{ka_1}{a_j} \right\rfloor + 1, \ldots, \left\lfloor \frac{ka_{j-1}}{a_j} \right\rfloor + 1, k, \left\lfloor \frac{ka_{j+1}}{a_j} \right\rfloor + 1, \ldots, \left\lfloor \frac{ka_n}{a_j} \right\rfloor + 1\right).$$

Finally, we must determine the vertex which will tile all vertices of translates of Π_j. Because Π_j is closed on all edges of the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{j-1}$, the point $(1, 1, \ldots, 1, 0, \ldots, 0)$, with the first 0 is in the j-th entry, will tile the vertices.

Putting all of this together we obtain the integer-point transform of K_j,

$$\sigma_{K_j}(\mathbf{u}) = \frac{u_1 u_2 \cdots u_{j-1} + \sum_{k=1}^{a_j-1} u_1^{\left\lfloor \frac{ka_1}{a_j} \right\rfloor + 1} \cdots u_{j-1}^{\left\lfloor \frac{ka_{j-1}}{a_j} \right\rfloor + 1} u_j^k u_{j+1}^{\left\lfloor \frac{ka_{j+1}}{a_j} \right\rfloor + 1} \cdots u_n^{\left\lfloor \frac{ka_n}{a_j} \right\rfloor + 1}}{(1-\mathbf{u}^{\mathbf{a}}) \prod_{\substack{1 \le k \le n \\ k \ne j}} (1-u_k)} \tag{3.1}$$

$$= \frac{u_1 u_2 \cdots u_{j-1} + \mathbf{u}^{\mathbf{a}}\, \mathrm{c}\left(u_j, u_{j+1}, \ldots, u_n, u_1, \ldots, u_{j-1}; a_j, a_{j+1}, \ldots, a_n, a_1, \ldots, a_{j-1}\right)}{(1-\mathbf{u}^{\mathbf{a}}) \prod_{\substack{1 \le k \le n \\ k \ne j}} (1-u_k)}. \tag{3.2}$$

We are now ready to establish a geometric proof of Theorem 1.5. We begin with the equation

$$\sigma_{K_1}(\mathbf{u}) + \sigma_{K_2}(\mathbf{u}) + \cdots + \sigma_{K_n}(\mathbf{u}) = \sigma_{K_{Q_1^{(n)}}}(\mathbf{u}).$$

Substituting the expression from (3.1) gives:

$$\sum_{j=1}^{n} \frac{u_1 u_2 \cdots u_{j-1} + \mathbf{u}^{\mathbf{a}}\, \mathrm{c}\left(u_j, u_{j+1}, \ldots, u_n, u_1, \ldots, u_{j-1}; a_j, a_{j+1}, \ldots, a_n, a_1, \ldots, a_{j-1}\right)}{(1-\mathbf{u}^{\mathbf{a}}) \prod_{\substack{1 \le k \le n \\ k \ne j}} (1-u_k)}$$
$$= \frac{1}{\prod_{i=1}^{n} (1-u_i)}.$$

Now, by multiplying the equation by $(1-\mathbf{u}^{\mathbf{a}})$ and isolating the Carlitz polynomials on the left-hand side of the equation we simplify to obtain:

$$
\begin{aligned}
&u_1 u_2 \cdots u_n \sum_{j=1}^{n} (u_j - 1)\,\mathrm{c}\left(u_j, u_{j+1}, \ldots, u_n, u_1, \ldots, u_{j-1}; a_j, a_{j+1}, \ldots, a_n, a_1, \ldots, a_{j-1}\right) \\
&\quad = \mathbf{u}^{\mathbf{a}} - 1 - (u_1 - 1) - u_1 (u_2 - 1) - u_1 u_2 (u_3 - 1) - \cdots - u_1 \cdots u_{n-1} (u_n - 1) \\
&\quad = \mathbf{u}^{\mathbf{a}} - u_1 u_2 \cdots u_n.
\end{aligned}
$$

Dividing by $u_1 u_2 \cdots u_n$ yields Theorem 1.5. $\square$

Chapter 4

Two Rays in Two Dimensions

The geometry used in the previous section is very simple: one ray in space. We now turn to a slightly more complex geometric picture: two rays in space. To simplify further, we consider only the case of two rays in the first quadrant of the plane. Let a, b, c, d be positive integers and draw infinite rays through the points (a, b) and (c, d) in the first quadrant. This construction decomposes the first quadrant into three rational cones.

The integer-point transform of each of the two exterior cones, K_1 and K_3, is easily computed in the same manner as in Chapter 2. However, the cone in the middle, K_2 is bounded on either side by a non-unit vector, and therefore the interior points of the fundamental parallelogram of this cone are not trivial to list. To avoid this problem, we add the constraint that the fundamental parallelogram should have no interior

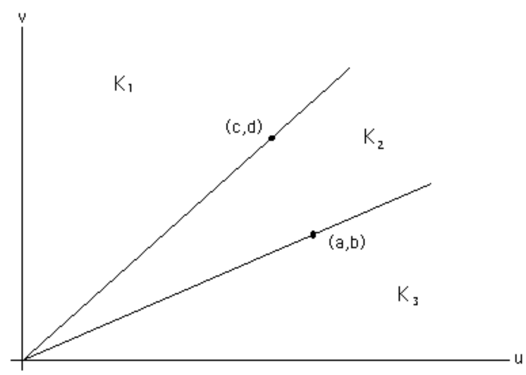

Figure 4.1: Two-ray decomposition of the first quadrant.

lattice points. This is equivalent to saying that it has area 1. The fundamental parallelogram Π_2 is described by the rays (a, b) and (c, d) and therefore has area

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Thus we have the constraint $ad - bc = 1$. This construction leads to the following new theorem.

Theorem 4.1. *If a, b, c, d are such that $ad - bc = 1$, then*

$$\begin{aligned}(u-1)\left(u^a v^b - 1\right) \mathrm{c}\left(u, v; c, d\right) &+ (v-1)\left(u^c v^d - 1\right) \mathrm{c}\left(v, u; b, a\right) \\ &= u^{a+c-1} v^{b+d-1} - u^a v^b - u^c v^d + u^{a-1} v^b + u^c v^{d-1} - u^{a-1} v^{b-1} - u^{c-1} v^{d-1} + 1.\end{aligned}$$

Proof. Again, we prove the theorem geometrically. To simplify our computations, define

$$K_1 = \{\lambda_2 \mathbf{e}_2 + \lambda_{cd}(c,d) : \lambda_2 > 0, \lambda_{cd} \geq 0\} \subset \mathbb{R}^2,$$
$$K_2 = \{\lambda_{ab}(a,b) + \lambda_{cd}(c,d) : \lambda_{ab}, \lambda_{cd} \geq 0\} \subset \mathbb{R}^2,$$
$$K_3 = \{\lambda_1 \mathbf{e}_1 + \lambda_{ab}(a,b) : \lambda_1 > 0, \lambda_{ab} \geq 0\} \subset \mathbb{R}^2,$$

so that K_2 is closed and K_1 and K_3 are half-open. Thus $K_1 \cup K_2 \cup K_3 = Q_1^{(2)}$ and K_1, K_2, K_3 are pairwise disjoint. With the method introduced in Chapter 2, the integer-point transform of each cone becomes the following:

$$\begin{aligned}
\sigma_{K_1}(u,v) &= \frac{v + \sum_{k=1}^{c-1} u^k v^{\left\lfloor \frac{kd}{c} \right\rfloor + 1}}{(1-v)(1-u^c v^d)} = \frac{v + uv\,\mathrm{c}(u,v;c,d)}{(v-1)(u^c v^d - 1)},\\
\sigma_{K_2}(u,v) &= \frac{1}{(1-u^a v^b)(1-u^c v^d)} = \frac{1}{(u^a v^b - 1)(u^c v^d - 1)},\\
\text{and} \quad \sigma_{K_3}(u,v) &= \frac{u + \sum_{k=1}^{b-1} v^k u^{\left\lfloor \frac{ka}{b} \right\rfloor + 1}}{(1-u)(1-u^a v^b)} = \frac{u + uv\,\mathrm{c}(v,u;b,a)}{(u-1)(u^a v^b - 1)}.
\end{aligned}$$

