TWO PROBLEMS ON LATTICE POINT ENUMERATION OF RATIONAL POLYTOPES

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Chapter 1

Introduction

Ehrhart theory is a well-established field that studies lattice points in dilations of polytopes. There are several excellent sources that can serve as exposure or further reading to this area, including, but certainly not limited to [2, 13, 29].

Much of the structure connecting the discrete volume of a dilated polytope to the number of lattice points it contains is encoded in its Ehrhart polynomial. An object of interest is the discrete volume $\mathrm{ehr}_{\mathcal{P}}(m) := \#(m\mathcal{P} \cap Z^d)$ of the mth dilate of a polytope $\mathcal{P}$ that is invariant under the action of a group G. The counting function that arises in this work for polytopes is known as the *Ehrhart quasipolynomial* of $\mathcal{P}$. The Ehrhart quasipolynomial is encoded in the Ehrhart series, written as a rational function, where we call the numerator the *h^*-polynomial*:

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \sum_{m \geq 0} \mathrm{ehr}_{\mathcal{P}}(m) z^m = \frac{h^*_{\mathcal{P}}(z)}{(1 - z^p)^{\dim(\mathcal{P})+1}}.$$

This thesis is a result of two different projects within the realm of Ehrhart theory and discrete geometry. The first part of this thesis was done with a goal of obtaining an equivariant analogue of Ehrhart theory for the permutahedron. The second part of this thesis was motivated by finding rational analogues to the works of Betke and McMullen in [4] and Alan Stapledon in [30].

1.1 Part I

The first part of this thesis expands the work of Stapledon, where we make progress towards answering one of his open problems in [31], specifically, determining the equivariant Ehrhart theory of the permutahedron. This project is joint work with Anna Maria Schindler.

Alan Stapledon described an equivariant analogue to Ehrhart theory as an extension of the theory with group actions. For the purposes of this thesis, we consider our group to be the symmetric group S_d and our polytope $\mathcal{P}$ to be the permutahedron Π_d. The permutahedron is a $(d-1)$-dimensional polytope embedded in d-dimensional space, the vertices of which are formed by permuting the coordinates of the vector $(1, 2, 3, \ldots, d)$. An element $\sigma \in S_d$ acts on $\mathbb{Z}^d$ by permuting the coordinates. Let $(\Pi_d)_\sigma$ denote the set of lattice points of Π_d that are fixed by a permutation $\sigma \in S_d$; we call $(\Pi_d)_\sigma$ the fixed subpolytope of Π_d. Stapledon proved that P_σ is a rational polytope. Our work expands this theory where $\mathcal{P} = \Pi_d$ and $P_\sigma = (\Pi_d)_\sigma$ is the fixed subpolytope of the d-permutahedron. We prove the following theorem:

Theorem 3.8. Let Π_d denote the d-permutahedron, and $\sigma \in S_d$, where σ is composed of disjoint cycles, i.e, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$. Let $(\Pi_d)_\sigma$ be the subpolytope of Π_d fixed by σ. Let $V(\sigma)$ denote the set of vertices $v = (x_1, \ldots, x_d)$ of Π_d that satisfy the following property: for each cycle $\sigma_j = (j_1 j_2 \ldots j_r)$ of σ, $x_{j_1}, \ldots, x_{j_r}$ is a consecutive sequence of integers. Let $e_{\sigma_j} = \sum_{i\in\sigma_j} e_i$.

The following are equal:

(A) $(\Pi_d)_\sigma = \{x \in \Pi_d : \sigma \cdot x = x\}$,

(B) $V_\sigma = \operatorname{conv}\left\{\frac{1}{|\sigma|}\sum_{i=1}^{|\sigma|}\sigma^i v : v \in V(\sigma)\right\}$,

(C) $M_\sigma = \sum_{j\neq k}\left[l_j e_{\sigma_k}, l_k e_{\sigma_j}\right] + \sum_k \frac{l_k+1}{2}e_{\sigma_k}$.

Theorem 3.8 establishes that $(\Pi_d)_\sigma$ is a zonotope, the translation of a Minkowski sum of line segments. Also, it shows that $(\Pi_d)_\sigma = V_\sigma$, providing a vertex description of $(\Pi_d)_\sigma$. Stapledon's results tell us that $(\Pi_d)_\sigma = \operatorname{conv}(W(\sigma))$, where W is the set of all vertices of Π_d and $W(\sigma) = \left\{\frac{1}{|\sigma|}\sum_{i=1}^{|\sigma|}\sigma^i w : w \in W\right\}$. As $V(\sigma)$ is a proper subset of $W(\sigma)$, our theorem provides a refinement of Stapledon's definition. Additionally, the fixed subpolytopes $(\Pi_d)_\sigma$ are combinatorially equivalent to Π_m, where m is the number of cycles of σ, and its vertex set is $V(\sigma)$.

The theory of lattice zonotopes provides a formula for the Ehrhart series of a zonotope. Therefore, by proving that the fixed subpolytopes are zonotopes we hoped to determine its Ehrhart series, where we would have concluded our goal of determining the equivariant Ehrhart theory of the permutahedron. Unfortunately, our fixed subpolytopes are not integral, and it is not known whether a similar formula holds for rational zonotopes, which our fixed subpolytopes are.

1.2 Part II

The second part of this thesis introduces a decomposition formula of the h^*-polynomial for rational polytopes. In 1985 Ulrich Betke and Peter McMullen [4] proved that the h^*-polynomial of a lattice polytope can be decomposed as

$$\sum_{\Delta\in\mathcal{T}} h_{\mathrm{link}(\Delta)}(z)B_\Delta(z),$$

where $h_{\mathrm{link}(\Delta)}$ and $B_\Delta(z)$ are defined in Section 2.2. This result is particularly interesting because it brings together arithmetic data from the simplices of the triangulation and combinatorial information from the face structure of the triangulation. In Chapter 4 we prove the analogue of this theorem for rational polytopes:

Theorem 4.3. Fix a triangulation $\mathcal{T}$ of the rational d-polytope $\mathcal{P}$ such that p is the least common multiple of all the vertex coordinate denominators of every $\Delta \in \mathcal{T}$. Then

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\sum_{\Delta \in \mathcal{T}} h_{\mathrm{link}(\Delta)}(z^p) B_{\Delta,p}(z)}{(1 - z^p)^{d+1}}.$$

Chapter 2

Background

In this chapter we discuss the foundational topics that are required to reach our main results.

2.1 Polytopes

We begin with a discussion of convex polytopes. For a more thorough introduction to polytopes, see [2], [12], and [34]. For our pursuits, we focus only on objects that are convex, so we begin with defining convexity.

Definition 2.1. A subset X of $\mathbb{R}^n$ is *convex* if $\{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\} \subseteq X$ for every $x, y \in X$. If we consider a finite set C in $\mathbb{R}^n$, the *convex hull* of C, which we denote $\operatorname{conv}(C)$, is the intersection of all convex sets containing C. Equivalently,

$$\operatorname{conv}(C) = \left\{ \sum_{c \in C} \lambda_c c : \lambda_c \in \mathbb{R}_{\geq 0}, \sum_{c \in C} \lambda_c = 1 \right\}.$$

The d-dimensional analogue of a convex polygon is a convex polytope, i.e., polytopes are geometric objects with flat faces and extreme points called vertices.

Definition 2.2. One way to define a *convex polytope* $\mathcal{P}$ is as the convex hull of points, $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$:

$$\mathcal{P} = \operatorname{conv}\{v_1, v_2, \ldots, v_n\} := \left\{ \lambda_1 v_1 + \cdots + \lambda_k v_n : \lambda_j \geq 0, \sum_{j=1}^{n} \lambda_j = 1 \right\}.$$

This is known as the *vertex-description ($\mathcal{V}$-description)* of $\mathcal{P}$, and we consider $\mathcal{P}$ to be a $\mathcal{V}$-polytope.

Definition 2.3. The set $\{x \in \mathbb{R}^d : Ax = b\}$ is called a *hyperplane*. The sets $\{x \in \mathbb{R}^d : Ax \leq b\}$ and $\{x \in \mathbb{R}^d : Ax \geq b\}$ are called *closed halfspaces*. We say that the hyperplane $H = \{x \in \mathbb{R}^d : Ax = b\}$ is a *supporting hyperplane* of $\mathcal{P}$ if $\mathcal{P}$ lies entirely on one side of H.

Definition 2.4. A polytope can be also defined as the bounded intersection of finitely many half-spaces and hyperplanes. This is known as the *hyperplane description ($\mathcal{H}$-description)* of a polytope, where we consider $\mathcal{P}$ to be a $\mathcal{H}$-polytope.

Theorem 2.5. *A polytope $\mathcal{P}$ is a $\mathcal{V}$-polytope if and only if it is a $\mathcal{H}$-polytope.*

For a proof of this theorem see [34].

Now that we know that these descriptions are equivalent, we need not specify whether a polytope is an $\mathcal{H}$-polytope or $\mathcal{V}$-polytope.

Definition 2.6. The *dimension* of a polytope $\mathcal{P}$ is the dimension of the affine space

$$\operatorname{affspan} \mathcal{P} := \{x + \lambda(y - x) : x, y \in \mathcal{P}, \lambda \in \mathbb{R}\}$$

spanned by $\mathcal{P}$. If $\mathcal{P}$ has dimension d, we denote it as $\dim \mathcal{P} = d$ and refer to $\mathcal{P}$ a *d-polytope*.

The following are several useful definitions regarding polytopes.

Definition 2.7. The integer points $\mathbb{Z}^d$ form a lattice in $\mathbb{R}^d$ and we refer to integer points as *lattice points*. Let $\mathcal{P}$ be a d-polytope, then

- a *face* of $\mathcal{P}$ is a set of the form $\mathcal{P} \cap H$, where H is a supporting hyperplane of $\mathcal{P}$,
- the $(d-1)$-dimensional faces are called *facets*,
- the 1-dimensional faces are called *edges*,
- the 0-dimensional faces are called *vertices* of $\mathcal{P}$ and are the "extreme points" of $\mathcal{P}$,
- a polytope $\mathcal{P}$ is called *integral* if all of its vertices have integer coordinates (also called a *lattice polytope*),
- a polytope $\mathcal{P}$ is called *rational* if all of its vertices have rational coordinates,
- we are also interested in the dilations of $\mathcal{P}$, which we denote $m\mathcal{P} := \{mx : x \in \mathcal{P}\}$.

One particular group of polytopes that will be helpful are simplices, which we now define:

Definition 2.8. A convex d-polytope with exactly $d + 1$ vertices is called a *d-simplex* and is denoted Δ.

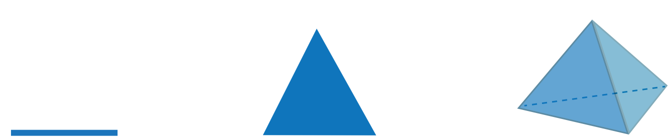


Figure 2.1: From left to right: A 1-simplex, 2-simplex, and 2-simplex.

Observe that a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

Definition 2.9. A *triangulation* of a convex d-polytope $\mathcal{P}$ is a finite collection $\mathcal{T}$ of simplices with the following properties:

- $\mathcal{P} = \cup_{\Delta \in \mathcal{T}} \Delta$
- For every $\Delta_1, \Delta_2 \in \mathcal{T}$, $\Delta_1 \cap \Delta_2$ is a face common to both Δ_1 and Δ_2.

Theorem 2.10. [2] *Every convex polytope can be triangulated using no new vertices, i.e., there exists a triangulation $\mathcal{T}$ such that the vertices of every $\Delta \in \mathcal{T}$ are vertices of $\mathcal{P}$.*

For a proof of this theorem, refer to [2].

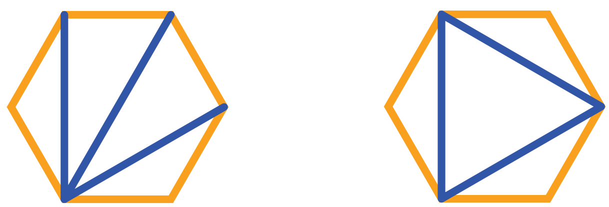


Figure 2.2: Two triangulations of the regular hexagon.

Definition 2.11. A *subdivision* $\mathcal{D}$ of a polytope $\mathcal{P}$ is a non-empty, finite collection of polytopes such that

i) $\mathcal{Q} \in \mathcal{D}$ implies all faces of $\mathcal{Q}$ are also in $\mathcal{D}$,

ii) $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{D}$ implies $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is a common face of both $\mathcal{Q}_1$ and $\mathcal{Q}_2$,

iii) $\cup_{\mathcal{Q} \in \mathcal{D}} \mathcal{Q} = \mathcal{P}$.

Definition 2.12. Consider a polytope $\mathcal{P} \subset \mathbb{R}^d$.

- Given a vector $w \in \mathbb{R}^d$, let $\text{face}_w(\mathcal{P}) = \{x \in \mathcal{P} : x \cdot w \leq y \cdot w, \text{ for all } y \in \mathcal{P}\}$. Then $\text{face}_w(\mathcal{P})$ is a polytope because it is the intersection of $\mathcal{P}$ with the hyperplane $x \cdot w = \min\{x \cdot w : x \in \mathcal{P}\}$. Such a polytope is called a *face* of $\mathcal{P}$.

