Discrete Volume Computations for Polyhedra

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Graduate Student Meeting
Applied Algebra & Combinatorics
Themes

Combinatorial polynomials

Generating functions

Computation (complexity)

Combinatorial structures

Polyhedra
Themes

Combinatorial polynomials
Generating functions
Polyhedra

Combinatorial structures

Nonlinear Algebra
Computation (complexity)

Linear Algebra
Motivation I: Birkhoff–von Neumann Polytope

\[ B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n}_{\geq 0} : \sum_{j} x_{jk} = 1 \text{ for all } 1 \leq k \leq n \right\} \]

\[ \sum_{k} x_{jk} = 1 \text{ for all } 1 \leq j \leq n \]
Theorem [Appel & Haken 1976] The chromatic number of any planar graph is at most 4.

This theorem had been a conjecture (conceived by Guthrie when trying to color maps) for 124 years.

Birkhoff [1912] says: Try polynomials!

Four-Color Theorem Rephrased For a planar graph $G$, we have $\chi_G(4) > 0$, that is, 4 is not a root of the polynomial $\chi_G(k)$. 
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Stanley [EC 1] says: Try monomial algebras and generating functions!

Four-Color Theorem Rephrased For a planar graph $G$, we have $\chi_G(4) > 0$, that is, 4 is not a root of the polynomial $\chi_G(k)$. 
Discrete Volumes

Rational polyhedron \( \mathcal{P} \subset \mathbb{R}^d \) – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand \( \mathcal{P} \cap \mathbb{Z}^d \) . . .

\[
\sum_{m \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}
\]

(count) \( |\mathcal{P} \cap \mathbb{Z}^d| \)
Discrete Volumes

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\]

\[
|\mathcal{P} \cap \mathbb{Z}^d|
\]

\[
\operatorname{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^d} \left| \mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d \right|
\]
Discrete Volumes

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand $\mathcal{P} \cap \mathbb{Z}^d$ . . .

- (list) $\sum_{m \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

- (count) $|\mathcal{P} \cap \mathbb{Z}^d|$

- (volume) $\text{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^d} |\mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d|$}

Ehrhart function $L_{\mathcal{P}}(t) := \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right| = \left| t\mathcal{P} \cap \mathbb{Z}^d \right|$ for $t \in \mathbb{Z}_{>0}$
Why Polyhedra?

- Linear systems are everywhere, and so polyhedra are everywhere.
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- In applications, the volume of the polytope represented by a linear system measures some fundamental data of this system ("average").
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- Many discrete problems in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.

- Much discrete geometry can be modeled using polynomials and, conversely, many combinatorial polynomials can be modeled geometrically.
A Warm-Up Ehrhart Function

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_P(t) := |t\mathcal{P} \cap \mathbb{Z}^d|$

Example 1:

$\Delta = \text{conv} \{(0, 0), (1, 0), (0, 1)\}$

$= \{(x, y) \in \mathbb{R}^2_{\geq 0} : x + y \leq 1\}$
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Example 1:

$\Delta = \text{conv} \{(0, 0), (1, 0), (0, 1)\} = \{(x, y) \in \mathbb{R}_{\geq 0}^2 : x + y \leq 1\}$

Example 2:

$\Box = [0, 1]^d$ (the unit cube in $\mathbb{R}^d$)
Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in $t$ of degree $\dim \mathcal{P}$ with leading coefficient $\text{vol } \mathcal{P}$ and constant term 1.

Equivalently, $\text{Ehr}_\mathcal{P}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t$ is rational:

$$\text{Ehr}_\mathcal{P}(z) = \frac{h^*(z)}{(1 - z)^{\dim \mathcal{P} + 1}}$$

where the Ehrhart $h$-vector $h^*(z)$ satisfies $h^*(0) = 1$ and $h^*(1) = (\dim \mathcal{P})! \text{vol}(\mathcal{P})$. 
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Seeming dichotomy: $\text{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^{\dim \mathcal{P}}} L_{\mathcal{P}}(t)$ can be computed discretely via a finite amount of data.
Ehrhart Polynomials

**Theorem** (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_\mathcal{P}(t)$ is a polynomial in $t$ of degree $d := \dim \mathcal{P}$ with leading coefficient $\text{vol} \mathcal{P}$ and constant term 1.