Because the fundamental parallelogram Π_2 has no interior integer points, the generating function σ_{K_2} is relatively simple. Then the sum of the integer-point transforms of each of the cones is equal to the integer-point transform of $Q_1^{(2)}$ so we have

$$\sigma_{K_1}(u,v) + \sigma_{K_2}(u,v) + \sigma_{K_3}(u,v) = \sigma_{Q_1^{(2)}}(u,v).$$

Replacing these with the expressions above we have

$$\frac{v+uv\,\mathrm{c}\,(u,v;c,d)}{(v-1)\,(u^cv^d-1)}+\frac{1}{(1-u^av^b)\,(1-u^cv^d)}+\frac{u+uv\,\mathrm{c}\,(v,u;b,a)}{(u-1)\,(u^av^b-1)}=\frac{1}{(u-1)(v-1)}.$$

By clearing the denominator and isolating the Carlitz sums on the left,

$$\begin{aligned}&(u-1)\left(u^av^b-1\right)\mathrm{c}\,(u,v;c,d)+(v-1)\left(u^cv^d-1\right)\mathrm{c}\,(v,u;b,a)\\&\quad=u^{a+c-1}v^{b+d-1}-u^av^b-u^cv^d+u^{a-1}v^b+u^cv^{d-1}-u^{a-1}v^{b-1}-u^{c-1}v^{d-1}+1.\end{aligned}$$

Thus we have proved Theorem 4.1. □

Recall that we are able to obtain classical Dedekind reciprocity from Carlitz reciprocity. It is interesting to consider what would happen if we performed the same computations on Theorem 4.1. In the classical case, we applied the operators $u\,\partial u$ twice and then $v\,\partial v$ once and then set $u=v=1$. If we perform these exact operations, we obtain a trivial result, but we lose the greatest integer functions and therefore can not obtain, through conversion to the fractional-part function, the Dedekind sums that we are seeking. However, applying the operators $u\,\partial u$ and $v\,\partial v$ twice each, and then setting $u=v=1$ yields a nontrivial result.

Corollary 4.2 (Rademacher)**.** *If a,b,c,d are positive integers such that $ad-bc=1$, then*

$$\mathrm{s}(a,b)+\mathrm{s}(d,c)=-\frac{1}{2}+\frac{1}{12}\left(\frac{a}{b}+\frac{a}{c}+\frac{d}{b}+\frac{d}{c}\right).$$

Proof. Let a, b, c, d be relatively prime positive integers such that $ad - bc = 1$. Derivate Theorem 4.1 using the operators $u\,\partial u$ and $v\,\partial v$ twice each, and then set $u = v = 1$ to obtain:

$$\begin{aligned}
&2a\sum_{k=1}^{c-1}\left\lfloor\frac{kd}{c}\right\rfloor^2 + 4b\sum_{k=1}^{c-1}k\left\lfloor\frac{kd}{c}\right\rfloor + 2b(2a-1)\sum_{k=1}^{c-1}\left\lfloor\frac{kd}{c}\right\rfloor + 2b^2\sum_{k=1}^{c-1}k + (2a-1)\sum_{k=1}^{c-1}b^2\\
&+ 2d\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor^2 + 4c\sum_{k=1}^{b-1}k\left\lfloor\frac{ka}{b}\right\rfloor + 2c(2d-1)\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor + 2c^2\sum_{k=1}^{b-1}k + (2d-1)\sum_{k=1}^{b-1}c^2\\
&= (a+c-1)^2(b+d-1)^2 - a^2b^2 - c^2d^2 + (a-1)^2b^2 + c^2(d-1)^2\\
&\qquad - (a-1)^2(b-1)^2 - (c-1)^2(d-1)^2 + 1.
\end{aligned}$$

Then substituting the identities from Propositions 1.3 and 1.4 yields the simplification:

$$\begin{aligned}
&12(ad+bc)\left(\mathrm{s}(a,b) + \mathrm{s}(d,c)\right)\\
&\quad = 3 - 3a - 3b - 3c - 3d + 3a^2d - 3abc + 3abd - 3b^2c + 3acd - 3bc^2 + 3ad^2\\
&\qquad - 3bcd + 3b^2c^2 - 3a^2d^2 + 4ab^2c - 4a^2bd + 4bc^2d - 4acd^2 + 6ab - 6ad\\
&\qquad + \frac{a}{c} + \frac{d}{b} + \frac{ad^2}{c} + bd + \frac{a^2d}{b} + ac + 6cd\\
&\quad = (ad+bc)\left(\frac{a}{b} + \frac{a}{c} + \frac{d}{b} + \frac{d}{c} - 6\right).
\end{aligned}$$

We use the assumption that $ad - bc = 1$ to obtain the second equality, and the theorem follows. $\square$

Corollary 4.2 was given by Hans Rademacher in 1956 [9]. It was derived algebraically from classical Dedekind reciprocity using the following properties of Dedekind sums for the relation $ad - bc = 1$:

$$\mathrm{s}(d,b) = \mathrm{s}(a,b) \qquad \text{and} \qquad \mathrm{s}(b,d) = \mathrm{s}(-c,d) = -\,\mathrm{s}(c,d).$$

Taking the sum of the two reciprocity identities

$$\mathrm{s}(d,c) + \mathrm{s}(c,d) + \mathrm{s}(d,b) + \mathrm{s}(b,d) = -\frac{1}{4} + \frac{1}{12}\left(\frac{c}{d} + \frac{d}{c} + \frac{1}{cd}\right) - \frac{1}{4} + \frac{1}{12}\left(\frac{b}{d} + \frac{d}{b} + \frac{1}{bd}\right)$$

and substituting the above properties gives Corollary 4.2.

Chapter 5

Perpendicular Rays in the Plane

We now turn our focus back to a single ray in the plane, the ray through the points $(0,0)$ and (a,b). Extending the ray to the line through these points, and drawing the perpendicular line through the origin, we decompose each quadrant of the plane into two cones.

If we label each of these cones counter-clockwise by convention, beginning with the positive x-axis, then each cone has its vertex at the origin and K_1 is the cone defined by the rays $(1,0)$ and (a,b), K_2 is defined by (a,b) and $(0,1)$, and so on.

5.1 Reciprocity From Any Quadrant in the Plane

We showed in Section 2.1 that Carlitz reciprocity can be derived from the geometric picture of a ray in the first quadrant of the plane. In fact, we can do the same for

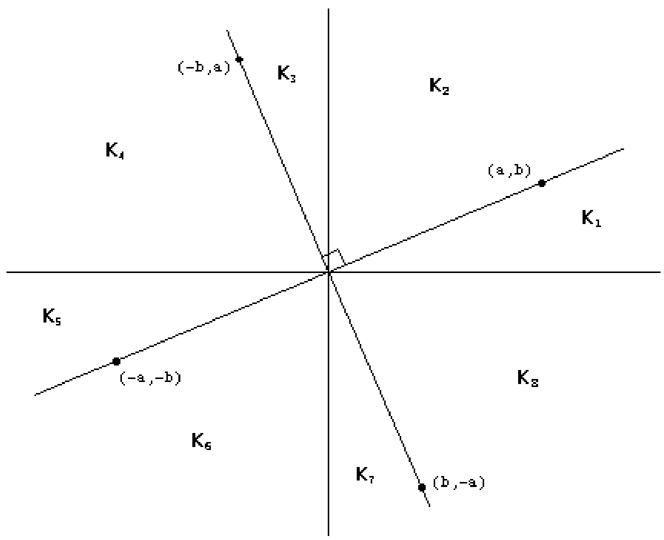

Figure 5.1: Perpendicular rays in the plane.

any quadrant. In the second quadrant, letting K_7 be closed and K_8 half-open and using the integer-point transform of each cone we have:

$$(u-1)\,\mathrm{c}\left(u, \frac{1}{v}; b, a\right) + \left(\frac{1}{v} - 1\right)\mathrm{c}\left(\frac{1}{v}, u; a, b\right) = u^{b-1}\left(\frac{1}{v}\right)^{a-1} - 1, \tag{5.1}$$

which is Carlitz reciprocity in the variables u and $1/v$.