- For F a face of polytope $\mathcal{P}$, we define the set

$$N_{\mathcal{P}}(F) = \{w \in \mathbb{R}^d : \text{face}_w(\mathcal{P}) = F\},$$

  the *normal cone* of $\mathcal{P}$ at F.

- The *normal fan* of $\mathcal{P}$ is the collection of all the normal cones of $\mathcal{P}$,

$$N(\mathcal{P}) = \{N_{\mathcal{P}}(F) : F \text{ is a face of } \mathcal{P}\}.$$

- Two polytopes are normally equivalent if their normal fans coincide.

A polytope of particular interest to us is the permutahedron. We will see that the numerous combinatorial properties of the permutahedron make it a natural object to study.

Definition 2.13. The *permutahedron*

$$\Pi_d := \text{conv}\{(\pi(1), \pi(2), ..., \pi(d)) : \pi \in S_d\}$$

is the convex hull of $(1, ..., d)$ and all points formed by permuting its entries.

The permutahedron has the following properties [32]:

i) Π_d is a $(d-1)$-dimensional polytope in $\mathbb{R}^d$.

ii) Π_d has $d!$ vertices.

iii) All the vectors v_σ lie on the hyperplane $\sum x_i = \frac{d(d+1)}{2}$, where v_σ is the vertex vector whose coordinates are the elements of σ.

iv) Each v_σ is a vertex of Π_d, and two vertices v_σ and $v_{\sigma'}$ are adjacent (i.e., connected by an edge), if and only if the permutations σ and σ' are adjacent (i.e., differ by the exchange of two consecutive numbers).

Note that a polytope $\mathcal{P} \subset \mathbb{R}^d$ does not necessarily have dimension d, as is the case with the permutahedron. As of now, we have seen the $\mathcal{V}$-description of the permutahedron. The following defines the $\mathcal{H}$-description of the permutahedron.

Proposition 2.14. [5, 32, 34] *The $\mathcal{H}$-description of the permutahedron is*

$$x_1 + \cdots + x_d = \frac{d(d+1)}{2},$$

$$x_{i_1} + \cdots + x_{i_n} \geq \frac{n(n+1)}{2}$$

for $\emptyset \subsetneq \{i_1, \ldots, i_n\} \subsetneq [d]$.

The permutahedron is a nice polytope to study because it falls under a special class of polytopes called zonotopes. Zonotopes are special polytopes that have several equivalent definitions; they are Minkowski sums of line segments and projections of cubes, see below. The following figure depicts the 3-permutahedron as concrete examples of the aforementioned theory.

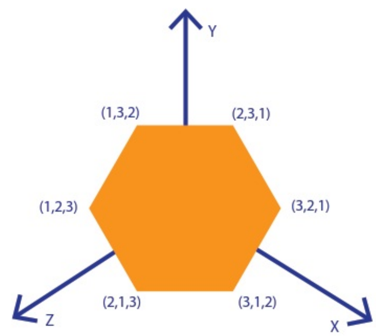


Figure 2.3: Π_3 is a 2-polytope in 3-space.

Definition 2.15. Given d linearly independent vectors $v_1, \ldots, v_d \in \mathbb{Z}^m$, we define the d-dimensional *lattice parallelepiped* as

$$P(v_1, \ldots, v_d) := \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_d v_d : 0 \le \lambda_i \le 1\}.$$

Definition 2.16. Consider the polytopes $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \subset \mathbb{R}^d$. We define the *Minkowski sum* of the n polytopes as

$$\mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_n := \{x_1 + x_2 + \cdots + x_n : x_j \in \mathcal{P}_j \text{ for } 1 \le j \le n\}.$$

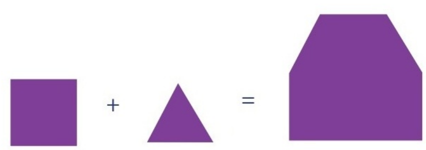


Figure 2.4: Minkowski sum of a square and a triangle.

Definition 2.17. The *unit d-cube* $C^d := [0,1]^d$ is a d-dimensional polytope that takes the standard unit basis vectors $e_1, e_2, \ldots, e_d$ as its generating vectors:

$$C^d = \{\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_d e_d : 0 \le \lambda_i \le 1\}.$$

A zero-dimensional parallelepiped is a point, a line segment is a one-dimensional parallelepiped, and a two-dimensional parallelepiped is a parallelogram. We consider a parallelepiped as a higher-dimensional parallelogram. The unit d-cube is probably the simplest example of a d-dimensional parallelepiped.

Definition 2.18. Take n line segments in $\mathbb{R}^d$, where each line segment has one endpoint at the origin and other endpoint is located at the vector $u_j \in \mathbb{R}^d$, for $j = 1, \ldots, n$. We then define a *zonotope* as the Minkowski sum of these line segments:

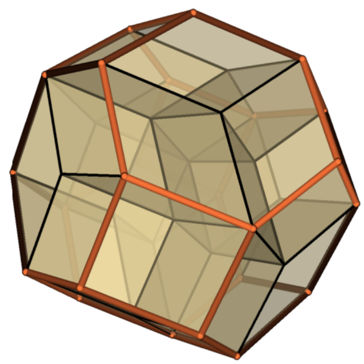
$$Z(u_1, u_2, \ldots, u_n) = \{\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n : 0 \le \lambda_j \le 1\}.$$

Note that any translation of a zonotope is itself a zonotope.

An interesting property of zonotopes is observed in the following theorem, which was proved by Geoffrey Shephard [25].

Theorem 2.19. [25] *Every zonotope admits a subdivision into parallelepipeds.*

(a) Π_3 tiled by parallelograms.



(b) Π_4 tiled by parallelepipeds [23].

Figure 2.5: Examples of different permutahedron subdivided into parallelepipeds.

We stated earlier that the permutahedron is a zonotope, this means it can be written as a Minkowski sum of line segments. That is what the following theorem states.

Theorem 2.20. [2] *Let Π_d be the d-permutahedron. Then*

$$\Pi_d = \sum_{j \neq k} [e_j, e_k] + \sum_j e_j \ ,$$

i.e., Π_d is the Minkowski sum of the line segments between each pair of unit vectors in $\mathbb{R}^d$ and a translation vector.

2.2 Ehrhart Theory

2.2.1 Ehrhart Theory in Dimension 2

The power of Ehrhart theory is best illustrated by examples.

Example 2.21. Let $S = \mathrm{conv}\{(-1, 1), (-1, -1), (1, 1), (1, -1)\}$. This gives the following square in $\mathbb{R}^2$.

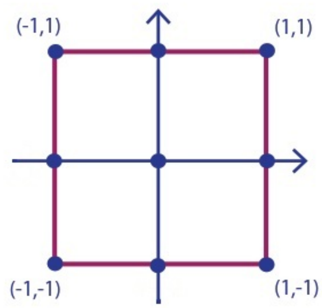


Figure 2.6: $S = \text{conv}\{(-1,1),(-1,-1),(1,1),(1,-1)\}$.

The following properties of S can be easily verified from the image:

1. S has area 4.
2. S has 8 lattice boundary points.
3. S has a total of 9 lattice points including the boundary and interior.

These three properties are tied together by the expression

$$9 = 4 + \frac{8}{2} + 1.$$

That is, the number of lattice points in S is equal to the sum of the area of S, one-half the number of boundary points and one. In 1899, the Austrian mathematician Georg Pick proved that this relationship holds for any convex lattice polygon [21].

Theorem 2.22 (Pick 1899)**.** *For any convex lattice polygon P*

$$L_P = A_P + \frac{B_P}{2} + 1$$

where A_P, B_P, and L_P denote the area of P, number of lattice points on the boundary of P, and total number of lattice points of P, respectively.

If we dilate the polygon by a factor of m, is there a polynomial that counts the number of lattice points? The answer to this is yes, and is the result of Ehrhart's theorem [10].

Theorem 2.23 (Ehrhart's Theorem in Dimension 2)**.** *Let P be a convex lattice polygon and let m be a positive integer. The following equality holds:*

$$L_P(m) = A_P m^2 + \frac{B_P}{2} m + 1.$$

Example 2.24. Taking S as defined in the previous example, we dilate by a factor of m to obtain

$$L_S(m) = 4m^2 + 4m + 1.$$

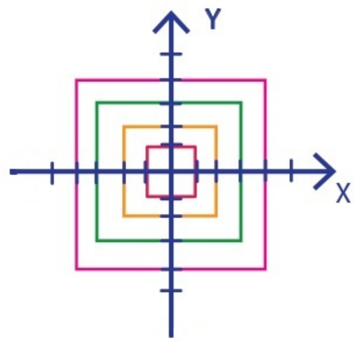


Figure 2.7: Dilations of S.

Recall that the permutahedron is a $(d-1)$-dimensional polytope, so let us look at another example, the 3-permutahedron, which is a hexagon.

Example 2.25. Let Π_3 denote the following hexagon on a triangular lattice:

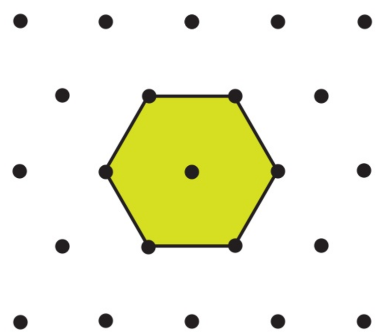


Figure 2.8: A hexagon on a triangular lattice.

Similar to the other examples, we can gather that the lattice points in dilations of the hexagon can be counted by the following polynomial:

$$L_{\Pi_3} = 3m^2 + 3m + 1.$$

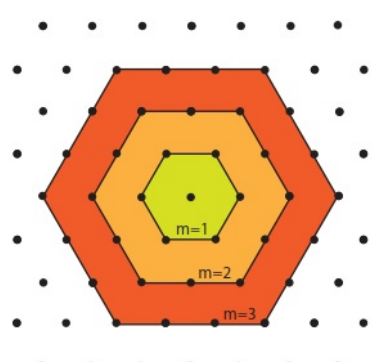


Figure 2.9: A hexagon and its dilates.

Next, we provide the tools to explore Ehrhart theory in higher dimensions.

2.2.2 Ehrhart Theory in General Dimension

We have seen in dimension two that the Ehrhart polynomial of a polygon is a counting function for the number of integer points in the polygon. In general, we consider a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ and its Ehrhart function

$$\operatorname{ehr}_{\mathcal{P}}(m) := \left| m\mathcal{P} \cap \mathbb{Z}^d \right|.$$

We call the corresponding generating function

$$\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{m \geq 1} \operatorname{ehr}_{\mathcal{P}}(m) z^m$$

the *Ehrhart series* of $\mathcal{P}$. The Ehrhart function turns out to be a polynomial in any dimension. Before we continue detailing facts about the Ehrhart polynomial and its Ehrhart series, we continue by stating some facts about triangulations, cones, and other topics, which will become useful as we continue.

Definition 2.26. A *pointed cone* $\mathcal{K} \subseteq \mathbb{R}^d$ is a set of the form

$$\mathcal{K} = \{v + \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m : \lambda_1, \ldots, \lambda_m \geq 0\},$$

where $v, w_1, w_2, \ldots, w_m \in \mathbb{R}^d$ are such that there exists a hyperplane H for which $H \cap \mathcal{K} = \{v\}$, i.e., $\mathcal{K} \backslash \{v\}$ lies strictly on one side of H. In this setting

- the vector v is called the *apex* of $\mathcal{K}$,
- the w_k are the *generators* of $\mathcal{K}$,
- the cone is *rational* if $v, w_1, \ldots, w_m \in \mathbb{Q}^d$,
- the *dimension* of $\mathcal{K}$ is the dimension of the affine space spanned by $\mathcal{K}$,
- if $\mathcal{K}$ is of dimension d it is called a *d-cone*,
- a d-cone $\mathcal{K}$ is *simplicial* if $\mathcal{K}$ has precisely d linearly independent generators.

We study the arithmetic of a set $\mathcal{S} \subseteq \mathbb{R}^d$ by computing a multivariate generating function, which we call the integer-point transform. We want to encode the information contained by the lattice points in the set $\mathcal{S}$.

Definition 2.27. If S is a rational cone or polytope and $\vec{z} = (z_1, \ldots, z_d)$, then the *integer-point transform* is

$$\sigma_{\mathcal{S}}(\vec{z}) = \sigma_{\mathcal{S}}(z_1, z_2, \ldots, z_d) := \sum_{m \in \mathcal{S} \cap \mathbb{Z}^d} \vec{z}^{\,m},$$

the generating function that lists all integer points in S as a formal sum of Laurent monomials. Here $\vec{z}^m = z_1^{m_1} \cdots z_d^{m_d}$.

Generating functions of this form are extremely helpful for lattice-point problems. When we are interested in the lattice-point count, we evaluate $\sigma_{\mathcal{S}}$ at $z = (1, 1, \dots, 1)$:

$$\sigma_{\mathcal{S}}(1, 1, \dots, 1) = \sum_{m \in \mathcal{S} \cap \mathbb{Z}^d} \vec{1}^m = \sum_{m \in \mathcal{S} \cap \mathbb{Z}^d} 1 = \#(\mathcal{S} \cap \mathbb{Z}^d),$$

where $\vec{1}$ denotes the vector whose components are all 1.

