Show complete work—that is, all the steps needed to completely justify your answer. You may refer to theorems proved in class and in the text.

(1) Suppose $G \subseteq \mathbb{C}$ is open, $f : G \to \mathbb{C}$, and $z \in G$. Define:

(a) $f$ is differentiable at $z$.
(b) $f$ is analytic at $z$.

Solution:

(a) $f$ is differentiable at $z_0$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

(b) $f$ is analytic at $z_0$ if $f$ is differentiable for all points in $\{z \in \mathbb{C} : |z - z_0| < r\}$ for some $r > 0$.

(2) For one of the following functions, give the subset of $\mathbb{C}$ where the function is differentiable, respectively analytic, and find its derivative. (As usual, $z = x + iy$.)

(a) $f(z) = e^x (\cos y + i \sin y)$
(b) $f(z) = |z|^2 = x^2 + y^2$
(c) $f(z) = \frac{1}{\sin z}$
(d) $f(z) = (z - 1 + 2i)^3i$
(e) $f(z) = (3i)^z - 1 + 2i$

Solution:

(a) $f(z) = e^x (\cos y + i \sin y) = e^x e^{iy} = \exp z$ is entire (as proved in class).
(b) The real part of $f$ is $u = x^2 + y^2$ and the imaginary part is $v = 0$. For the Cauchy-Riemann equations to hold, we need

\[ u_x = 2x = 0 = v_y \quad \text{and} \quad u_y = 2y = 0 = -v_x , \]

and these equations are only satisfied for $x = 0$ and $y = 0$. Hence the first part of the Cauchy-Riemann Theorem 2.4 says that $f$ is not differentiable for all $z \in \mathbb{C} \setminus \{0\}$. Since $u_x, u_y, v_x, v_y$ are continuous, the second part of the Cauchy-Riemann Theorem 2.4 implies that $f$ is differentiable at 0. Since a point contains no disk, $f$ is nowhere analytic.

(c) $\sin$ is entire, so the only points where $1/\sin z$ is not analytic are the zeros of the sine. These can be computed as follows:

\[
\sin z = \frac{1}{2i} (\exp(iz) - \exp(-iz)) = \frac{1}{2i} (e^{-y}e^{ix} - e^{y}e^{-ix}) \\
= \frac{1}{2i} (\cos x (e^{-y} - e^{y}) + i \sin x (e^{-y} + e^{y})) = 0
\]
means (after cancelling $2i$)

$$\cos x (e^{-y} - e^y) = 0 \quad \text{and} \quad \sin x (e^{-y} + e^y) = 0.$$ 

Since $e^{-y} + e^y > 0$, the second equation implies $x = \pi k$, $k \in \mathbb{Z}$. For any of those $x$, $\cos x = \pm 1 \neq 0$, so that the first equation can only hold if $e^{-y} - e^y = 0$, which means $y = 0$. Hence the zeros of the sine are precisely at $z = \pi k$, $k \in \mathbb{Z}$, which in turn means that $1/ \sin z$ is analytic on $\mathbb{C} \setminus \{\pi k : k \in \mathbb{Z}\}$.

(d) By definition,

$$(z - 1 + 2i)^{3i} = \exp ((3i) \log (z - 1 + 2i)).$$

$\exp$ is an entire function, so $(z - 1 + 2i)^{3i}$ is analytic wherever $\log (z - 1 + 2i)$ is. As we showed many times, $\log$ is analytic everywhere but the nonpositive real axis ($z = x + iy$ with $x \leq 0$ and $y = 0$), which implies that $\log (z - 1 + 2i)$ is analytic everywhere but when $z - 1 + 2i$ is real and nonpositive, that is, for $z = x + iy$ with $x \leq 1$ and $y = -2$. Hence $(z - 1 + 2i)^{3i}$ is analytic on $\mathbb{C} \setminus \{x + iy \in \mathbb{C} : x \leq 1, y = -2\}$.

(e) By definition,

$$(3i)^{z - 1 + 2i} = \exp ((z - 1 + 2i) \log (3i)) = \exp ((z - 1 + 2i) (\ln |3i| + \arg (3i)))$$

$$= \exp ((z - 1 + 2i) (\ln 3 + i\pi/2)).$$

This is the exponential function applied to a polynomial. Both are entire functions, so $(3i)^{z - 1 + 2i}$ is entire.

(3) Prove: If $f$ is entire and real valued (that is, $\text{Im}(f(z)) = 0$ for all $z \in \mathbb{C}$) then $f$ is constant.

**Solution:** Let $f = u + iv$, then the conditions imply that $v = 0$. Hence by the Cauchy-Riemann equations,

$$u_x = v_y = 0 \quad \text{and} \quad u_y = -v_x = 0,$$

so $u$ and thus $f = u + iv$ have to be constant.

(4) Integrate one of the following functions over the circle $|z| = 2$, oriented counterclockwise.

(a) $z$

(b) $\frac{1}{z^4}$

(c) $\left(\frac{\exp z}{z}\right)^2$

(d) $\log (z + 3)$

(e) $\frac{\sin z}{(z - 1)(z - 3)}$

(f) $\frac{1}{z^3 + 34z}$
Solution: A parametrization of the circle $\gamma$ is $\gamma(t) = 2 e^{it} = 2 \cos t + 2i \sin t$, $0 \leq t \leq 2\pi$. Note that $\gamma'(t) = 2i e^{it}$.