5.2 Reflected Cones

We would like to have a relation between the cones $\sigma_{K_3}(u,v)$ and $\sigma_{K_7}(u,v)$. The (half-open) cones that make up a half-space have the property that the sum of their integer-point transfoms is zero [3, Theorem 9.2]. For example, suppose that K_1 is closed, K_2 and K_8 are open, and K_3 is half-open: closed on the side that it shares with K_2. Then K_3, K_2, K_1 and K_8 make up an (open) half-space and

$$\sigma_{K_3}(u,v) + \sigma_{K_2}(u,v) + \sigma_{K_1}(u,v) + \sigma_{K_8}(u,v) = 0.$$

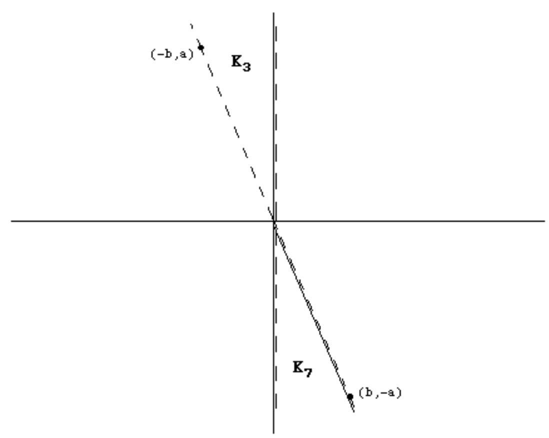

Figure 5.2: The cones K_3 and K_7.

Similarly, if we define K_1 to be closed, K_2 and K_8 open as above, and K_7 to be half-open where it is closed on the side that it shares with K_8, then K_2, K_1, K_8 and

K_7 make up an (open) half-space so that

$$\sigma_{K_2}(u,v) + \sigma_{K_1}(u,v) + \sigma_{K_8}(u,v) + \sigma_{K_7}(u,v) = 0.$$

Putting these two equations together we have

$$\sigma_{K_3}(u,v) = \sigma_{K_7}(u,v), \tag{5.2}$$

where

$$K_3 = \{\lambda_1(0,1) + \lambda_2(-b,a) : \lambda_1 > 0, \lambda_2 \geq 0\} \subset \mathbb{R}^2,$$
$$K_7 = \{\lambda_1(0,-1) + \lambda_2(b,-a) : \lambda_1 \geq 0, \lambda_2 > 0\} \subset \mathbb{R}^2.$$

This construction gives rise to an interesting version of Carlitz reciprocity.

Theorem 5.1. *Let a and b be relatively prime positive integers. Then for the indeterminates u and v,*

$$\begin{aligned} uv^{-1}(v-1)\left(u^b v^a - 1\right) \mathrm{c}\left(v^{-1}, u; a, b\right) + u^{-1}v(u-1)\left(u^b v^{-a} - 1\right) \mathrm{c}\left(u^{-1}, v; b, a\right) \\ = u\left(u^{-b}v^a - 1\right) + v\left(u^b v^{-a} - 1\right), \end{aligned} \tag{5.3}$$

or equivalently,

$$-uv^{a-1}(v-1)\,\mathrm{c}\left(v^{-1},u;a,b\right)+u^{b-1}v(u-1)\,\mathrm{c}\left(u^{-1},v;b,a\right)=-uv^{a}+u^{b}v. \quad (5.4)$$

Before we prove the theorem, we first state a useful identity.

Proposition 5.2. *If a and b are relatively prime positive integers and $k \in \mathbb{Z}_{\geq 0}$, such that $1 \leq k \leq b-1$, then $\left\lfloor -\frac{ka}{b} \right\rfloor = -\left\lfloor \frac{ka-1}{b} \right\rfloor - 1$.*

Proof. First note that for any integers a and b, $\left\lfloor -\frac{a}{b} \right\rfloor \leq -\left\lfloor \frac{a}{b} \right\rfloor$ since $\left\lfloor \frac{a}{b} \right\rfloor \leq \frac{a}{b}$ and $\left\lfloor -\frac{a}{b} \right\rfloor \leq -\frac{a}{b}$. Now, let $a, b \in \mathbb{Z}_{\geq 0}$ and let $1 \leq k \leq b-1$. Clearly $\frac{ka-1}{b} < \frac{ka}{b}$, thus $-\frac{ka}{b} < -\frac{ka-1}{b}$. Therefore, $\left\lfloor -\frac{ka}{b} \right\rfloor \leq \left\lfloor -\frac{ka-1}{b} \right\rfloor \leq -\left\lfloor \frac{ka-1}{b} \right\rfloor$, with equality between the first two expressions if and only if $\frac{ka}{b} \in \mathbb{Z}$. Since $\frac{ka}{b}$ and $\frac{ka-1}{b}$ differ by no more than 1, we have that $\left\lfloor -\frac{ka}{b} \right\rfloor + 1 = -\left\lfloor \frac{ka-1}{b} \right\rfloor$. □

Proof of Theorem 5.1. Let us define the cones K_3 and K_7 as the (half-open) cones as they appear in (5.2). Then the integer-point transform of each cone is written:

$$\sigma_{K_3}(u,v)=\frac{v+\sum_{k=1}^{b-1}u^{-k}v^{\left\lfloor \frac{ka}{b}\right\rfloor+1}}{(1-v)(1-u^{-b}v^{a})}=\frac{v+\frac{1}{u}v\,\mathrm{c}\left(\frac{1}{u},v;b,a\right)}{(v-1)\left(u^{-b}v^{a}-1\right)},$$

$$\sigma_{K_7}(u,v)=\frac{u^{b}v^{-a}+\sum_{k=1}^{b-1}u^{k}v^{\left\lfloor -\frac{ka}{b}\right\rfloor}}{\left(1-v^{-1}\right)\left(1-u^{b}v^{-a}\right)}=\frac{u^{b}v^{-a}+uv^{-1}\,\mathrm{c}\left(u,\frac{1}{v};b,a\right)}{\left(v^{-1}-1\right)\left(u^{b}v^{-a}-1\right)}.$$

Putting this into (5.2), we have

$$\frac{v+\frac{1}{u}v\,\mathrm{c}\left(\frac{1}{u},v;b,a\right)}{(v-1)\left(u^{-b}v^{a}-1\right)}=\frac{u^{b}v^{-a}+uv^{-1}\,\mathrm{c}\left(u,\frac{1}{v};b,a\right)}{\left(v^{-1}-1\right)\left(u^{b}v^{-a}-1\right)},$$

and thus,

$$u\left(1-u^{-b}v^{a}\right)\mathrm{c}\left(u,v^{-1};b,a\right)=u^{-1}v\left(u^{b}v^{-a}-1\right)\mathrm{c}\left(u^{-1},v;b,a\right). \tag{5.5}$$