Theorem 2.28. *Suppose*

$$\mathcal{K} := \{\lambda_1 w_1, \lambda_2 w_2 + \cdots + \lambda_d w_d : \lambda_1, \lambda_2, \dots, \lambda_d \geq 0\}$$

is a simplicial d-cone, where $w_1, \dots, w_d \in \mathbb{Z}^d$. Then for $v \in \mathbb{R}^d$, the integer-point transform $\sigma_{v+\mathcal{K}}$ of the shifted cone $v + \mathcal{K}$ is the rational function

$$\sigma_{v+\mathcal{K}} = \frac{\sigma_{v+\Pi}(\vec{z})}{(1 - \vec{z}^{\,w_1})(1 - \vec{z}^{\,w_2}) \cdots (1 - \vec{z}^{\,w_d})},$$

where Π is the half-open parallelepiped

$$\Pi := \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_d w_d : 0 \leq \lambda_1, \lambda_2, \dots, \lambda_d < 1\}.$$

The half-open parallelepiped Π is called the *fundamental parallelepiped* of $\mathcal{K}$.

Corollary 2.29. *For a pointed cone*

$$\mathcal{K} = \{v + \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_1 w_m : \lambda_1, \lambda_2, \dots, \lambda_m \geq 0\}$$

with $v \in \mathbb{R}^d$, $w_1, w_2, \dots, w_m \in \mathbb{Z}^d$, the integer-point transform $\sigma_{\mathcal{K}}(z)$ evaluates to a rational function in the coordinates of z.

For a proof of Theorem 2.28 and Corollary 2.29, refer to [2].

Now, you may be wondering: why are cones important? The answer to that question in our setting is that we can cone over a polytope. If we are given a polytope $\mathcal{P} \subset \mathbb{R}^d$ with vertices $v_1, v_2, \dots, v_n$, we lift these vertices into $\mathbb{R}^{d+1}$ by adding a 1 to their last coordinate. We then proceed by letting the generators of cone($\mathcal{P}$) be

$$w_1 = (v_1, 1), w_2 = (v_2, 1), \dots, w_n = (v_n, 1).$$

With this, now we define our cone over a polytope.

Definition 2.30. The *cone over* $\mathcal{P} = \operatorname{conv}(v_1, \dots, v_n)$ is defined to be

$$\operatorname{cone}(\mathcal{P}) = \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_n w_n : \lambda_1, \lambda_2, \dots, \lambda_n \geq 0\} \subset \mathbb{R}^{d+1},$$

where $w_i = (v_i, 1)$ for $i = 1, \dots, n$

This pointed cone has the origin as its apex, and we can obtain our original polytope $\mathcal{P}$ (to be more precise, its translated copy $\{(x,1) : x \in \mathcal{P}\}$) by cutting cone($\mathcal{P}$) with the hyperplane $x_{d+1} = 1$, as can be seen in Figure 2.10. We can then compute the Ehrhart series of $\mathcal{P}$ as

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \sum_{m\geq 0} \#(\text{lattice points in } \operatorname{cone}(\mathcal{P}) \text{ at height } m) z^m.$$

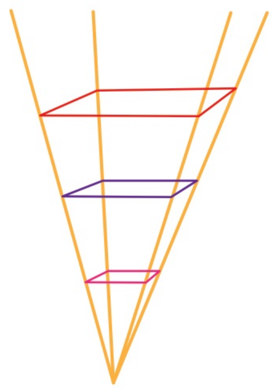


Figure 2.10: Cone over a square.

This leads us to the following theorem.

Theorem 2.31. [2] *For a lattice polytope $\mathcal{P}$*

$$\sigma_{\operatorname{cone}(\mathcal{P})}(1,1,\ldots,1,z) = 1 + \sum_{t\geq 1} \mathrm{ehr}_{\mathcal{P}}(m) z^m = \mathrm{Ehr}_{\mathcal{P}}(z).$$

Definition 2.32. The *normalized volume* of a d-polytope $\mathcal{P}$ on a lattice M is its d-dimensional volume divided by the volume of any fundamental parallelepiped of M, i.e., the parallelepiped generated by any lattice basis. The *surface volume* of P is the sum of $(d-1)$-dimensional volumes of its facets.

The following result is due to the French mathematician Eugene Ehrhart [10].

Theorem 2.33 (Ehrhart 1962)**.** *Let $\mathcal{P}$ be a d-polytope with integer vertices in $\mathbb{R}^n$. Then*

$$\mathrm{ehr}_{\mathcal{P}}(m) = c_d m^d + c_{d-1} m^{d-1} + \cdots + c_1 m + c_0$$

is a polynomial in m of degree d, which we call the Ehrhart polynomial. Furthermore,

$$c_d = \textit{volume of } \mathcal{P},\ c_{d-1} = \frac{1}{2}\left(\textit{surface volume of } \mathcal{P}\right),\ \textit{and } c_0 = 1.$$

Now we encode the Ehrhart polynomial in a generating function. As mentioned earlier, this is the Ehrhart series.

Corollary 2.34. *For a lattice polytope $\mathcal{P}$ in $\mathbb{R}^n$, the Ehrhart series of $\mathcal{P}$ is*

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{P}}(z) = \sum_{m\geq 0} \mathrm{ehr}_{\mathcal{P}}(m) z^m &= \mathrm{ehr}_{\mathcal{P}}(0)z^0 + \mathrm{ehr}_{\mathcal{P}}(1)z^1 + \mathrm{ehr}_{\mathcal{P}}(2)z^2 + \cdots \\ &= \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_1^* z + h_0^*}{(1-z)^{d+1}},\end{aligned}$$

where h_k^* equals the number of integer points in the fundamental parallelepiped Π of $\mathrm{cone}(\mathcal{P})$ that are on the hyperplane $x_{d+1} = k$.

For proofs and further reading, refer to [2, 3, 6, 10]

We call the numerator of $\mathrm{Ehr}_{\mathcal{P}}$ the h^*-*polynomial*, and the vector $(h_0^*, \ldots, h_d^*)$ of its coefficients is called the h^*-*vector* for $\mathcal{P}$. One theorem about this numerator is due to Richard Stanley and is sometimes called Stanley's non-negativity theorem [26].

Theorem 2.35 (Stanley 1980)**.** *If $\mathcal{P}$ is a lattice d-polytope, then $(h_0^*, \ldots, h_d^*) \in \mathbb{Z}_{\geq 0}^{d+1}$.*

Definition 2.36. A *graph* is a pair of sets (V, E), where V is the set of nodes and E is the set of edges, formed by pairs of vertices. Furthermore, a *forest* is a graph with no cycles.

Theorem 2.37 (Ehrhart Polynomial of the Permutahedron)**.** *The coefficient c_k of the Ehrhart polynomial*

$$\mathrm{ehr}_{\Pi_d}(m) = c_{d-1}m^{d-1} + \cdots + c_1 m + c_0$$

of the permutahedron Π_d equals the number of labeled forests on d nodes with k edges.

For more on this theorem including its proof, refer to [2, 29].

Going back to our original example, we can compute the Ehrhart polynomial seen in Figure 2.11 for the hexagon using the information from the above theorem. We see that there are 3 labeled forests with 2 edges, 3 with 1 edge, and 1 with 0 edges. This is consistent with the Ehrhart polynomial in Example 2.25: $\mathrm{ehr}_{\Pi_3}(m) = 3m^2 + 3m + 1$.

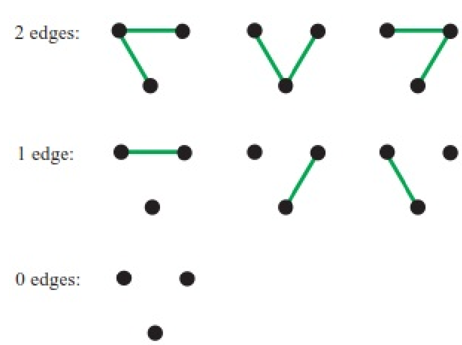


Figure 2.11: The $3 + 3 + 1$ labeled forests on three nodes.

We can also use this method to find the Ehrhart polynomial for Π_4. On 4 vertices there are 16 labeled forests wth 3 edges, 15 with 2 edges, 6 with one edge, and 1 with 0 edges, giving us

$$\mathrm{ehr}_{\Pi_4}(m) = 16m^3 + 15m^2 + 6m + 1.$$

Let us now look at a theorem that shows a decomposition of the h^*-polynomial of a lattice polytope which we later generalize for rational polytopes in Chapter 4. First, we provide some set-up and definitions.

Given a lattice d-polytope $\mathcal{P}$, we fix a lattice triangulation $\mathcal{T}$. Given a simplex $\Delta \in \mathcal{T}$, let $\Pi(\Delta)^\circ$ be the fundamental parallelepiped of cone(Δ) and let $B_\Delta(z) := \sigma_{\Pi(\Delta)^\circ}(1, 1, \ldots, 1, z)$. Thus, $B_\Delta(z)$ is an "open"variant of $h^*_\Delta(z) = \sigma_{\Pi(\Delta)}(1, 1, \ldots, 1, z)$. We include $\emptyset$ in the collection of faces of a triangulation of a polytope and by convention we define $\dim(\emptyset) := -1$; we consider $\emptyset$ the empty simplex and define $B_\emptyset(z) := 1$. We need one more concept before introducing the theorem:

Definition 2.38. Given a simplex $\Delta \in \mathcal{T}$, let

$$\mathrm{link}_\mathcal{T}(\Delta) := \{\Omega \in \mathcal{T} : \Omega \cap \Delta = \emptyset, \Omega \subseteq \Phi \text{ for some } \Phi \in \mathcal{T} \text{ with } \Delta \subseteq \Phi\},$$

the *link* of Δ. In words, this is the set of simplices that are disjoint from Δ, yet are still a face of a simplex in $\mathcal{T}$ containing Δ. We then encode the combinatorial data coming from the simplices in the h-polynomial (to not deviate too much, for further reading on h-polynomials, see [2, Theorem 10.2] or [29]) of the link as

$$h_{\mathrm{link}(\Delta)}(z) = (1-z)^{d-\dim(\Delta)} \sum_{\Phi \supseteq \Delta} \left(\frac{z}{1-z}\right)^{\dim(\Phi)-\dim(\Delta)}, \tag{2.1}$$

where the sum is over all simplices $\Phi \in \mathcal{T}$ containing Δ, and d is the dimension of the polytope $\mathcal{P}$.

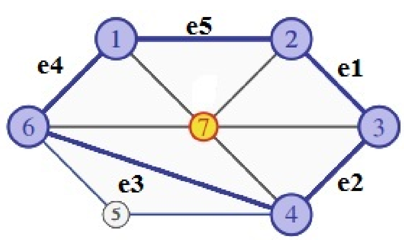


Figure 2.12: $\mathrm{link}(7) = \{1, 2, 3, 5, 6, e1, e2, e3, e4, e5\}$.

In Figure 2.12 the hexagon is a simplicial complex. The link of 7 in the simplicial complex is $\mathrm{link}(7) = \{1, 2, 3, 5, 6, e1, e2, e3, e4, e5\}$. The aforementioned set-up was necessary to introduce the following theorem. It decomposes the h^*-polynomial and is due to Ulrich Betke and Peter McMullen [4].

Theorem 2.39 (Betke & McMullen 1985)**.** *Fix a lattice triangulation $\mathcal{T}$ of the lattice d-polytope $\mathcal{P}$. Then*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\sum_{\Delta \in \mathcal{T}} h_{\mathrm{link}(\Delta)}(z) B_{\Delta}(z)}{(1-z)^{d+1}}.$$

Consider an example we have seen before, $S = \mathrm{conv}\{(-1,1),(-1,-1),(1,1),(1,-1)\}$. The following are two ways to compute its Ehrhart series: from its Ehrhart polynomial and then by using Theorem 2.39.

Example 2.40. Take the polytope S defined in Example 2.21. Then the Ehrhart Series of S is

$$\begin{aligned} \mathrm{Ehr}_S(z) &= \sum_{m \geq 0} \mathrm{ehr}_S(m) z^m \\ &= \sum_{m \geq 0} (4m^2 + 4m + 1) z^m \\ &= \frac{z^2 + 6z + 1}{(1-z)^3}. \end{aligned}$$

Next we use Theorem 2.39. Fix the following triangulation.

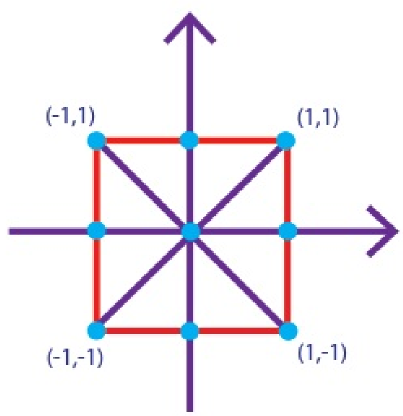


Figure 2.13: A triangulations of S.