(a) $\int_{\gamma} \bar{z} \, dz = \int_{0}^{2\pi} 2 e^{-it} 2i e^{it} \, dt = \int_{0}^{2\pi} 4i \, dt = 8\pi i$.

(b) $\frac{1}{z^3}$ has the antiderivative $-\frac{1}{3z^3}$ on $\mathbb{C} \setminus \{0\}$, which contains $\gamma$, and thus the integral is zero.

(c) Here we use the “extended Cauchy Formula” Theorem 5.1.: exp is entire, so we can choose $G = \mathbb{C}$, then $\gamma$ is $G$-contractible, and $w = 0$ is inside $\gamma$. Hence

$$\int_{\gamma} \left( \frac{\exp z}{z} \right)^2 \, dz = \int_{\gamma} \frac{\exp^2 z}{z^2} \, dz = 2\pi i \left( \exp^2 z \right)' \bigg|_{z=0} = 2\pi i (2 \exp z \cdot \exp z) \bigg|_{z=0} = 4\pi i.$$

(d) The integrand $f(z) = \log(z + 3)$ is analytic in $G = \mathbb{C} \setminus \{z \in \mathbb{C} : \Re z \leq -3\}$. However, $\gamma$ is $G$-contractible, so by Corollary 4.5 (to Cauchy’s Theorem)

$$\int_{\gamma} \log(z + 3) \, dz = 0.$$

(e) A partial fraction expansion gives

$$\frac{1}{(z - 1)(z - 3)} = \frac{1/2}{z - 3} - \frac{1/2}{z - 1},$$

so

$$\int_{\gamma} \frac{\sin z}{(z - 1)(z - 3)} \, dz = \frac{1}{2} \int_{\gamma} \frac{\sin z}{z - 3} \, dz - \frac{1}{2} \int_{\gamma} \frac{\sin z}{z - 1} \, dz.$$

For the first integral, we can use Corollary 4.5 (to Cauchy’s Theorem) with $G = \mathbb{C} \setminus \{3\}$, $f(z) = \sin z/(z - 3)$: note that $f$ is analytic in $G$ and $\gamma$ is closed and $G$-contractible, and so

$$\int_{\gamma} \frac{\sin z}{z - 3} \, dz = 0.$$

For the second integral, we can use Cauchy’s Integral Formula (Theorem 4.7) with $G = \mathbb{C}$, $f(z) = \sin z$, and $w = 1$ (note that $w$ is inside $\gamma$ and that $\gamma$ is $G$-contractible):

$$\int_{\gamma} \frac{\sin z}{z - 1} \, dz = 2\pi i \sin 1.$$

Putting it all together, we get

$$\int_{\gamma} \frac{\sin z}{(z - 1)(z - 3)} \, dz = -\pi i \sin 1.$$

(f) We write

$$\int_{\gamma} \frac{1}{z^3 + 34z} \, dz = \int_{\gamma} \frac{1}{(z^2 + 34)z} \, dz = \int_{\gamma} \frac{1}{z^2 + 34} \, dz,$$

and use Cauchy’s Integral Formula (Theorem 4.7) with $f(z) = \frac{1}{z^2 + 34}$, $G = \mathbb{C} \setminus \{\pm i\sqrt{34}\}$, and $w = 0$ (note that 0 is inside $\gamma$ and that $\gamma$ is $G$-contractible). Hence

$$\int_{\gamma} \frac{1}{z^3 + 34z} \, dz = \int_{\gamma} \frac{1}{z^2 + 34} \, dz = 2\pi i \frac{1}{0^2 + 34} = \frac{\pi i}{17}.$$
(5) Suppose \((f_n(z))_{n=1}^\infty\) is a sequence of functions defined on \(G \subseteq \mathbb{C}\). Define:

(a) \((f_n(z))_{n=1}^\infty\) converges pointwise on \(G\).
(b) \((f_n(z))_{n=1}^\infty\) converges uniformly on \(G\).

Solution: There exists a function \(f : G \rightarrow \mathbb{C}\) such that:

(a) \(\forall x \in G \ \forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ |f_n(z) - f(z)| < \epsilon\);
(b) \(\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ \forall x \in G \ |f_n(z) - f(z)| < \epsilon\).

(6) Prove that the function sequence \((z^n)_{n=1}^\infty\) converges

(a) pointwise on \(\{z \in \mathbb{C} : |z| < 1\}\),
(b) uniformly on \(\{z \in \mathbb{C} : |z| \leq 1/2\}\).

Solution: We claim the limit function is \(f(z) = 0\) for \(|z| < 1\).

(a) Given any \(z\) with \(|z| < 1\) and any \(\epsilon > 0\), choose \(N > \log \epsilon / \log |z|\). Then for all \(n \geq N\)

\[|z^n - 0| = |z|^n \leq |z|^N < \epsilon.\]

(b) Given any \(\epsilon > 0\), choose \(N\) such that for all \(n \geq N\)

\[|(\frac{1}{2})^n - 0| = (\frac{1}{2})^n < \epsilon.\]

(This \(N\) exists because of part (a).) Hence for all \(z\) with \(|z| \leq 1/2\) and for all \(n \geq N\)

\[|z^n - 0| = |z|^n \leq (\frac{1}{2})^n < \epsilon.\]