Now we multiply this by $(u-1)$ so that we can use Carlitz reciprocity to replace $\mathrm{c}\left(u,v^{-1};b,a\right)$:

$$\begin{aligned} &u\left(1-u^{-b}v^{a}\right)\left(u^{b-1}v^{-(a-1)}-1+v^{-1}(v-1)\,\mathrm{c}\left(u^{-1},v;a,b\right)\right)\\ &\quad=u^{-1}v(u-1)\left(u^{b}v^{-a}-1\right)\mathrm{c}\left(u^{-1},v;b;a\right), \end{aligned}$$

so that

$$\begin{aligned} &uv^{-1}(v-1)\left(u^{-b}v^{a}-1\right)\mathrm{c}\left(u^{-1},v;a,b\right)+u^{-1}v(u-1)\left(u^{b}v^{-a}-1\right)\mathrm{c}\left(u^{-1},v;b;a\right)\\ &\quad=u\left(u^{-b}v^{a}-1\right)\left(u^{b-1}v^{-(a-1)}-1\right), \end{aligned}$$

and finally,

$$\begin{aligned} &uv^{-1}(v-1)\left(u^{b}v^{a}-1\right)\mathrm{c}\left(v^{-1},u;a,b\right)+u^{-1}v(u-1)\left(u^{b}v^{-a}-1\right)\mathrm{c}\left(u^{-1},v;b,a\right)\\ &\quad=u\left(u^{-b}v^{a}-1\right)+v\left(u^{b}v^{-a}-1\right). \end{aligned}$$

$\square$

It is interesting to note that Equation (5.5) can be derived algebraically by a simple change of variables.

Proposition 5.3. *The Carlitz polynomials* $\mathrm{c}\left(u, v^{-1}; b, a\right)$ *and* $\mathrm{c}\left(u^{-1}, v; b, a\right)$ *are related in the following way:*

$$u\left(1-u^{-b}v^{a}\right)\mathrm{c}\left(u, v^{-1}; b, a\right) = u^{-1}v\left(u^{b}v^{-a}-1\right)\mathrm{c}\left(u^{-1}, v; b, a\right).$$

Proof. Beginning with the right hand side of the equation,

$$u^{-1}v\left(u^{b}v^{-a}-1\right)\mathrm{c}\left(u^{-1}, v; b, a\right) = \sum_{k=1}^{b-1} u^{-k+b}v^{\left\lfloor\frac{ka}{b}\right\rfloor+1-a} - \sum_{k=1}^{b-1} u^{-k}v^{\left\lfloor\frac{ka}{b}\right\rfloor+1},$$

replace k with $b-k$ in both sums and the expression becomes

$$\sum_{k=1}^{b-1} u^{k}v^{\left\lfloor-\frac{ka}{b}\right\rfloor+1} - \sum_{k=1}^{b-1} u^{k-b}v^{\left\lfloor-\frac{ka}{b}\right\rfloor+1+a}.$$

Here note that $\left\lfloor-\frac{ka}{b}\right\rfloor+1 = -\left\lfloor\frac{ka-1}{b}\right\rfloor$ by Proposition 5.2. Then, because $\frac{ka-1}{b} < \frac{ka}{b} \notin \mathbb{Z}$ and there does not exist any integer x such that $\frac{ka-1}{b} < x < \frac{ka}{b}$, we have $\left\lfloor\frac{ka-1}{b}\right\rfloor = \left\lfloor\frac{ka}{b}\right\rfloor$ and hence $\left\lfloor-\frac{ka}{b}\right\rfloor+1 = -\left\lfloor\frac{ka}{b}\right\rfloor$. Thus,

$$u^{-1}v\left(u^{b}v^{-a}-1\right)\mathrm{c}\left(u^{-1}, v; b, a\right) = \sum_{k=1}^{b-1} u^{k}v^{-\left\lfloor\frac{ka}{b}\right\rfloor} - \sum_{k=1}^{b-1} u^{k-b}v^{-\left\lfloor\frac{ka}{b}\right\rfloor+a}.$$

This is equal to the left-hand side of the equation

$$u\left(1-u^{-b}v^{a}\right)\mathrm{c}\left(u,v^{-1};b,a\right)=-\sum_{k=1}^{b-1}u^{k-b}v^{-\left\lfloor\frac{ka}{b}\right\rfloor+a}+\sum_{k=1}^{b-1}u^{k}v^{-\left\lfloor\frac{ka}{b}\right\rfloor}.$$

$\square$

Note, however, that the construction of the equation itself is not intuitive. We emphasize that the set-up of the equation itself is a result of the geometry.

Chapter 6

Brion Decompositions

6.1 The Triangle

Until now, we have only considered pointed cones with the vertex at the origin. Let us now turn to a different object. A *convex polytope* is the bounded intersection of finitely many half-spaces, and is called *rational* if all of its vertices have rational coordinates. A well-known theorem relating to convex polytopes is Brion's theorem [5] which gives an expression for the number of integer points in a rational polytope.

Definition 6.1. If $\mathbf{v}$ is a vertex of a closed polytope P, then a *vertex cone* $K_{\mathbf{v}}$ is the smallest cone with apex $\mathbf{v}$ that contains P.

Theorem 6.1 (Brion's theorem)**.** *Suppose P is a rational convex polytope, and let $K_{\mathbf{v}}$ be a vertex cone with vertex $\mathbf{v}$ for each vertex $\mathbf{v}$ of P. Then we have the following* *identity of rational functions:*

$$\sigma_P(\mathbf{z}) = \sum_{\mathbf{v}} \sigma_{K_{\mathbf{v}}}(\mathbf{z}),$$

where the sum is over all vertices of P.

We construct a triangle in two dimensions using the positive quadrant and the half-space $y \leq \frac{b}{a}x$ for relatively prime a and b as illustrated in Figure 6.1.

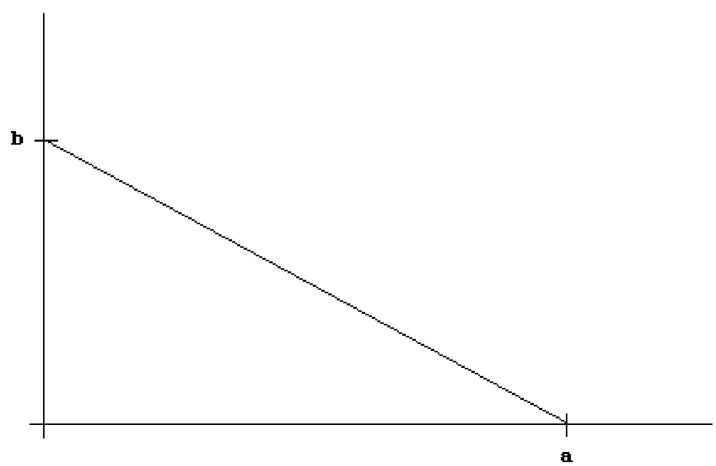

Figure 6.1: A simple polytope: a triangle.

This polytope P is the convex hull of the points $(0,0),(a,0)$, and $(0,b)$. If we call the (closed) vertex cones K_1, K_2 and K_3 with vertices at each of these three points, respectively, then Brion's theorem says that

$$\sigma_P(\mathbf{z}) = \sigma_{K_1}(\mathbf{z}) + \sigma_{K_2}(\mathbf{z}) + \sigma_{K_3}(\mathbf{z}).$$

This construction allows us to give a novel expression for the Carlitz sum as the integer-point transform of a triangle.