Through careful computation, we obtain:

dimension	**convex hull**	$h_{\mathrm{link}(\Delta)}(z^p)$	$B_\Delta(z)$
2	$\{(-1,1)\,,(1,1)\,,(0,0)\}$	1	z
2	$\{(1,1)\,,(1,-1)\,,(0,0)\}$	1	z
2	$\{(-1,-1,(1,-1)\,,(0,0)\}$	1	z
2	$\{(-1,-1,(-1,1)\,,(0,0)\}$	1	z
1	$\{(-1,1),(1,1)$	1	0
1	$\{(1,1),(1,-1)\}$	1	0
1	$\{(-1,-1),(1,-1)\}$	1	0
1	$\{(-1,-1),(-1,1)\}$	1	0
1	$\{(-1,1),(0,0)\}$	$z+1$	0
1	$\{(1,1),(0,0)\}$	$z+1$	0
1	$\{(1,-1),(0,0)\}$	$z+1$	0
1	$\{(1,-1),(0,0)\}$	$z+1$	0
0	$\{(-1,1)\}$	$3z+1$	0
0	$\{(1,1)\}$	$3z+1$	0
0	$\{(1,-1)\}$	$3z+1$	0
0	$\{(-1,-1)\}$	$3z+1$	0
0	$\{(0,0)\}$	z^2+2z+1	0
∅		z^2+2z+1	1

Applying Theorem 2.39, we obtain

$$\mathrm{Ehr}_S(z) = \frac{z^2+6z+1}{(1-z)^3},$$

which agrees with the Ehrhart series above.

To see how to compute $B_\Delta(z)$, you may refer to Example 4.4.

From the decomposition of the h^*-polynomial, provided by Betke and McMullen, Alan Stapledon was able to prove the following relationship involving the h^*-polynomial.

Theorem 2.41 (Stapledon 2009)**.** [30] *If $h^*_{\mathcal{P}}(z)$ has degree s, then there exist unique polynomials $a(z)$ and $b(z)$ with nonnegative coefficients such that*

$$(1+z+\cdots+z^{d-s})h^*_{\mathcal{P}}(z) = a(z) + z^{d+1-s}b(z),$$

$a(z) = z^d a(\frac{1}{z}), b(z) = z^{d-l}b(\frac{1}{z})$, and, writing $a(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0$,

$$1 = a_0 \le a_1 \le a_j \text{ for } 2 \le j \le d-1.$$

2.2.3 Ehrhart Theory for Rational Polytopes

For this subsection, let $\mathcal{P}$ be a d-dimensional convex polytope in $\mathbb{R}^n$ whose vertices have rational coordinates. As stated earlier, we call $\mathcal{P}$ a rational polytope. Here we see some results that parallel the results for lattice polytopes.

Definition 2.42. A *quasipolynomial* Q is an expression of the form

$$Q(m) = c_n(m)m^n + \cdots + c_1(m)m + c_0(m),$$

where $c_0, \ldots, c_n$ are periodic functions in m. The *degree* of Q is n and the least common multiple of $c_0, \ldots, c_n$ is the *period* of Q. An alternate way of looking at a quasipolynomial Q is that there exist a positive integer k and polynomials $p_0, p_1, \ldots, p_{k-1}$ such that

$$Q(m) = \begin{cases} p_0(m) & \text{if } m \equiv 0 \bmod k, \\ p_1(m) & \text{if } m \equiv 1 \bmod k, \\ \vdots & \\ p_{k-1}(m) & \text{if } m \equiv k-1 \bmod k. \end{cases}$$

The minimal such k is the period of Q, and for this minimal k, the polynomials $p_0, p_1, \ldots, p_{k-1}$ are the *constituents* of Q.

Theorem 2.43 (Ehrhart's Theorem for Rational Polytopes)**.** *If $\mathcal{P}$ is a rational d-polytope, then $\mathrm{ehr}_{\mathcal{P}}(m)$ is a quasipolynomial in m of degree d. Its period divides the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$.*

Corollary 2.44. *If $\mathcal{P}$ is a rational d-polytope in $\mathbb{R}^n$ and $p \in \mathbb{Z}_{>0}$ is such that $p\mathcal{P}$ is a lattice polytope, the Ehrhart series of $\mathcal{P}$ is*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{m \geq 1} \mathrm{ehr}_{\mathcal{P}}(m) z^m = \frac{h^*_{\mathcal{P},p}(z)}{(1-z^p)^{d+1}},$$

where the numerator is called the h^*-polynomial and is of degree less than $p(d+1)$. Note that the h^*-polynomial depends on the choice of p.

One possible p is any multiple of the denominators of the vertex coordinates of $\mathcal{P}$. The proofs for the previous theorems can be found in [2, 16, 27]. The following theorem is in the class of reciprocity theorems and is thanks to Ian Macdonald [16].

Theorem 2.45 (Ehrhart–Macdonald Reciprocity Theorem)**.** *Suppose $\mathcal{P}$ is a convex rational d-polytope and $m \in \mathbb{Z}_{\geq 0}$. Then the evaluation of the quasipolynomial $\mathrm{ehr}_{\mathcal{P}}$ at negative integers yields*

$$\mathrm{ehr}_{\mathcal{P}}(-m) = (-1)^d \, \mathrm{ehr}_{\mathcal{P}^\circ}(m),$$

where $\mathcal{P}^\circ$ is the relative interior of $\mathcal{P}$.

Now that we have this information about rational polytopes, let us look at an example.

Example 2.46. Let $\mathcal{P} = \mathrm{conv}\{\left(\frac{-1}{2}, 1\right), \left(\frac{-1}{2}, -1\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right)\}$.

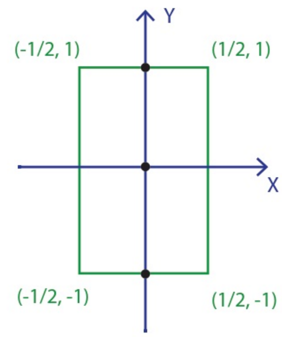


Figure 2.14: $\mathcal{P} = \text{conv}\{\left(\frac{-1}{2}, 1\right), \left(\frac{-1}{2}, -1\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right)\}$.

The Ehrhart polynomial of $\mathcal{P}$ can be shown to be

$$\text{ehr}(m) = \begin{cases} 2m^2 + 3m + 1 & \text{when } m \text{ is even,} \\ 2m^2 + m & \text{when } m \text{ is odd.} \end{cases}$$

We now compute the Ehrhart series:

$$\begin{aligned}
\text{Ehr}(z) &= \sum_{m \geq 0} \text{ehr}(m) z^m \\
&= \sum_{\substack{m \geq 0 \\ m \text{ even}}} (2m^2 + 3m + 1) z^m + \sum_{\substack{m \geq 1 \\ m \text{ odd}}} (2m^2 + m) z^m \\
&= \sum_{k \geq 0} [2(2k)^2 + 3(2k) + 1] z^{2k} + \sum_{k \geq 0} [2(2k+1)^2 + (2k+1)] z^{2k+1} \\
&= \frac{3z^4 + 12z^2 + 1}{(1 - z^2)^3} + \frac{z^5 + 12z^3 + 3z}{(1 - z^2)^3} \\
&= \frac{z^5 + 3z^4 + 12z^3 + 12z^2 + 3z + 1}{(1 - z^2)^3},
\end{aligned}$$

which is in agreement with Corollary 2.44

2.3 Representation Theory

Before we get to equivariant Ehrhart theory, we recall in this section some fundamental facts from representation theory. For an introduction to this field of mathematics and proofs of the statements made, we refer to [11] and [24].

Definition 2.47. Let G be a finite group with identity e, and let S be a finite set. A *group action* of G on S is an operation $\cdot : G \times S \to S$ such that, for all $g, h \in G$ and all $s \in S$:

1. $g \cdot s \in S$

2. $(gh) \cdot s = g \cdot (h \cdot s)$

3. $e \cdot s = s$.

Definition 2.48. Let V be a d-dimensional vector space over $\mathbb{C}$. Let Mat_d be the set of all $d \times d$ matrices with complex entries and define GL_d to be the complex general linear group of degree d,

$$\mathrm{GL}_d = \{X : X = (x_{i,j})_{d\times d} \in \mathrm{Mat}_d \text{ is invertible}\}.$$

Definition 2.49. A *representation* of G is a multiplication $\cdot : G \times V \to V$ such that, for all $g, h \in G$, all $v, w \in V$, and all $c, d \in \mathbb{C}$:

1. $g \cdot v \in V$

2. $g \cdot (cv + dw) = c(g \cdot v) + d(g \cdot w)$

3. $(gh) \cdot v = g \cdot (h \cdot v)$

4. $e \cdot v = v$.

Observe that each group element is associated with a linear transformation of V. If we choose a basis for V, we can consider the representation as a group homomorphism $X : G \to \mathrm{GL}_d$ mapping each group element to the matrix associated with this linear transformation. We summarize this idea in the following definition.

Definition 2.50. A *matrix representation* of a group G is a group homomorphism

$$X \,:\, G \to \mathrm{GL}_d\,.$$

From this definition it follows that $X(e) = I$ is the identity matrix, where e is the identity element in the group G, and $X(gh) = X(g)X(h)$, for all $g, h \in G$.

Let us look at some examples of representations on the symmetric group S_d.

Example 2.51. The trivial representation of S_d:

$$X \,:\, S_d \to \mathrm{GL}_1$$

$$X(g) = 1 \;\; \forall g \in S_d$$

Example 2.52. The sign representation of S_d:

$$X \,:\, S_d \to \mathrm{GL}_1$$

$$X(g) = \mathrm{sign}(g) \;\; \forall g \in S_d$$

where $\mathrm{sign}(g)$ is the function yielding -1 if g is an odd permutation, and 1 if g is an even permutation. Any permutation may be written as a product of transpositions. If the number of transpositions is even then it is an even permutation, otherwise it is an odd permutation.

Definition 2.53. Let S be a finite set and G a group. Let $\mathbb{C}S$ denote the vector space over $\mathbb{C}$ generated by S. If G has a group action on S, then this action can be extended to $\mathbb{C}S$, giving us a representation called a *permutation representation.*

Example 2.54. Let X be a permutation representation. Then, for all $g \in G$, $X(g)$ has all entries equal to either 0 or 1. If we let $X(g) = (x_{i,j})_{d\times d}$, where the rows and columns are indexed with basis elements $s_1, \dots, s_d$ of $\mathbb{C}S$, then

$$x_{i,j} = \begin{cases} 1 & \text{if } g \cdot s_j = s_i, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.55. Consider S_3 and the permutation representation

$$X \,:\, S_3 \to \mathrm{GL}_3 \,.$$

Then

$$X(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad X((1,2)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$X((1,3)) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad X((2,3)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$
$$X((1,2,3)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad X((3,2,1)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Definition 2.56. The *group character* of a representation X is the map $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{tr}(X(g)) = \sum_{i=1}^{d} x_{i,i}$.

If $X(g)$ is a permutation representation, then $x_{i,i} = 1$ if $g \cdot s_i = s_i$, and $x_{i,i} = 0$ otherwise. Thus $\chi(g)$ counts the number of basis elements of $\mathbb{C}S$ that are fixed under the action of g. We will call these *fixed points.*

Definition 2.57. A *class function* is a function on a group G that is constant on the conjugacy classes of G.

Note that the character of a representation of G over a field is always a class function.

Example 2.58. One important example of a permutation representation is that associated with the left action of G on itself, called the *standard representation* of G, or X_{std}. Note that, if $g \neq e$, $\{h \in G : g \cdot h = h\} = \emptyset$, i.e., there are no points fixed by g. On the other hand, all points in G are fixed by e. Thus, if χ_{std} denotes the character associated with the standard representation,

$$\chi_{\mathrm{std}}(g) = \begin{cases} |G| & \text{if } g = e, \\ 0 & \text{otherwise.} \end{cases}$$

2.4 Equivariant Ehrhart Theory

In 2011, Alan Stapledon described an equivariant analogue to Ehrhart theory; refer to [31] for further reading and proofs of theorems stated in this section. We will see that equivariant Ehrhart theory allows us to encode the action of a group G on a polytope $\mathcal{P}$. In particular, the theory organizes and quantifies the points of P that are fixed by the action of the various elements of G. Let

- G be a finite group,
- $\mathcal{P}$ be a lattice polytope that is invariant under the action of G,
- $m\mathcal{P}$ be the mth dilate of $\mathcal{P}$,
- $X_{m\mathcal{P}}$ denote the permutation representation induced by the action of G on the lattice points $m\mathcal{P} \cap \mathbb{Z}^d$.

Our first object of interest is the character associated with $X_{m\mathcal{P}}$, which we denote by $\chi_{m\mathcal{P}}$. For a fixed element $g \in G$, recall that a permutation character evaluated at g counts the number of points fixed by the group action. It is the case that $\chi_{m\mathcal{P}}(g)$ is the character that counts the number of lattice points in mP that are fixed by the group action.

Theorem 2.59 (Stapledon 2011)**.** *Let $\mathcal{P}_g$ denote the set of lattice points of $\mathcal{P}$ that are fixed by g, i.e., $\mathcal{P}_g = \{x \in \mathcal{P} : g \cdot x = x\}$. Then*

$$\mathcal{P}_g = \operatorname{conv}\left\{ \frac{1}{|g|} \sum_{i=1}^{|g|} g^i \cdot v : v \text{ is a vertex of } \mathcal{P} \right\}$$

is a rational polytope.

Corollary 2.60. *Let $m \in \mathbb{Z}$ be the dilation factor of $\mathcal{P}$, then $m\mathcal{P}_g$ is a rational polytope.*

Computing $\chi_{m\mathcal{P}}$ is equivalent to enumerating the lattice points contained in $m\mathcal{P}_g$ for each $g \in G$. We can look to Ehrhart theory to address this enumeration.

Corollary 2.61. *If $\mathrm{ehr}_{\mathcal{P}_g}(m)$ denotes the Ehrhart quasi-polynomial for $\mathcal{P}_g$, then $\chi_{m\mathcal{P}}(g) = \mathrm{ehr}_{\mathcal{P}_g}(m)$.*

Since $m\mathcal{P}_g$ is a rational polytope, it follows that the characters $\chi_{m\mathcal{P}}(g)$ are quasi-polynomials, i.e.,

$$\chi_{m\mathcal{P}}(g) = \mathrm{ehr}_{\mathcal{P}_g}(m) = c_n(m)m^n + \cdots + c_1(m)m + c_0(m),$$

where $c_0, \ldots, c_n$ are periodic functions in m.

Recall that, as a class function, χ_{mP} is constant on conjugacy classes (for a fixed m). Thus we study χ_{mP} by looking at the character individually on each conjugacy class. Taken together, these characters give us a full picture of how G acts on the dilates of $\mathcal{P}$.

Note that if $g = e$, then $\chi_{m\mathcal{P}}(g)$ is simply the ordinary Ehrhart quasi-polynomial $\mathrm{ehr}_{\mathcal{P}}(m)$. Furthermore, if G is the trivial group, equivariant Ehrhart theory reduces to ordinary Ehrhart theory.

Corollary 2.62. *The leading coefficient* $c_n(m)$ *of* $\chi_{m\mathcal{P}}(g)$ *is equal to* $\frac{\operatorname{vol}\mathcal{P}}{|G|}\chi_{\text{std}}$.

The second leading coefficient $c_{n-1}(m)$ has similar properties:

Corollary 2.63.

$$c_{n-1}(m)(g) = \begin{cases} \dfrac{\textit{surface volume } \mathcal{P}}{2} & \textit{if } g = e \textit{ and } \mathcal{P} \textit{ is a lattice polytope,} \\ \\ \textit{volume } \mathcal{P}_g & \textit{if } g \textit{ is a transposition and } m \equiv 0 \bmod \operatorname{ind}(\mathcal{P}_g), \\ 0 & \textit{otherwise,} \end{cases}$$

where the index of $\mathcal{P}_g$*, denoted* $\operatorname{ind}(\mathcal{P}_g)$*, is the smallest positive integer* m *such that the affine span of* $m\mathcal{P}_g$ *contains a lattice point.*

Now, we turn our attention to the generating series for $\chi_{mP}(g)$.

Definition 2.64. Let ρ denote the permutation representation of G on the lattice M, i.e., $\rho : G \to \mathrm{GL}(M)$. Then, we define the equivariant Ehrhart series as

$$\sum_{m \geq 0} \chi_{mP}(g) z^m = \frac{\phi(z)}{(1-z)\det[I - \rho(g)z]}$$

for each $g \in G$.

Thus, $\phi(z)$ is the equivariant analog to the h^* polynomial of ordinary Ehrhart theory; $\phi(z)$ is a rational function [31].

Just as χ_{mP} restricts to the Ehrhart quasi-polynomial for $\mathcal{P}$ when G is the trivial group, in this case $\phi(z)$ restricts to the h^*-polynomial. This can be verified by noting that for a d-dimensional lattice, $I - \rho z$ is a $d \times d$ diagonal matrix with diagonal entries. For a lattice polytope, this means that its determinant is $(1-z)^d$. Comparing with the closed form of the Ehrhart series, we see that $\phi(z) = h^*(z)$. In the next chapter we compute some examples.

Chapter 3

Fixed Subpolytopes of the Permutahedron

In the following, we compute examples and prove certain theorems, which lead us closer to describing an equivariant Ehrhart theory for the permutahedron.

First, we compute $\chi_{m\mathcal{P}}(g)$ for each conjugacy class of S_3 and S_4 as well as compute the closed form of the generating series. By comparing our result to

$$\sum_{m\geq 0} \chi_{mP}(g) z^m = \frac{\phi(z)}{(1-z)\det[I-\rho(g)z]}$$

we will recover $\phi(z)$.

Example 3.1. Let $\mathcal{P} = \Pi_3$ be the 3-permutahedron and let $G = S_3$ act on the lattice points of $m\Pi_3$ by permuting their coordinates. Note that S_3 has three conjugacy classes: that of $e, (ij),$ and (ijk). Take $\sigma \in S_3$.

***e*:**
For $\sigma = e$, $(\Pi_3)_e = \Pi_3$, so $\chi_{m\Pi_3}(e)$ is the ordinary Ehrhart polynomial for the 3-permutahedron, which we saw in Example 2.25:

$$\chi_{m\Pi_3}(e) = \mathrm{ehr}_{\Pi_3}(m) = 3m^2 + 3m + 1.$$

The corresponding Ehrhart series is

$$\sum_{m\geq 0} \left(3m^2+3m+1\right) z^m = \frac{z^2+4z+1}{(1-z)^3}.$$

In this case the denominator of the equivariant series will be $(1-z)^3$, so we conclude that $\phi(z) = z^2 + 4z + 1$, which is, as expected, the ordinary h^*-polynomial for Π_3.

***(ij)*:**
For $\sigma = (ij)$, $(\Pi_3)_\sigma$ is the intersection of Π_3 with the plane $x_i = x_j$, which is the axis of reflection for the transformation enacted by (ij). Thus, $(\Pi_3)_\sigma$ is a line segment with rational

vertices, see Figure 3.1. By counting lattice points in dilates of $(\Pi_3)_\sigma$, it can be checked that its quasipolynomial is given by

$$\chi_{m\Pi_3}(\sigma) = \begin{cases} m+1 & \text{if } m \text{ is even,} \\ m & \text{if } m \text{ is odd.} \end{cases}$$

For $\sigma \in \{(ij)\}$ we then have

$$\sum_{\substack{m\geq 0 \\ m \text{ even}}} (m+1)\, z^m + \sum_{\substack{m\geq 0 \\ m \text{ odd}}} m z^m = \frac{z^3+z^2+z+1}{(1-z^2)^2}.$$

Comparing with

$$\frac{\phi(z)}{(1-z)\det\left[I-\rho((ij))z\right]} = \frac{\phi(z)}{(1-z)(1-z^2)},$$

we see that $\phi(z) = z^2+1$.

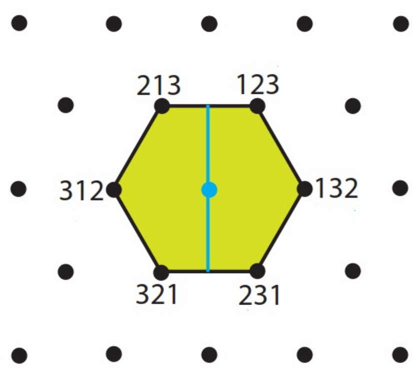


Figure 3.1: Π_3 (hexagon) and $(\Pi_3)_{(12)}$ (line segment).

***(ijk)*:**

Finally, for $\sigma = (ijk)$, $(\Pi_3)_{(ijk)}$ is the intersection of Π_3 with the line given by $x_1 = x_2 = x_3$. This consists only of one lattice point. Thus

$$\chi_{m\Pi_3}(\sigma) = 1.$$

Then for the associated generating function for $\sigma \in \{(ijk)\}$, we have

$$\sum_{m\geq 0} z^m = \frac{1}{1-z} = \frac{\phi(z)}{(1-z)\det\left[I-\rho((ijk))z\right]} = \frac{\phi(z)}{1-z^3}$$

where $\phi(z) = z^2+z+1$.

Example 3.2. Let $\mathcal{P} = \Pi_4$ be the 4-permutahedron and let $G = S_4$ act on the lattice points of $m\Pi_4$ by permuting their coordinates. S_4 has five conjugacy classes: those of $e, (ij), (ijk), (ij)(kl), (ijkl)$. Take $\sigma \in S_4$.

***e*:**
For $\sigma = e$, we have the ordinary Ehrhart polynomial for the 4-permutahedron:

$$\chi_{m\Pi_4}(e) = 16m^3 + 15m^2 + 6m + 1.$$

Computing the generating series for $\chi_{m\Pi_4}(e)$, we find that

$$\sum_{m\geq 0}(16m^3 + 15m^2 + 6m + 1)z^m = \frac{6z^3 + 55z^2 + 34z + 1}{(1-z)^4}.$$

Thus, $\phi(z) = 6z^3 + 55z^2 + 34z + 1$ is once again the h^*-polynomial for the Ehrhart series.

***(ij)*:**
Next, we find $\chi_{m\Pi_4}((ij))$. Note that $(\Pi_4)_{(ij)}$ is the intersection of Π_4 with the hyperplane $x_i = x_j$. This yields a two-dimensional cross section of Π_4. We observe that this is a hexagon whose edges bisect the facets of Π_4, see Figure 3.2. This is a rational polytope with denominator 2, and thus we expect a degree 2 quasi-polynomial with period 2. Enumerating the lattice points in several dilates, we deduce that

$$\chi_{m\Pi_4}((ij)) = \begin{cases} 4m^2 + 3m + 1 & \text{if } m \text{ is even,} \\ 4m^2 + 2m & \text{if } m \text{ is odd.} \end{cases}$$

To recover $\phi(z)$ we compute

$$\sum_{\substack{m\geq 0 \\ m \text{ odd}}} \left(4m^2 + 2m\right) z^m + \sum_{\substack{m\geq 0 \\ m \text{ even}}} \left(4m^2 + 3m + 1\right) z^m$$

$$= \sum_{m\geq 0} \left(4m^2 + 2m\right) z^m + \sum_{\substack{m\geq 0 \\ m \text{ even}}} (m+1)\, z^m$$

$$= \frac{2z^4 + 9z^3 + 15z^2 + 5z + 1}{(1-z^2)^3} = \frac{\phi(z)}{(1-z)\det\left[I - \rho((ij))z\right]} = \frac{\phi(z)}{(1-z)^3(1+z)}.$$

where $\phi(z) = 2z^3 + 7z^2 + 8z - 3 + \dfrac{4}{1+z}$.

Remark 3.3. *This example demonstrates that unlike for integral polytopes $\phi(z)$ is not always a polynomial.*

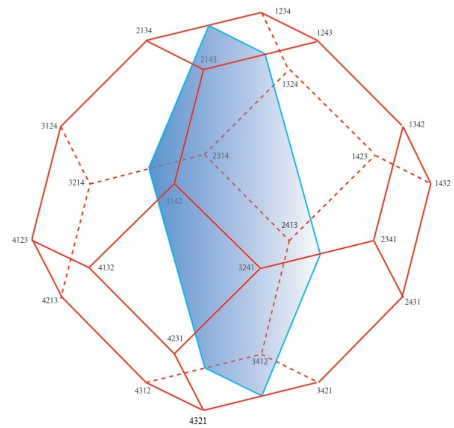


Figure 3.2: Π_4 (truncated octahedron) and $(\Pi_4)_{(12)}$ (hexagon).

***(ijk)*:**
Next we consider $\chi_{m\Pi_4}((ijk))$. Observe that $(\Pi_4)_{(ijk)}$ is the intersection of Π_4 with the plane $x_i = x_j = x_k$. This intersection yields a lattice polytope: a line segment through the center of Π_4 with endpoints on two of its hexagonal facets. One can check that the number of lattice points on $(\Pi_4)_{(ijk)}$ is given by the degree-one polynomial

$$\chi_{m\Pi_4}((ijk)) = m + 1.$$

We compute the generating series and find $\phi(z)$:

$$\sum_{m\geq 0}(m+1)z^m = \frac{1}{(1-z)^2} = \frac{\phi(z)}{(1-z)(1-z^3)}$$

where $\phi(z) = z^2 + z + 1$.

***(ij)(kl)*:**
Consider $\chi_{m\Pi_4}((ij)(kl))$. By similar reasoning as above, $(\Pi_4)_{(ij)(kl)}$ is also a line segment through the center of Π_4, but in this case it is the rational polytope with endpoints on two of the square facets of Π_4. $(\Pi_4)_{(ij)(kl)}$ has denominator 2, so once again we should have a quasi-polynomial of period 2 and degree one. By counting lattice points we conclude that

$$\chi_{m\Pi_4}((ij)(kl)) = \begin{cases} 2m+1 & \text{if } m \text{ is even,} \\ 2m & \text{if } m \text{ is odd.} \end{cases}$$

Once again we find the generating series and solve for $\phi(z)$:

$$\sum_{\substack{m\geq 0 \\ m \text{ even}}} (2m+1)\, z^m + \sum_{\substack{m\geq 0 \\ m \text{ odd}}} (2m)\, z^m = \frac{2z^2+z+1}{(1-z)^2(1+z)} = \frac{\phi(z)}{(1-z)^2(1+z)^2}$$

where $\phi(z) = 2z^3 + 3z^2 + 2z + 1$.