Theorem 6.2. *Let a and b be relatively prime positive integers and u and v be indeterminates. If P is the triangle with vertices $(0,0),(a,0)$ and $(0,b)$, then the Carlitz polynomial $\mathrm{c}\left(v,\frac{1}{u};b,a\right)$ and the integer-point transform of P are related in the following manner:*

$$(u-1)\,\sigma_P(u,v) = u^a v\,\mathrm{c}\left(v,u^{-1};b,a\right) + u\left(u^a + v^b\right) - \frac{v^{b+1}-1}{v-1}\,.$$

Proof. Formally, we define our cones

$$\begin{aligned} K_1 &= \{j(1,0) + k(0,1) : j,k \geq 0\}\,, \\ K_2 &= \{(a,0) + j(-1,0) + k(-a,b) : j,k \geq 0\}\,, \\ K_3 &= \{(b,0) + j(a,-b) + k(0,-1) : j,k \geq 0\}\,. \end{aligned}$$

Now let us compute the generating functions. The integer-point transform of the first cone, K_1, is simple; we have done it before:

$$\sigma_{K_1}(u,v) = \frac{1}{(1-u)(1-v)}\,.$$

To compute $\sigma_{K_3}(u,v)$, shift the cone K_3 down b units so that the vertex is at the origin and call this cone K_3'. K_3' is now a familiar object as well. We computed the integer-point transform of a very similar object (permute a and b) in Section 5.2. We have

$$\sigma_{K_3'}(u,v) = \frac{1+\sum_{k=1}^{a-1} u^k v^{\left\lfloor -\frac{kb}{a}\right\rfloor}}{\left(1-v^{-1}\right)\left(1-u^a v^{-b}\right)} = \frac{1+uv^{-1}\,\mathrm{c}\left(u,v^{-1};a,b\right)}{\left(1-v^{-1}\right)\left(1-u^a v^{-b}\right)}\,.$$

To shift the cone back up, multiply $\sigma_{K_3'}(u,v)$ by v^b to obtain

$$\sigma_{K_3}(u,v) = -\frac{v^{b+1}+uv^b\,\mathrm{c}\left(u,v^{-1};a,b\right)}{(v-1)\left(u^a v^{-b}-1\right)}\,.$$

Applying the same process to cone K_2 we have the integer-point transform

$$\sigma_{K_2}(u,v) = -\frac{u^{a+1}+u^a v\,\mathrm{c}\left(v,u^{-1};b,a\right)}{(u-1)\left(u^{-a} v^{b}-1\right)}\,.$$

Using the integer-point transform of each of these three cones in Brion's theorem we obtain

$$\sigma_P(u,v) = \frac{1}{(u-1)(v-1)} - \frac{u^{a+1} + u^a v \, \mathrm{c}\left(v, u^{-1}; b, a\right)}{(u-1)\left(u^{-a}v^b - 1\right)} - \frac{v^{b+1} + uv^b \, \mathrm{c}\left(u, v^{-1}; a, b\right)}{(v-1)\left(u^a v^{-b} - 1\right)} .$$

We begin simplifying by re-writing the generating functions in a slightly different form:

$$\sigma_P(u,v) = \frac{1}{(u-1)(v-1)} + \frac{u^{2a+1} + u^{2a} v \, \mathrm{c}\left(v, u^{-1}; b, a\right)}{(u-1)\left(u^a - v^b\right)} - \frac{v^{2b+1} + uv^{2b} \, \mathrm{c}\left(u, v^{-1}; a, b\right)}{(v-1)\left(u^a - v^b\right)} . \tag{6.1}$$

Clearing the denominator gives

$$\begin{aligned}(u-1)(v-1)\left(u^a - v^b\right)\sigma_P(u,v) &= u^a - v^b + u^{2a+1}(v-1) - v^{2b+1}(u-1) \\ &+ u^{2a}v(v-1)\,\mathrm{c}\left(v, u^{-1}; b, a\right) - uv^{2b}(u-1)\,\mathrm{c}\left(u, v^{-1}; a, b\right) .\end{aligned}$$

Note that all but the factor v^b in the final term of this equation can be replaced using reciprocity from a variation of (5.4) from Theorem 5.1. By replacing the variable u with $\frac{1}{v}$ and v with $\frac{1}{u}$, Theorem 5.1 becomes

$$uv^b(u-1)\,\mathrm{c}\left(u, v^{-1}; a, b\right) - u^a v(v-1)\,\mathrm{c}\left(v, u^{-1}; b, a\right) = u^a v - uv^b,$$

and substituting this into (6.1) we have

$$\begin{aligned}
&(u-1)(v-1)\left(u^a-v^b\right)\sigma_P(u,v)\\
&\quad = u^{2a}v(v-1)\,\mathrm{c}\left(v,u^{-1};b,a\right) - uv^{b+1}(v-1)\,\mathrm{c}\left(v,u^{-1};b,a\right)\\
&\qquad + u^a - v^b + u^{2a+1}(v-1) - v^{2b+1}(u-1) - u^a v^{b+1} + uv^{2b}. \qquad (6.2)
\end{aligned}$$

Notice that we now have only one Carlitz sum in the above equation! The right-hand side becomes

$$u^a v(v-1)\left(u^a-v^b\right)\mathrm{c}\left(v,u^{-1};b,a\right)+u^a-v^b+u^{2a+1}(v-1)-uv^{2b}(v-1)-v^{b+1}\left(u^a-v^b\right),$$

so that

$$\begin{aligned}
(u-1)(v-1)\left(u^a-v^b\right)\sigma_P(u,v) &= u^a v(v-1)\left(u^a-v^b\right)\mathrm{c}\left(v,u^{-1};b,a\right)\\
&\quad + u^a - v^b + u^{2a+1}(v-1) - uv^{2b}(v-1) - v^{b+1}\left(u^a-v^b\right). \qquad (6.3)
\end{aligned}$$

Dividing both sides of this equation by $(v-1)\left(u^a-v^b\right)$, we have Theorem 6.2. $\square$

6.2 The Tetrahedron

6.2.1 Dedekind-Rademacher-Carlitz Sums

It is natural to extend the application of Brion's theorem to a convex polytope in higher dimensions. We consider a simple polytope in the first octant $Q_1^{(3)}$: the tetrahedron P defined

$$P = \left\{(x, y, z) \in \mathbb{R}^3; x, y, z \geq 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1\right\}, \tag{6.4}$$

with vertices $(0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c)$ where a, b and c are positive integers.

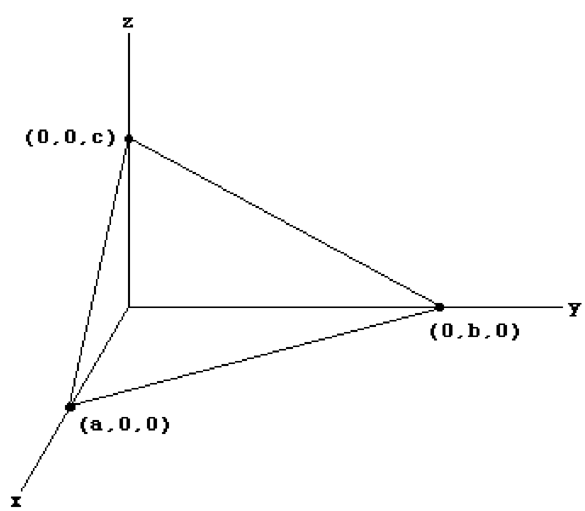

Figure 6.2: The tetrahedron P.

The tth dilate of a subset $S \subset \mathbb{R}^d$ is the set

$$\{(tx_1, tx_2, \ldots, tx_d) : (x_1, x_2, \ldots, x_d) \in S\}.$$

Let us consider the tth dilate of P for $t \in \mathbb{Z}_{>0}$. This gives us the dilated tetrahedron tP with vertices $(0,0,0), (ta,0,0), (0,tb,0)$, and $(0,0,tc)$. Note that $t=1$ gives the tetrahedron in Figure 6.2.

Using the motivation of the triangle in two dimensions and the tetrahedron tP, we will use Brion's theorem to find an expression for the integer-point transform of tP.

Definition 6.2 (Dedekind-Rademacher-Carlitz (DRC) sum)**.** If a, b, and c are positive integers, and u, v, w are indeterminates, then define

$$\bar{\mathrm{c}}(u, v, w; a, b, c) := \sum_{k=0}^{c-1} \sum_{j=0}^{b-1} u^{\left\lfloor \frac{ja}{b} + \frac{ka}{c} \right\rfloor} v^j w^k.$$

We discover that the integer-point transform of tP is an expression of DRC sums.