***(ijkl)*:**
Finally, for permutations $(ijkl)$, $(\Pi_4)_{(ijkl)}$ consists only of the point in Π_4 such that $x_1 = x_2 = x_3 = x_4$. This point has coordinates $(2.5, 2.5, 2.5, 2.5)$, so we can see that $(m\Pi_4)_{(ijkl)}$ is a lattice point if and only if m is even. Thus

$$\chi_{m\Pi_4}((ijkl)) = \begin{cases} 1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Here, we have the Ehrhart series

$$\sum_{\substack{m \geq 0 \\ m \text{ even}}} z^m = \frac{1}{1-z^2} = \frac{\phi(z)}{(1-z)(1+z)(z^2+1)}$$

where $\phi(z) = z^2 + 1$.

We can determine the dimensions of the fixed subpolytopes as a result of the following theorem.

Proposition 3.4. *Let $\sigma = \sigma_1 \cdots \sigma_m \in S_d$ act on Π_d by permuting coordinates, then $(\Pi_d)_\sigma$ has dimension $m - 1$, where m equals the number of disjoint cycles.*

Proof. Consider Π_d, which has dimension $d - 1$ and whose points are determined by d parameters. Suppose σ permutes the coordinates of Π_d via k transpositions. Then $(\Pi_d)_\sigma$ is the intersection of Π_d with the set of points satisfying $x_i = x_j$ for k pairs i, j (not necessarily disjoint). This means that the number of parameters in the points of Π_d has been reduced by k, yielding a polytope $(\Pi_d)_\sigma$ of dimension $d - 1 - k$. It is known that the minimum number of transpositions that can be used to write a permutation as a product of transpositions is precisely $d - m$, where m is number of disjoint cycles in the given permutation on d letters [17]. Letting k be the minimum number of transpositions, we have that $k = d - m$. Equivalently, $m = d - k$. Thus, $d - 1 - k = d - k - 1 = m - 1$, the result follows. Note that there is no redundancy because each $\sigma = \sigma_1 \cdots \sigma_m \in S_d$ is a product of disjoint cycles and each cycle has distinct elements from $1, \ldots, d$. $\square$

The following theorem establishes a vertex description of the fixed subpolytopes and that they are zonotopes. Furthermore, it establishes that three representations of the fixed subpolytope are equal.

Definition 3.5. Let Π_d denote the d-permutahedron, and $\sigma \in S_d$ with disjoint cycles $\sigma_1, \sigma_2, \ldots, \sigma_m$. If W is the set of all vertices of Π_d then

$$W(\sigma) = \left\{ \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i w : w \in W \right\}$$

is the set of points of W fixed by σ. Using Theorem 2.59, it follows that $(\Pi_d)_\sigma = \text{conv}(W_\sigma)$.

Lemma 3.6. *For any vertex $w = (w_1, w_2, \ldots, w_d)$ of Π_d, let w_σ be the corresponding vertex of $(\Pi_d)_\sigma$, where $w_\sigma = \frac{1}{|\sigma|}\sum_{i=1}^{|\sigma|} \sigma^i w$. Then*

$$w_\sigma = \sum_{j=1}^{m} \left(\frac{\sum_{i\in\sigma_j} w_i}{l_j} e_{\sigma_j} \right).$$

Proof. Let w_{σ_i} be the component wise product of w with e_{σ_i}, and $|\sigma_j| = l_j$. Since $\sigma_1, \sigma_2, \ldots \sigma_m$ are disjoint cycles affecting only $w_{\sigma_1}, w_{\sigma_2}, \ldots, w_{\sigma_m}$, respectively, we can break up the sum as follows:

$$\begin{aligned}
w_\sigma &= \frac{1}{|\sigma|}\sum_{i=1}^{|\sigma|} \sigma^i w \\
&= \frac{1}{|\sigma|}\sum_{i=1}^{|\sigma|} \sigma^i (w_{\sigma_1} + w_{\sigma_2} + \cdots + w_{\sigma_m}) \\
&= \frac{1}{|\sigma|}\sum_{j=1}^{m} \left(\left(\sum_{i=1}^{|\sigma|} \sigma^i w_{\sigma_j} \right) \right. \\
&= \frac{1}{|\sigma|}\sum_{j=1}^{m} \left(\left(\frac{|\sigma|}{|\sigma_j|} \sum_{i=1}^{|\sigma_j|} \sigma^i w_{\sigma_j} \right) \right. \\
&= \frac{1}{|\sigma|}\sum_{j=1}^{m} \left(\frac{|\sigma|}{|\sigma_j|} \left(\sum_{i\in\sigma_j} w_i \right) e_{\sigma_j} \right) \\
&= \sum_{j=1}^{m} \left(\frac{1}{|\sigma_j|} \left(\sum_{i\in\sigma_j} w_i \right) e_{\sigma_j} \right) \\
&= \sum_{j=1}^{m} \left(\frac{\sum_{i\in\sigma_j} w_i}{l_j} e_{\sigma_j} \right).
\end{aligned}$$

Note that the fourth equality follows from the fact that each time a cycle of σ acts on w_{σ_j} $|\sigma|$ times the permutations, $w_{\sigma_j}, \sigma w_{\sigma_j}, \sigma^2 w_{\sigma_j}, \ldots, \sigma^{|\sigma|} w_{\sigma_j}$ cycle through the same values $\frac{|\sigma|}{|\sigma_j|}$ times. □

Lemma 3.7.

$$\sum_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \frac{\sum_{i=l_m+\cdots+l_2+1}^{l_m+\cdots+l_1} i}{l_1} e_{\sigma_1} + \cdots + \frac{\sum_{i=1}^{l_m} i}{l_m} e_{\sigma_m} = \sum_{j\neq k}\left[l_j e_{\sigma_k}, l_k e_{\sigma_j}\right] + \sum_k \frac{l_k+1}{2} e_{\sigma_k}.$$

Proof. Let us start by noting that the numerator of the coefficient of e_{σ_k} is a sum of consecutive integers. In particular, it is the sum of the consecutive integers starting with $1 + \sum\limits_{j>k} l_j$ and ending with $\sum\limits_{j\geq k} l_k$. Thus the numerator of the coefficient of e_{σ_k} is equal to $\left(\sum\limits_{j=k+1}^{m} l_j + \dfrac{l_k+1}{2}\right) l_k$.

This means that the coefficient of e_{σ_k} is $\dfrac{\left(\sum\limits_{j=k+1}^{m} l_j + \dfrac{l_k+1}{2}\right) l_k}{l_k} = \sum\limits_{j=k+1}^{m} l_j + \dfrac{l_k+1}{2}$.

Furthermore, M_σ can be expressed as

$$M_\sigma = \sum_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \sum_{k=1}^{m}\left(\left(\sum_{j=k+1}^{m} l_j + \frac{l_k+1}{2}\right) e_{\sigma_k}\right).$$

Summing over k,

$$\begin{aligned}
&\sum_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \sum_{k}\left(\left(l_m + \cdots + l_{k+1} + \frac{l_{k+1}}{2}\right) e_{\sigma_k}\right) \\
&\qquad = \sum_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \sum_{j>k} l_j e_{\sigma_k} + \sum_{k} \frac{l_{k+1}}{2} e_{\sigma_k} \\
&\qquad = \sum_{j\neq k} \left[l_j e_{\sigma_k}, l_k e_{\sigma_j}\right] + \sum_{k} \frac{l_k+1}{2} e_{\sigma_k}.
\end{aligned}$$

□

In the following theorem, we provide three different descriptions for the subpolytopes fixed by an action of the symmetric group, and prove that they are in fact the same set. The three descriptions, which are detailed below, are denoted $(\Pi_d)_\sigma$, $\mathrm{conv}(V_\sigma)$, and M_σ. This proof provides a vertex description of the fixed subpolytopes and establishes that they are also zonotopes.

Theorem 3.8. *Let Π_d denote the d-permutahedron, and $\sigma \in S_d$, where σ is composed of disjoint cycles, i.e, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$. Let $(\Pi_d)_\sigma$ be the subpolytope of Π_d fixed by σ. Let $V(\sigma)$ denote the set of vertices $v = (x_1, \ldots, x_d)$ of Π_d that satisfy the following property: for each cycle $\sigma_j = (j_1 j_2 \ldots j_r)$ of σ, $x_{j_1}, \ldots, x_{j_r}$ is a consecutive sequence of integers. Let $e_{\sigma_j} = \sum_{i\in\sigma_j} e_i$.*

The following are equal:

(A) $(\Pi_d)_\sigma = \{x \in \Pi_d : \sigma \cdot x = x\}$,

(B) $V_\sigma = \mathrm{conv}\left\{\frac{1}{|\sigma|}\sum\limits_{i=1}^{|\sigma|} \sigma^i v : v \in V(\sigma)\right\}$,

$(C)\ M_\sigma = \sum_{j\neq k} \left[l_j e_{\sigma_k}, l_k e_{\sigma_j}\right] + \sum_k \frac{l_k+1}{2} e_{\sigma_k}$.

Proof. We prove that these three items are equal through containment as follows:

i) $(B) \subseteq (A)$

ii) $(C) = (B)$

iii) $(A) \subseteq (C)$

We begin:

i) $(B) \subseteq (A)$:

The first containment follows from refining the definitions that Stapledon provided in his work [31]. He originally defined the fixed subpolytopes as the convex hull of a larger set, $W(\sigma)$. Since $V(\sigma)$ is contained in $W(\sigma)$, the containment straightforwardly follows. That is, it was originally defined that $(\Pi_d)_\sigma = \text{conv}(W(\sigma))$. Note that by definition $V(\sigma) \subseteq W(\sigma)$. Therefore, it follows that $V_\sigma \subseteq \text{conv}(W(\sigma)) = (\Pi_d)_\sigma$.

ii) $(C) = (B)$:

Here we actually prove that $(C) = (B)$. The goal of this equality is to show that the vertices of M_σ are in bijection with the points in $V(\sigma)$. This part of the proof is developed in two parts. First, we take a direction such that in that direction there is a vertex of M_σ. Then we show that the vertex is in $V(\sigma)$. Secondly, we choose a vertex in V_σ and then produce a direction such that in that direction we obtain a maximal point of M_σ, which is equal to the chosen vertex of $V(\sigma)$.

We let c be the linear function defined by

$$c(x) = c(x_1, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Take a direction c such that $(M_\sigma)_c$ is a vertex. We now proceed by showing that $(M_\sigma)_c \in V(\sigma)$.

Let $c_{\sigma_k} = \dfrac{\sum\limits_{i\in\sigma_k} c_i}{l_k}$. We show that in order to find $(M_\sigma)_c$ it is enough to know the ordering of $c_{\sigma_1}, c_{\sigma_2}, \ldots, c_{\sigma_m}$. By the properties of Minkowski sums,

$$(M_\sigma)_c = \sum_{j\neq k} [l_j e_{\sigma_k}, l_k e_{\sigma_j}]_c + \sum_{k=1}^{m} \frac{l_k+1}{2} e_{\sigma_k}.$$

To find the point of M_σ that maximizes c, we find the point in each Minkowski summand of M_σ that maximizes c. Furthermore, the point that maximizes a line segment will always be one of its endpoints. Thus, to find $(M_\sigma)_c$ it suffices to find the endpoint of each interval that maximizes c.

We show that $c(l_k e_{\sigma_j}) < c(l_j e_{\sigma_k})$ if and only if $c_{\sigma_j} < c_{\sigma_k}$. Suppose that $c_{\sigma_j} < c_{\sigma_k}$. Then

$$c(l_k e_{\sigma_j}) = l_k c(e_{\sigma_j}) = l_k \left(\sum_{i \in \sigma_j} c_i \right) = l_k l_j c_{\sigma_j} < l_k l_j c_{\sigma_k} = l_j \left(\sum_{i \in \sigma_k} c_i \right) = l_j c(e_{\sigma_k}) = c(l_j e_{\sigma_k}).$$

Reversing our steps we can see that the converse also holds. So, to find which endpoint of each interval maximizes c, and as a result, which point of M_σ maximizes c, it is enough to know the ordering of $c_{\sigma_1}, c_{\sigma_2}, \ldots, c_{\sigma_m}$.

Now, suppose c has $c_{\sigma_1}, c_{\sigma_2}, \ldots, c_{\sigma_m}$ pairwise distinct. Since $c(l_k e_{\sigma_j}) < c(l_j e_{\sigma_k})$ if and only if $c_{\sigma_j} < c_{\sigma_k}$, this also means that $[l_j e_{\sigma_k}, l_k e_{\sigma_j}]_c = l_j e_{\sigma_k}$ if and only if $c_{\sigma_j} < c_{\sigma_k}$. Hence,

$$\begin{aligned} (M_\sigma)_c &= \sum_{k=1}^{m} \left(\sum_{\substack{j \\ c_{\sigma_j} < c_{\sigma_k}}} l_j \right) e_{\sigma_k} + \sum_{k=1}^{m} \frac{l_k + 1}{2} e_{\sigma_j} \\ &= \sum_{k=1}^{m} \left(\left(\sum_{\substack{j \\ c_{\sigma_j} < c_{\sigma_k}}} l_j \right) + \frac{l_k + 1}{2} \right) e_{\sigma_k}. \end{aligned}$$

Lastly, we show that $(M_\sigma)_c$ corresponds to a point v_σ in V_σ.
Recall that $V_\sigma = \left\{ \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i v : v \in V(\sigma) \right\}$, where $V(\sigma)$ is the set of vertices $v = (x_1, \ldots, x_d)$ of Π_n that satisfy the following property: for each cycle $\sigma_j = (j_1 j_2 \cdots j_r)$ of σ, $x_{j_1}, \ldots, x_{j_r}$ is a consecutive sequence of integers.