Theorem 6.3. *Let tP be the dilated tetrahedron with vertices $(0,0,0)$, $(ta,0,0)$, $(0,tb,0)$, $(0,0,tc)$ for $t \in \mathbb{Z}_{>0}$ and a, b and c relatively prime positive integers. Then* *for indeterminates* u, v, w*,*

$$\begin{aligned}
&(u-1)(v-1)(w-1)\left(u^a-v^b\right)\left(u^a-w^c\right)\left(v^b-w^c\right)\sigma_P(u,v,w)\\
&\quad = u^{(t+2)a}(v-1)(w-1)\left(v^b-w^c\right)\left[(u-1)+\bar{\mathrm{c}}\left(u^{-1},v,w;a,b,c\right)\right]\\
&\qquad - v^{(t+2)b}(u-1)(w-1)\left(u^a-w^c\right)\left[(v-1)+\bar{\mathrm{c}}\left(v^{-1},u,w;b,a,c\right)\right]\\
&\qquad + w^{(t+2)c}(u-1)(w-1)\left(u^a-v^b\right)\left[(w-1)+\bar{\mathrm{c}}\left(w^{-1},u,v;c,a,b\right)\right]\\
&\qquad - \left(u^a-v^b\right)\left(u^a-w^c\right)\left(v^b-w^c\right).
\end{aligned}$$

Proof. Let P be the tetrahedron defined above with $t = 1$ and define the vertex cones

$$\begin{aligned}
K_0 &= \{\lambda_1(1,0,0)+\lambda_2(0,1,0)+\lambda_3(0,0,1) : \lambda_1,\lambda_2,\lambda_3 \geq 0\},\\
K_1 &= \{(a-\lambda_1,0,0)+\lambda_2(0,b,0)+\lambda_3(0,0,c) : \lambda_1,\lambda_2,\lambda_3 \geq 0\},\\
K_2 &= \{\lambda_1(a,0,0)+(0,b-\lambda_2,0)+\lambda_3(0,0,c) : \lambda_1,\lambda_2,\lambda_3 \geq 0\},\\
K_3 &= \{\lambda_1(a,0,0)+\lambda_2(0,b,0)+(0,0,c-\lambda_3) : \lambda_1,\lambda_2,\lambda_3 \geq 0\}.
\end{aligned}$$

Brion's theorem states that the sum of the integer-point transforms of these cones is equal to the integer-point transform of P. We approach the definition of these generating functions as we did in the two-dimensional case by shifting the vertex of each cone K_i to the origin. If we call this shifted cone K_i', then the fundamental parallelepiped Π_i' for each $i = 0, 1, 2, 3$ is defined through

$$\Pi_0' = \{\lambda_1(1,0,0) + \lambda_2(0,1,0) + \lambda_3(0,0,1) : 0 \le \lambda_1, \lambda_2, \lambda_3 < 1\},$$

$$\Pi_1' = \{\lambda_1(-1,0,0) + \lambda_2(-a,b,0) + \lambda_3(-a,0,c) : 0 \le \lambda_1, \lambda_2, \lambda_3 < 1\},$$

$$\Pi_2' = \{\lambda_1(a,-b,0) + \lambda_2(0,-1,0) + \lambda_3(0,-b,c) : 0 \le \lambda_1, \lambda_2, \lambda_3 < 1\},$$

$$\Pi_3' = \{\lambda_1(a,0,-c) + \lambda_2(0,b,-c) + \lambda_3(0,0,-1) : 0 \le \lambda_1, \lambda_2, \lambda_3 < 1\}.$$

By construction, there exists exactly one integer point in the interior of Π_1' at each integer j along the y-axis and k along the z-axis. In other words, for $j, k \in \mathbb{Z}_{\ge 0}$, $(x, j, k) \in \Pi_1'$ is written as $(x, j, k) = (-\lambda_1 - \lambda_2 a - \lambda_3 a, \lambda_2 b, \lambda_c)$ and hence $\lambda_2 = \frac{j}{b}$ and $\lambda_3 = \frac{k}{c}$. Because $0 \le \lambda_i < 1$, we have that $j = 0, 1, \ldots, a-1$ and $k = 0, 1, \ldots, b-1$, and hence $(x, j, k) = \left(\left\lfloor -\frac{ja}{b} - \frac{ka}{c} \right\rfloor, j, k\right)$. Now, writing the integer-point transform of K_1 is as simple as multiplying the integer-point transform of K_1' by u^a. That is, $\sigma_{K_1}(u, v, w) = u^a \sigma_{K_1'}(u, v, w)$.

We would like to know how this relates to the dilated tetrahedron tP. Let tK_i denote the vertex cone related to tP (as we related K_i to P) for $i = 0, 1, 2, 3$. Then

$$tK_1 = (ta, 0, 0) + K_1', \quad tK_2 = (0, tb, 0) + K_2', \quad tK_3 = (0, 0, tc) + K_3',$$

and $tK_0 = K_0$. So, the cones K_i and tK_i, $i = 1, 2, 3$ are similar cones! That is, they have the same generators but different vertices. Therefore, for $t = 1, 2, 3$, writing the integer-point transform of tK_1 is as trivial as multiplying the integer-point transform of K_1' by u^{ta}. Similarly for tK_2 and tK_3, we have $\sigma_{tK_2}(u,v,w) = v^{tb}\sigma_{K_2'}(u,v,w)$ and $\sigma_{tK_3}(u,v,w) = w^{tc}\sigma_{K_3'}(u,v,w)$.

Note that $\left\lfloor -\frac{ja}{b} - \frac{ka}{c} \right\rfloor = -\left\lfloor \frac{ja}{b} + \frac{ka}{c} \right\rfloor - 1$ for $1 \le j \le b-1$ and $1 \le k \le c-1$. Therefore, the integer-point transform of tK_1 becomes

$$\begin{aligned}
\sigma_{tK_1}(u,v,w) &= u^{ta}\left(\frac{\sum_{k=0}^{c-1}\sum_{j=0}^{b-1} u^{\left\lfloor -\frac{ja}{b} - \frac{ka}{c} \right\rfloor} v^j w^k}{\left(1-u^{-1}\right)\left(1-u^{-a}v^b\right)\left(1-u^{-a}w^c\right)}\right) \\
&= u^{ta}\left(\frac{1+\sum_{k=0}^{c-1}\sum_{j=0}^{b-1} u^{-\left\lfloor \frac{ja}{b} + \frac{ka}{c} \right\rfloor - 1} v^j w^k - u^{-1}}{u^{-2a-1}\left(u-1\right)\left(u^a-v^b\right)\left(u^a-w^c\right)}\right) \\
&= \frac{u^{(t+2)a}\left[\left(u-1\right)+\bar{\mathrm{c}}\left(u^{-1},v,w;a,b,c\right)\right]}{\left(u-1\right)\left(u^a-v^b\right)\left(u^a-w^c\right)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sigma_{tK_2}(u,v,w) &= v^{tb}\left(\frac{\sum_{k=0}^{c-1}\sum_{j=0}^{a-1} u^j v^{\left\lfloor -\frac{jb}{a} - \frac{kb}{a} \right\rfloor} w^k}{\left(1-v^{-1}\right)\left(1-u^a v^{-b}\right)\left(1-v^{-b}w^c\right)}\right) \\
&= -\frac{v^{(t+2)b}\left[\left(v-1\right)+\bar{\mathrm{c}}\left(v^{-1},u,w;b,a,c\right)\right]}{\left(v-1\right)\left(u^a-v^b\right)\left(v^b-w^c\right)},
\end{aligned}$$

and

$$\sigma_{tK_3}(u,v,w) = w^{tc}\left(\frac{\sum_{k=0}^{b-1}\sum_{j=0}^{a-1} u^j v^k w^{\left\lfloor -\frac{jc}{a}-\frac{kc}{b}\right\rfloor}}{\left(1-w^{-1}\right)\left(1-u^a w^{-c}\right)\left(1-v^b w^{-c}\right)}\right)$$
$$= \frac{w^{(t+2)c}\left[(w-1)+\bar{\mathrm{c}}\left(w^{-1},u,v;c,a,b\right)\right]}{(w-1)\left(u^a-w^c\right)\left(v^b-w^c\right)}\,.$$

The integer-point transform for K_0 doesn't change when we dilate the tetrahedron so we have that the generators of K_0 are the three unit vectors and

$$\sigma_{K_0}(u,v,w) = \sum_{\substack{(m_1,m_2,m_3)=i(1,0,0)+j(0,1,0)+k(0,0,1)\\ i,j,k\geq 0}} u^{m_1}v^{m_2}w^{m_3}$$
$$= -\frac{1}{(u-1)(v-1)(w-1)}\,.$$

By Brion's theorem, we have that

$$\sigma_P(u,v,w) = \sigma_{K_0}(u,v,w) + \sigma_{tK_1}(u,v,w) + \sigma_{tK_2}(u,v,w) + \sigma_{tK_3}(u,v,w)$$

and hence

$$\sigma_P(u,v,w) =$$
$$\frac{u^{(t+2)a}\left[(u-1)+\bar{\mathrm{c}}\left(u^{-1},v,w;a,b,c\right)\right]}{(u-1)\left(u^a-v^b\right)\left(u^a-w^c\right)} - \frac{v^{(t+2)b}\left[(v-1)+\bar{\mathrm{c}}\left(v^{-1},u,w;b,a,c\right)\right]}{(v-1)\left(u^a-v^b\right)\left(v^b-w^c\right)}$$
$$+\frac{w^{(t+2)c}\left[(w-1)+\bar{\mathrm{c}}\left(w^{-1},u,v;c,a,b\right)\right]}{(w-1)\left(u^a-w^c\right)\left(v^b-w^c\right)} - \frac{1}{(u-1)(v-1)(w-1)}\,.$$

Clearing the denominator gives the theorem. □

6.2.2 The Relation to Dedekind Sums

It was perhaps unclear in the previous section the reason for calling the polynomial $\bar{\mathrm{c}}(u, v, w; a, b, c)$ the Dedekind-Rademacher-Carlitz sum. We hope that the following will help to clarify this. We begin by stating a well-known theorem [7, 8]; this was the first instant that an explicit formula for a lattice-point count in three-dimensional polytopes was given.

The *lattice-point enumerator* for the t^{th} dilate of $P \subset \mathbb{R}^d$ is denoted $L_P(t)$ and is equivalent to $\#\left(tP \cap \mathbb{Z}^d\right)$, the discrete volume of P.

Theorem 6.4 (Mordell, Pommersheim)**.** *Let P be given by* (6.4) *and let a, b and c be pairwise relatively prime. Then*

$$
\begin{aligned}
L_P(t) =& \frac{abc}{6}\, t^3 + \frac{ab + ac + bc + 1}{4}\, t^2 + \left(-\,\mathrm{s}(bc, a) - \mathrm{s}(ca, b) - \mathrm{s}(ab, c)\right) t \\
&+ \left(\frac{3}{4} + \frac{a + b + c}{4} + \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc}\right)\right) t + 1.
\end{aligned}
$$

We will prove that Theorem 6.3 implies Theorem 6.4. Before we prove this interesting result, we give some necessary identities.

Proposition 6.5. *If a, b and c are relatively prime positive integers, then*

$$\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}\left\{\frac{jc}{a}+\frac{kc}{b}\right\} = \frac{ab-1}{2}\,, \tag{6.5}$$

$$\sum_{k=0}^{b-1}\sum_{j=0}^{a-1} j\left\{\frac{jc}{a}+\frac{kc}{b}\right\} = c\,\mathrm{s}(ab,c) + \frac{ab(c-1)}{4}\,, \tag{6.6}$$

$$\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}\left\{\frac{jc}{a}+\frac{kc}{b}\right\}^2 = \frac{(ab-1)(2ab-1)}{6ab}\,. \tag{6.7}$$

Proof. A Dedekind–Rademacher sum is like a Dedekind sum with a shift in the sawtooth argument. We begin by noting the following invaluable property of one such function:

$$\sum_{k=0}^{a-1}\left(\left(x+\frac{kc}{a}\right)\right) = ((ax))\,, \tag{6.8}$$

which is given as an exercise in [3]. Using this, we can simplify the single sum

$$\sum_{k=0}^{a-1}\left\{\frac{jc}{a}+\frac{kc}{b}\right\} = \sum_{k=0}^{a-1}\left[\left(\left(\frac{jc}{a}+\frac{kc}{b}\right)\right)+\frac{1}{2}\right] = \left(\left(\frac{jac}{b}\right)\right)+\frac{a}{2} = \left\{\frac{jac}{b}\right\}+\frac{a-1}{2}\,. \tag{6.9}$$

We use this result to prove the first two identities as follows. (6.5): Combining (6.9) with Identity (1.6) we have that

$$\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}\left\{\frac{jc}{a}+\frac{kc}{b}\right\} = \sum_{j=0}^{b-1}\left(\left\{\frac{jac}{b}\right\}+\frac{a-1}{2}\right) = \frac{ab-1}{2}\,.$$

(6.6): Now, we use (6.9) to show

$$\sum_{k=0}^{b-1}\sum_{j=0}^{a-1} j\left\{\frac{jc}{a}+\frac{kc}{b}\right\} = a\sum_{j=1}^{a-1}\frac{j}{a}\ \sum_{k=0}^{b-1}\left\{\frac{jac}{b}\right\} = a\sum_{j=1}^{a-1}\left\{\frac{j}{a}\right\}\left(\left\{\frac{jbc}{a}\right\}+\frac{b-1}{2}\right)$$
$$= a\sum_{j=1}^{a-1}\left[\left(\left(\frac{j}{a}\right)\right)+\frac{1}{2}\right]\left[\left(\left(\frac{jbc}{a}\right)\right)+\frac{b}{2}\right] = a\,\mathrm{s}(bc,a)+\frac{ab(a-1)}{4}\,,$$

where the final expression was obtained by expanding the product and using (6.8) with $x=0$. (6.7): We use Identity (1.7) and the second Bernoulli polynomial, $B_2(x) := x^2 - x + \frac{1}{6}$ to obtain the final equation. The Bernoulli function $\bar{B}_k(x) := B_k\left(\{x\}\right)$ has the property [2] that $\sum_{k=0}^{a-1}\bar{B}_k(x) = a^{-1}\bar{B}_k(ax)$. Therefore, we re-write $\left\{\frac{jc}{a}+\frac{kc}{b}\right\}^2$ in terms of Bernoulli polynomials and replace this in the double sum as follows:

$$\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}\left\{\frac{jc}{a}+\frac{kc}{b}\right\}^2 = \sum_{k=0}^{b-1}\sum_{j=0}^{a-1}\left(\bar{B}_2\left(\frac{jc}{a}+\frac{kc}{b}\right)+\left\{\frac{jc}{a}+\frac{kc}{b}\right\}-\frac{1}{6}\right)$$
$$= \sum_{k=0}^{b-1}a^{-1}\bar{B}_2\left(\frac{kac}{b}\right)+\frac{ab-1}{2}-\frac{ab}{6} = \frac{(ab-1)(2ab-1)}{6ab}\,.$$