Observe that each vertex in $V(\sigma)$ corresponds to a linear ordering of $\sigma_1, \ldots, \sigma_m$. This ordering determines which set of consecutive integers is placed in the coordinates corresponding to each cycle. In particular, if $\sigma_j = (j_1 j_2 \cdots j_r)$ is the smallest cycle in the order, the corresponding vertex $v = (x_1, x_2, \ldots, x_d) \in \Pi_d$ will have $x_{j_1} = 1,\ x_{j_2} = 2, \ldots,\ x_{j_r} = l_j$. If $\sigma_k = (k_1 k_2 \cdots k_s)$ is the next cycle in the order, v will have $x_{k_1} = l_j + 1,\ x_{k_2} = l_j + 2, \ldots, x_{k_s} = l_j + l_k$. Continuing in this manner, every coordinate of v is determined. Recalling that

$$\frac{(l_1 + \cdots + l_j + 1) + (l_1 + \cdots + l_j + 2) + \cdots + (l_1 + \cdots + l_j + l_k)}{l_k} = l_1 + \cdots + l_j + \frac{l_k + 1}{2},$$

it follows that v_σ is equal to $(M_\sigma)_c$ if and only if $\sigma_1, \ldots, \sigma_m$ have the same ordering as $c_{\sigma_1}, \ldots, c_{\sigma_m}$. Therefore, the vertices of M_σ are precisely the points in V_σ, proving that $M_\sigma = V_\sigma$.

iii) $(A) \subseteq (C)$:

The last and third containment demonstrates that $(\Pi_d)_\sigma$ is a subset of M_σ. Knowing that $(\Pi_d)_\sigma$ is defined to be the convex hull of the set $W(\sigma)$, we prove that the set $W(\sigma)$ is contained in M_σ, which implies that the convex hull of the set is also in M_σ. This is done by showing that any element of the fixed subpolytope is a part of the Minkowski sum. We know that $(\Pi_d)_\sigma = \text{conv}(W(\sigma))$, where $W(\sigma)$ is defined in Definition 3.5. As mentioned earlier, for any vertex w of Π_d we have $w_\sigma = \frac{1}{|\sigma|}\sum_{i=1}^{|\sigma|}\sigma^i w$.

This part of the proof is done in three main parts. First, we start with a vertex of Π_d with coordinates $(d, d-1, \ldots, 2, 1)$. We prove that if we average the points on the σ-orbit of that vertex, we obtain a point t_σ on M_σ. Secondly, we demonstrate that if any two vertices of Π_d differ by a transposition of two coordinates, the corresponding fixed points with respect to σ are only a small translation, which we compute, away from one another. Lastly, using the previous idea and that any vertex of Π_d can be obtained by a number of transpositions of consecutive integers on the vertex with coordinates $(d, d-1, \ldots, 2, 1)$, we prove that any point fixed by σ acting on a vertex of Π_d is equal to t_σ plus a translation we have control over; this fixed point is shown to be in M_σ.

Let t denote the vertex of Π_d with coordinates $(d, d-1, \ldots, 2, 1)$. Then

$$t_\sigma = \frac{1}{|\sigma|}\sum_{i=1}^{|\sigma|}\sigma^i t = \frac{\sum_{i=l_d+\cdots+l_2+1}^{l_d+\cdots+l_2+l_1} i}{l_1} e_{\sigma_1} + \cdots + \frac{\sum_{i=l_d+1}^{l_d+l_{d-1}} i}{l_d} e_{\sigma_{d-1}} + \frac{\sum_{i=1}^{l_d} i}{l_d} e_{\sigma_d}, \tag{3.1}$$

where by Lemma 3.7

$$\begin{aligned} M_\sigma &= \sum_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \frac{\sum_{i=l_d+\cdots+l_2+1}^{l_d+\cdots+l_2+l_1} i}{l_1} e_{\sigma_1} + \cdots + \frac{\sum_{i=l_d+1}^{l_d+l_{d-1}} i}{l_d} e_{\sigma_{d-1}} + \frac{\sum_{i=1}^{l_d} i}{l_d} e_{\sigma_d} \\ &= \sum_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + t_\sigma. \end{aligned}$$

Hence, it follows that $t_\sigma \in M_\sigma$.

We now show that if two vertices $w = (w_1, w_2, \ldots, w_d)$, $w' = (w'_1, w'_2, \ldots, w'_d)$ of Π_d differ by a transposition of two coordinates w_a and w_b, i.e., $w'_a = w_b$ and $w'_b = w_a$, where $a \in \sigma_j$, $b \in \sigma_k$ with $j \geq k$, then

$$w'_\sigma = w_\sigma + \begin{cases} \dfrac{w_b - w_a}{l_j l_k}(l_k e_{\sigma_j} - l_j e_{\sigma_k}) & \text{if } j > k, \\ 0 & \text{if } j = k. \end{cases}. \tag{3.2}$$

Let $w = (w_1, w_2, \ldots, w_n)$ be a vertex of Π_d. Then

$$w_\sigma = \frac{\sum_{i\in\sigma_1} w_i}{l_1} e_{\sigma_1} + \frac{\sum_{i\in\sigma_2} w_i}{l_2} e_{\sigma_2} + \cdots + \frac{\sum_{i\in\sigma_d} w_i}{l_d} e_{\sigma_d}.$$

The coefficient of e_{σ_j} is an average of the coordinates of w whose positions are in σ_j.

Let $w' = (w'_1, w'_2, \ldots, w'_d)$, and suppose that w and w' differ by a transposition of two coordinates w_a and w_b, i.e. $w'_a = w_b$ and $w'_b = w_a$, where $a \in \sigma_j$, $b \in \sigma_k$, with $j \geq k$. Then w'_σ differs from w_σ only in the coefficients of e_{σ_j} and e_{σ_k}. If $j = k$, then the only difference is in the ordering of the terms in the numerator of the coefficient of $e_{\sigma_j} = e_{\sigma_k}$. Thus $w'_\sigma = w_\sigma$.

If $j > k$, then the coefficient of e_{σ_j} in w'_σ differs from that of w_σ by $\frac{w_b - w_a}{l_j}$ and the coefficient of e_{σ_k} in w'_σ differs from that of w_σ by $\frac{w_a - w_b}{l_k}$. This means that

$$\begin{aligned} w'_\sigma &= w_\sigma + \frac{w_b - w_a}{l_j} e_{\sigma_j} + \frac{w_a - w_b}{l_k} e_{\sigma_k} \\ &= w_\sigma + \frac{w_b - w_a}{l_j l_k} (l_k e_{\sigma_j} - l_j e_{\sigma_k}). \end{aligned}$$

Now, let w be any vertex of Π_d. Then we can consider the coordinates of w as an element $\pi_w \in S_d$. Similarly, let $\pi_t \in S_d$ be the permutation associated with the vertex t defined above. Recall that the number of transpositions of consecutive integers required to change the identity permutation to π_w is equal to the number of inversions in π_w [14]. Similarly, the number of transpositions of consecutive integers required to turn π_t into π_w is the number of correctly ordered pairs of integers in π_w.

By (3.2) each transposition of consecutive integers t_a, t_b (where $t_a = t_b + 1$) to π_t corresponds to adding $\frac{1}{l_j l_k}(l_k e_{\sigma_j} - l_j e_{\sigma_k})$ to t_σ if $j > k$, and no change if $j = k$. Thus, if $o_{j,k}$ is the number of correctly ordered pairs of integers x_a, x_b in w, where $a \in \sigma_j$ and $b \in \sigma_k$,

$$w_\sigma = t_\sigma + \sum_{j>k} \frac{o_{j,k}}{l_j l_k} (l_k e_{\sigma_j} - l_j e_{\sigma_k})$$

Now, note that for any choice of j and k, the number of pairs of integers x_a and x_b with $a \in \sigma_j$ and $b \in \sigma_k$ is $l_j l_k$. Thus the number of correctly ordered pairs of such integers is at most $l_j l_k$, and we have $0 \leq \dfrac{o_{j,k}}{l_j l_k} \leq 1$. This means that, for each choice of j and k, $\frac{o_{j,k}}{l_j l_k}\left(l_k e_{\sigma_j} - l_j e_{\sigma_k}\right)$ is a point in the segment $[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}]$, and thus $\sum_{j>k} \frac{o_{j,k}}{l_j l_k}\left(l_k e_{\sigma_j} - l_j e_{\sigma_k}\right)$ is an element of the Minkowski sum $\sum_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}]$. Recall that $t_\sigma + \sum_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] = M_\sigma$. Thus, we conclude that $w_\sigma \in M_\sigma$. $\square$

Observe that in the examples of the fixed subpolytopes of $(\Pi_d)_\sigma$ are all similar in shape to lower dimensional permutahedron. That observation is made precise by the following theorem.

Theorem 3.9. *The fixed subpolytopes $(\Pi_d)_\sigma$ are combinatorially equivalent to an m-permutahedron, where m is the number of disjoint cycles of σ.*

Proof. Consider the Minkowski sum of line segments $\sum\limits_{j>k}[0, e_j - e_k]$. This Minkowski sum is a translation of Π_m, implying that it is combinatorially equivalent to Π_m.

Now consider the map $f : \mathbb{R}^m \to (\mathbb{R}^d)_\sigma \subseteq \mathbb{R}^d$ defined by $f(e_i) = \frac{1}{l_i} e_{\sigma_i}$ and extended linearly. Note that f is bijective since it does not have elements getting mapped to the same image because the map takes and e_i and maps it to a factor of a unit vector. Also, any element in the range is attained. Moreover,

$$f\left(\sum_{j>k}[0, e_j - e_k]\right) = \sum_{j>k}\left[0, \frac{1}{l_j} e_{\sigma_j} - \frac{1}{l_k} e_{\sigma_k}\right].$$

This implies that $\sum\limits_{j>k}[0, e_j - e_k]$ is linearly equivalent, and thus combinatorially equivalent, to $\sum\limits_{j>k}[0, \frac{1}{l_j} e_{\sigma_j} - \frac{1}{l_k} e_{\sigma_k}]$.

If we scale each summand in the Minkowski sum, the normal fan of the resulting polytope does not change [1]. This tells us that $\sum\limits_{j>k}[0, \frac{1}{l_j} e_{\sigma_j} - \frac{1}{l_k} e_{\sigma_k}]$ is normally equivalent to

$$\sum_{j>k} l_j l_k [0, \frac{1}{l_j} e_{\sigma_j} - \frac{1}{l_k} e_{\sigma_k}] = \sum_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}].$$

Furthermore, normal equivalence implies combinatorial equivalence [1], so $\sum\limits_{j>k}[0, \frac{1}{l_j} e_{\sigma_j} - \frac{1}{l_k} e_{\sigma_k}]$ is combinatorially equivalent to $\sum\limits_{j>k}[0, l_k e_{\sigma_j} - l_j e_{\sigma_k}]$. Bringing together these equivalences, we conclude that $(\Pi_d)_\sigma$ is combinatorially equivalent to Π_m. $\square$

We had hoped to determine the Ehrhart theory of the fixed subpolytopes, which are rational polytopes. The theory of rational zonotopes is not as largely researched and no formula is known that provides the Ehrhart series for rational zonotopes. In the near future, we plan to determine a formula for the volume of the fixed subpolytopes in hopes that it will lead us to developing the rest of the equivariant Ehrhart theory of the permutahedron.

Chapter 4

A Decomposition of the h^*-Polynomial for Rational Polytopes

For this section we adjust the set-up for Theorem 2.39 to satisfy rational polytopes.

Definition 4.1. Consider a rational d-simplex Δ with vertices $v_1, v_2, \ldots, v_{d+1} \in \mathbb{Q}^d$. Fix any integer p such that $p\Delta$ is a lattice simplex. We let $w_1 = (v_1, 1), w_2 = (v_2, 1), \ldots, w_{d+1} = (v_{d+1}, 1)$ and define

$$\operatorname{cone}(\Delta) := \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_{d+1} w_{d+1} : \lambda_1, \lambda_2, \ldots, \lambda_{d+1} \geq 0\} \subset \mathbb{R}^{d+1}$$

and

$$\Pi(\Delta, p)^\circ := \{\lambda_1 p w_1 + \lambda_2 p w_2 + \cdots + \lambda_{d+1} p w_{d+1} : 0 < \lambda_1, \lambda_2, \ldots, \lambda_{d+1} < 1\}.$$

We call this open parallelepiped $\Pi(\Delta, p)^\circ$ the *fundamental open parallelepiped* of $\operatorname{cone}(\Delta)$.