□

Proof that Theorem 6.3 implies Theorem 6.4. Let tP be the dilated tetrahedron given in Theorem 6.3. If we divide both sides of the equation in the theorem by

$$(u-1)(v-1)(w-1)\left(u^a-v^b\right)\left(u^a-w^c\right)\left(v^b-w^c\right),$$

then we have a rational expression for $\sigma_{tP}(u,v,w)$ where the numerator is

$$\begin{aligned}
&u^{(t+2)a}(v-1)(w-1)\left(v^b-w^c\right)\left[(u-1)+\bar{\mathrm{c}}\left(u^{-1},v,w;a,b,c\right)\right]\\
&-v^{(t+2)b}(u-1)(w-1)\left(u^a-w^c\right)\left[(v-1)+\bar{\mathrm{c}}\left(v^{-1},u,w;b,a,c\right)\right]\\
&+w^{(t+2)c}(u-1)(w-1)\left(u^a-v^b\right)\left[(w-1)+\bar{\mathrm{c}}\left(w^{-1},u,v;c,a,b\right)\right]\\
&-\left(u^a-v^b\right)\left(u^a-w^c\right)\left(v^b-w^c\right),
\end{aligned}$$

and the denominator is

$$(u-1)(v-1)(w-1)\left(u^a-v^b\right)\left(u^a-w^c\right)\left(v^b-w^c\right).$$

The lattice-point enumerator of tP is the number of integer points in tP. We obtain this from the integer-point transform of tP by setting $u=v=w=1$. Because this operation reduces both numerator and denominator to zero, we use L'Hospital's rule to reduce. Because we have three indeterminates, the number of times we use L'Hospital and the indeterminates with respect to which we derivate is not immediately clear. However, both numerator and denominator are symmetric, reducing the problem to the number of derivations. It turns out that taking the partial derivative with respect to u once, v twice, and w three times does the trick.

Beginning with the denominator, we derivate as described and set $u = v = w = 1$. The result is very simple: $-12bc^2$.

Turning to the numerator, after differentiating and setting $u = v = w = 1$ we obtain a complex expression in t. Fortunately there is a way to further simplify the expression! Because the constant term of L_P is 1 [3, Corollary 3.15], we are only concerned with the non-constant terms of the expression:

$$
\begin{aligned}
&- 2ab^2c^3t^3 - 6ab^2c^3t^2 - 6bc^2t^2 - 4ab^2c^3t - 18bc^2t \\
&+ 6bct \sum_{j=0}^{a-1} \sum_{k=0}^{c-1} \left\lfloor \frac{jb}{a} + \frac{kb}{c} \right\rfloor - 6bc^2t \sum_{j=0}^{a-1} \sum_{k=0}^{c-1} \left\lfloor \frac{jb}{a} + \frac{kb}{c} \right\rfloor \\
&- 12bct \sum_{j=0}^{a-1} \sum_{k=0}^{c-1} k \left\lfloor \frac{jb}{a} + \frac{kb}{c} \right\rfloor - 6bct \sum_{j=0}^{a-1} \sum_{k=0}^{b-1} \left\lfloor \frac{jc}{a} + \frac{kc}{b} \right\rfloor \\
&+ 24bc^2t \sum_{j=0}^{a-1} \sum_{k=0}^{b-1} \left\lfloor \frac{jc}{a} + \frac{kc}{b} \right\rfloor + 6bc^2t^2 \sum_{j=0}^{a-1} \sum_{k=0}^{b-1} \left\lfloor \frac{jc}{a} + \frac{kc}{b} \right\rfloor \\
&- 6bct \sum_{j=0}^{a-1} \sum_{k=0}^{b-1} \left\lfloor \frac{jc}{a} + \frac{kc}{b} \right\rfloor \left(1 + \left\lfloor \frac{jc}{a} + \frac{kc}{b} \right\rfloor \right).
\end{aligned}
$$

As before, we replace all greatest-integer functions with fractional-part functions to obtain

$$
\begin{aligned}
&-2ab^2c^3t^3 - 3b^2c^3t^2 - 3abc^3t^2 - 6bc^2t^2 - \frac{b^2c^3}{a}t - 6abc^3t - 3bc^3t - ac^3t \\
&+ 6ab^2c^2t + 6abc^2t - 18bc^2t - 5ab^2ct - 6bct\sum_{k=0}^{c-1}\sum_{j=0}^{a-1}\left\{\frac{jb}{a}+\frac{kb}{c}\right\} \\
&+ 6bc^2t\sum_{k=0}^{c-1}\sum_{j=0}^{a-1}\left\{\frac{jb}{a}+\frac{kb}{c}\right\} + 12bct\sum_{k=0}^{c-1}\sum_{j=0}^{a-1}k\left\{\frac{jb}{a}+\frac{kb}{c}\right\} \\
&+ 12bct\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}\left\{\frac{jc}{a}+\frac{kc}{b}\right\} - 24bc^2t\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}\left\{\frac{jc}{a}+\frac{kc}{b}\right\} \\
&- 6bc^2t^2\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}\left\{\frac{jc}{a}+\frac{kc}{b}\right\} + \frac{12bc^2}{a}t\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}j\left\{\frac{jc}{a}+\frac{kc}{b}\right\} \\
&+ 12c^2t\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}k\left\{\frac{jc}{a}+\frac{kc}{b}\right\} - 6bct\sum_{k=0}^{b-1}\sum_{j=0}^{a-1}\left\{\frac{jc}{a}+\frac{kc}{b}\right\}^2.
\end{aligned}
$$

Now we replace the fractional-part functions in this expression with identities from Proposition 6.5. Simplifying the expression gives:

$$
\begin{aligned}
&-2ab^2c^3t^3 - 3b^2c^3t^2 - 3abc^3t^2 - 3bc^2t^2 - \frac{b^2c^3}{a}t - 3bc^3t - ac^3t - 3b^2c^2t \\
&- 3abc^2t - 9bc^2t - ab^2ct - \frac{c}{a}t + 12bc^2\left(\mathrm{s}(ab,c)+\mathrm{s}(ac,b)+\mathrm{s}(bc,a)\right)t.
\end{aligned}
$$

Recall that this is only the non-constant part of the numerator of the lattice-point enumerator of P. We divide by the denominator $-12bc^2$ and after rearranging we have,

$$\frac{abc}{6}t^3 + \frac{ab+ac+bc+1}{4}t^2 + \left(-\operatorname{s}(bc,a) - \operatorname{s}(ca,b) - \operatorname{s}(ab,c)\right) t$$
$$+ \left(\frac{3}{4} + \frac{a+b+c}{4} + \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc}\right)\right) t.$$

The final step is to add to this the constant term 1 of the lattice-point enumerator, which completes the proof. $\square$

Bibliography

[1] M. Beck, *Geometric proofs of polynomial reciprocity laws of Carlitz, Berndt, and Dieter*, Diophantine Analysis and Related Fields 2006 (M. Katsurada, T. Komatsu, and H. Nakad, eds.), 2006.

[2] Matthias Beck, *Dedekind cotangent sums*, Acta Arith. **109** (2003), no. 2, 109–130.

[3] Matthias Beck and Sinai Robins, *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Undergraduate Texts in Mathematics, Springer, New York, 2007.

[4] Bruce C. Berndt and Ulrich Dieter, *Sums involving the greatest integer function and Riemann-Stieltjes integration*, J. Reine Angew. Math. **337** (1982), 208–220.

[5] Michel Brion, *Points entiers dans les polyèdres convexes*, Ann. Sci. École Norm. Sup. (4) **21** (1988), no. 4, 653–663.

[6] L. Carlitz, *Some polynomials associated with Dedekind sums*, Acta Math. Acad. Sci. Hungar. **26** (1975), no. 3-4, 311–319.

[7] Louis J. Mordell, *Lattice points in a tetrahedron and generalized Dedekind sums*, J. Indian Math. Soc. (N.S.) **15** (1951), 41–46.

[8] James E. Pommersheim, *Toric varieties, lattice points and Dedekind sums*, Math. Ann. **295** (1993), no. 1, 1–24.

[9] Hans Rademacher, *Zur Theorie der Dedekindschen Summen*, Math. Z. **63** (1956), 445–463.