Given a rational d-polytope $\mathcal{P}$, we fix a triangulation $\mathcal{T}$. Given a simplex $\Delta \in \mathcal{T}$, let $\Pi(\Delta, p)^\circ$ be the open fundamental parallelepiped of $\operatorname{cone}(\Delta)$ and let $B_{\Delta,p}(z) := \sigma_{\Pi(\Delta,p)^\circ}(1, 1, \ldots, 1, z)$. We include $\emptyset$ in the collection of faces of a triangulation of a polytope; we consider $\emptyset$ the empty simplex.

Proposition 4.2. $B_{\Delta,p}(z) = (z^p)^{\dim(\Delta)+1} B_{\Delta,p}\left(\frac{1}{z}\right)$.

Proof. Note that $\Pi(\Delta, p)^\circ = -\Pi(\Delta, p)^\circ + p w_1 + p w_2 + \cdots + p w_{d+1}$:

$$\begin{aligned}
-\Pi(\Delta, p)^\circ + p w_1 + \cdots + p w_{d+1} &= \{-\mu_1 p w_1 - \cdots - \mu_{d+1} p w_{d+1} : 0 < \mu_i < 1\} \\
&\quad + p w_1 + \cdots + p w_{d+1} \\
&= \left\{ -\sum_{i=1}^{d+1} \mu_i p w_i : -1 < \mu_i - 1 < 0 \right\} + p \sum_{i=1}^{d+1} w_i \\
&= \left\{ p \sum_{i=1}^{d+1} w_i \left(1 - \mu_i\right) : 0 < 1 - \mu_i < 1 \right\} \\
&= \left\{ p \sum_{i=1}^{d+1} \lambda_1 w_i : 0 < \lambda_i < 1 \right\} \\
&= \Pi(\Delta, p)^\circ.
\end{aligned}$$

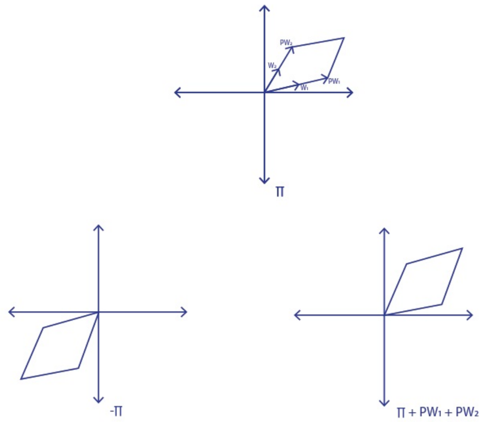


Figure 4.1: $\Pi(\Delta, p)^\circ = -\Pi(\Delta, p)^\circ + pw_1 + pw_2$.

We take the integer-point transform of the open parallelepiped,

$$\begin{aligned}\sigma_{\Pi(\Delta,p)^\circ}(\vec{z}) &= \sigma_{-\Pi(\Delta,p)^\circ}(\vec{z})\vec{z}^{\,pw_1}\vec{z}^{\,pw_2}\cdots\vec{z}^{\,pw_{d+1}} \\ &= \sigma_{\Pi(\Delta,p)^\circ}\left(\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_{d+1}}\right)\vec{z}^{\,pw_1}\vec{z}^{\,pw_2}\cdots\vec{z}^{\,pw_{d+1}}\end{aligned}$$

Now, recall that we want to specialize at $\vec{z} = (1, 1, \ldots, z)$. Therefore, we arrive at $\sigma_{\Pi(\Delta,p)^\circ}(1, 1, \ldots, z) = \sigma_{\Pi(\Delta,p)^\circ}\left(1, 1, \ldots, \frac{1}{z}\right)(z^p)^{d+1}$. This is equivalent to saying that $B_{\Delta,p}(z) = (z^p)^{\dim(\Delta,p)+1}B_{\Delta,p}\left(\frac{1}{z}\right)$. □

Theorem 4.3. *Fix a triangulation $\mathcal{T}$ of the rational d-polytope $\mathcal{P}$ such that p is the least common multiple of all the vertex coordinate denominators of every $\Delta \in \mathcal{T}$. Then*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\sum_{\Delta\in\mathcal{T}} h_{\mathrm{link}(\Delta)}(z^p)B_{\Delta,p}(z)}{(1-z^p)^{d+1}}.$$

Proof. We write $\mathcal{P}$ as the disjoint union of all open nonempty simplices in $\mathcal{T}$ and use Ehrhart-Macdonald reciprocity,

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{P}}(z) &= 1 + \sum_{\Delta\in\mathcal{T}\setminus\{\emptyset\}} \mathrm{Ehr}_{\Delta^\circ}(z) \\ &= 1 + \sum_{\Delta\in\mathcal{T}\setminus\{\emptyset\}} (-1)^{\dim(\Delta)+1}\,\mathrm{Ehr}_{\Delta}\left(\frac{1}{z}\right) \\ &= 1 + \sum_{\Delta\in\mathcal{T}\setminus\{\emptyset\}} (-1)^{\dim(\Delta)+1}\frac{h^*_{\Delta,p}(\frac{1}{z})}{(1-\frac{1}{z^p})^{\dim(\Delta)+1}} \\ &= 1 + \sum_{\Delta\in\mathcal{T}\setminus\{\emptyset\}} \frac{(z^p)^{\dim(\Delta)+1}(1-z^p)^{d-\dim(\Delta)}h^*_{\Delta,p}(\frac{1}{z})}{(1-z^p)^{d+1}}\end{aligned}$$

Next, we use the fact that

$$\sigma_{\Pi(\Delta,p)}(\vec{z}) = 1 + \sum_{\emptyset \neq \Omega \subseteq \Delta} \sigma_{\Pi(\Omega,p)^\circ}(\vec{z}),$$

where the sum is over all nonempty faces of Δ [2]. Taking $\vec{z} = (1, 1, \ldots, 1, z)$, the identity specializes to

$$h^*_{\Delta,p}(z) = \sum_{\Omega \subseteq \Delta} B_{\Omega,p}(z),$$

where this sum is over all faces of Δ, including $\emptyset$. Using this, we can rewrite our series as

$$\begin{aligned}
\mathrm{Ehr}_{\mathcal{P}}(z) &= 1 + \sum_{\Delta \in \mathcal{T} \setminus \emptyset} \frac{(z^p)^{\dim(\Delta)+1}(1-z^p)^{d-\dim(\Delta)} \sum_{\Omega \subseteq \Delta} B_{\Omega,p}(\frac{1}{z})}{(1-z^p)^{d+1}} \\
&= \frac{\sum_{\Delta \in \mathcal{T}} (z^p)^{\dim(\Delta)+1}(1-z^p)^{d-\dim(\Delta)} \sum_{\Omega \subseteq \Delta} B_{\Omega,p}(\frac{1}{z})}{(1-z^p)^{d+1}}.
\end{aligned}$$

By Proposition 4.2,

$$\begin{aligned}
h^*_{\mathcal{P}}(z) &= \sum_{\Delta \in \mathcal{T}} (z^p)^{\dim(\Delta)+1}(1-z^p)^{d-\dim(\Delta)} \sum_{\Omega \subseteq \Delta} B_{\Omega,p}\left(\frac{1}{z}\right) \\
&= \sum_{\Delta \in \mathcal{T}} (z^p)^{\dim(\Delta)+1}(1-z^p)^{d-\dim(\Delta)} \sum_{\Omega \subseteq \Delta} (z^p)^{-\dim(\Omega)-1} B_{\Omega,p}(z) \\
&= \sum_{\Omega \in \mathcal{T}} \sum_{\Delta \supseteq \Omega} (z^p)^{\dim(\Delta)-\dim(\Omega)}(1-z^p)^{d-\dim(\Delta)} B_{\Omega,p}(z) \\
&= \sum_{\Omega \in \mathcal{T}} (1-z^p)^{d-\dim(\Omega)} B_{\Omega,p}(z) \sum_{\Delta \supseteq \Omega} \left(\frac{z^p}{1-z^p}\right)^{\dim(\Delta)-\dim(\Omega)}.
\end{aligned}$$

Using Definition 2.38, (2.1), the theorem follows. □

Example 4.4. Let $\mathcal{P}$ be as defined in Example 2.46. We compute the Ehrhart series of $\mathcal{P}$ using Theorem 4.3.

Fix the following triangulation of $\mathcal{P}$:

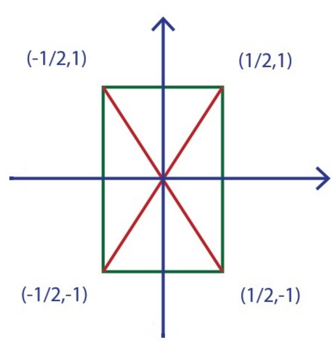


Figure 4.2: Triangulated $\mathcal{P}$.

Here we see a computation of a 2-simplex, $\Delta_1 = \text{conv}\{\left(\frac{-1}{2},1\right), \left(\frac{1}{2},1\right), (0,0)\}$.

$$\begin{aligned}
\Pi(\Delta_1,2)^\circ &= \{2\lambda_1\left(\frac{-1}{2},1,1\right) + 2\lambda_2\left(\frac{1}{2},1,1\right) + 2\lambda_3(0,0,1) : 0 < \lambda_1,\lambda_2,\lambda_3 < 1\} \\
&= \{(-\lambda_1, 2\lambda_1, 2\lambda_1) + (\lambda_2, 2\lambda_2, 2\lambda_2) + (0,0,2\lambda_3) : 0 < \lambda_1,\lambda_2,\lambda_3 < 1\} \\
&= \{(\lambda_2 - \lambda_1, 2(\lambda_1+\lambda_2), 2(\lambda_1+\lambda_2+\lambda_3)) : 0 < \lambda_1,\lambda_2,\lambda_3 < 1\} \\
&= \{(A,B,C)\}
\end{aligned}$$

(A): For (A) to be an integer, it must equal 0. $\Rightarrow \lambda_2 - \lambda_1 = 0 \Rightarrow \lambda_2 = \lambda_1$.

(B): $2(\lambda_1+\lambda_2) = 2(2\lambda_1) = 4\lambda_1 = m$ for $m \in \mathbb{Z}$. Then $m = \frac{\lambda_1}{4}$. Thus, $\lambda_1 = \lambda_2 = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$.

(C): $2(\lambda_1+\lambda_2+\lambda_3) = 2(2\lambda_1+\lambda_3) = n$ for $n \in \mathbb{Z}$. So, $1 + 2\lambda_3 = n$ or $2 + 2\lambda_3 = n$ or $3 + 2\lambda_3 = n$, which all imply that $\lambda_3 = \frac{1}{2}$.

Therefore, the integer points are: $(0,1,2), (0,2,3), (0,3,4)$. This makes the integer-point transform

$\sigma_{\Pi(\Delta_1,2)^\circ}(1,1,z) = z_1^0 z_2^1 z_3^2 + z_1^0 z_2^2 z_3^3 + z_1^0 z_2^2 z_3^4 = (1)^0(1)^1 z^2 + (1)^0(1)^2 z^3 + (1)^0(1)^2 z^4 = z^2 + z^3 + z^4.$

Through further careful computation, we obtain:

dimension	**convex hull**	$h_{\text{link}(\Delta)}(z^p)$	$B_{\Delta,p}(z)$
2	$\{\left(\frac{-1}{2},1\right), \left(\frac{1}{2},1\right), (0,0)\}$	1	$z^2+z^3+z^4$
2	$\{\left(\frac{1}{2},1\right), \left(\frac{1}{2},-1\right), (0,0)\}$	1	$3z^3$
2	$\{(\frac{-1}{2},-1, \left(\frac{1}{2},-1\right), (0,0)\}$	1	$z^2+z^3+z^4$
2	$\{(\frac{-1}{2},-1, \left(\frac{-1}{2},1\right), (0,0)\}$	1	$3z^3$
1	$\{(\frac{-1}{2},1), \left(\frac{1}{2},1\right)$	1	$z+z^2+z^3$
1	$\{(\frac{1}{2},1), \left(\frac{1}{2},-1\right)\}$	1	$z+z^2+z^3$
1	$\{(\frac{-1}{2},-1), \left(\frac{1}{2},-1\right)\}$	1	$3z^2$
1	$\{(\frac{-1}{2},-1), \left(\frac{-1}{2},1\right)\}$	1	$3z^2$
1	$\{(\frac{-1}{2},1),(0,0)\}$	z^2+1	0
1	$\{(\frac{1}{2},1),(0,0)\}$	z^2+1	0
1	$\{(\frac{1}{2},-1),(0,0)\}$	z^2+1	0
1	$\{(\frac{-1}{2},-1),(0,0)\}$	z^2+1	0
0	$\{(\frac{-1}{2},1)\}$	z^2+1	0
0	$\{(\frac{1}{2},1)\}$	z^2+1	0
0	$\{(\frac{1}{2},-1)\}$	z^2+1	0
0	$\{(\frac{-1}{2},-1)\}$	z^2+1	0
0	$\{(0,0)\}$	z^4+2z^2+1	z
∅		z^4+2z^2+1	1

Applying Theorem 4.3,

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{z^5 + 3z^4 + 12z^3 + 12z^2 + 3z + 1}{(1 - z^2)^3},$$

which agrees with Example 2.25.

This is only the beginning to exploring the h^*-polynomial for rational polytopes. One direction to take this work is to prove a rational analogue to Theorem 2.41.

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