$$\text{Ehr}_\mathcal{P}(z) := 1 + \sum_{t \geq 1} L_\mathcal{P}(t) z^t = \frac{h^*(z)}{(1 - z)^{d+1}}$$

Equivalent descriptions of an Ehrhart polynomial:

- $L_\mathcal{P}(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$

- via roots of $L_\mathcal{P}(t)$

- $\text{Ehr}_\mathcal{P}(z) \quad \rightarrow \quad L_\mathcal{P}(t) = h_0^* \binom{t + d}{d} + h_1^* \binom{t + d - 1}{d} + \cdots + h_d^* \binom{t}{d}$

$h^*(z)$ is the **binomial transform** of $L_\mathcal{P}(t)$
Ehrhart Polynomials

**Theorem** (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_P(t)$ is a polynomial in $t$ of degree $d := \dim \mathcal{P}$ with leading coefficient $\text{vol} \mathcal{P}$ and constant term 1.

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- via roots of $L_P(t)$
- $\text{Ehr}_P(z) \rightarrow \quad L_P(t) = h_0^*(t^d) + h_1^*(t^{d-1}) + \cdots + h_d^*(t)$

**Open Problem** Classify Ehrhart polynomials.
Two-dimensional Ehrhart Polynomials

Essentially due to Pick (1899) and Scott (1976)
**Ehrhart Polynomials**

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$$
\text{Ehr}_\mathcal{P}(z) := 1 + \sum_{t \geq 1} L_\mathcal{P}(t) z^t = \frac{h^*(z)}{(1 - z)^{d+1}}
$$

$$
\longrightarrow L_\mathcal{P}(t) = h^*_0(t+d) + h^*_1(t+d-1) + \cdots + h^*_d(t)
$$

**Theorem** (Macdonald 1971) $(-1)^dL_\mathcal{P}(-t)$ enumerates the interior lattice points in $t\mathcal{P}$. Equivalently,

$$
L_{\mathcal{P}^\circ}(t) = h^*_d(t+d-1) + h^*_{d-1}(t+d-2) + \cdots + h^*_0(t-1)
$$
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$$Ehr_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{h^*(z)}{(1 - z)^{d+1}}$$

$$\rightarrow \quad L_{\mathcal{P}^\circ}(t) = h_d^*(\frac{t+d-1}{d}) + h_{d-1}^*(\frac{t+d-2}{d}) + \cdots + h_0^*(\frac{t-1}{d})$$

**Theorem** (Stanley 1980) $h_0^*, h_1^*, \ldots, h_d^*$ are nonnegative integers.
Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in $t$ of degree $d := \dim \mathcal{P}$ with leading coefficient $\text{vol} \mathcal{P}$ and constant term 1.

$$Ehr_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{h^*(z)}{(1 - z)^{d+1}}$$

$$L_{\mathcal{P}^o}(t) = h^*_d \binom{t+d-1}{d} + h^*_{d-1} \binom{t+d-2}{d} + \cdots + h^*_0 \binom{t-1}{d}$$

Theorem (Stanley 1980) $h^*_0, h^*_1, \ldots, h^*_d$ are nonnegative integers.

Corollary If $h^*_{d+1-k} > 0$ then $k\mathcal{P}^o$ contains an integer point.
Interlude: Graph Coloring a la Ehrhart

\[ \chi_{K_2}(k) = 2 \binom{k}{2} \ldots \]

\[ K_2 \]

(Blass–Sagan)
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Nonequation Algebra

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Similarly, for any given graph \( G \) on \( d \) nodes, we can write

\[ \chi_G(k) = \chi_0^* \left( \binom{k + d}{d} \right) + \chi_1^* \left( \binom{k + d - 1}{d} \right) + \cdots + \chi_d^* \left( \binom{k}{d} \right) \]

for some (meaningful) nonnegative integers \( \chi_0^*, \ldots, \chi_d^* \)
Interlude: Graph Coloring a la Ehrhart

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Half-Open Problem  Prove that \( \chi_j^* > 0 \) for some \( 0 \leq j \leq 4 \) if \( G \) is planar.
Ehrhart $h^*$ Positivity Refined

$$\text{Ehr}_P(z) := 1 + \sum_{t \geq 1} |tP \cap \mathbb{Z}^d| z^t = \frac{h^*(z)}{(1 - z)^{d+1}}$$

**Theorem** (Stanley 1980) $h^*_0, h^*_1, \ldots, h^*_d$ are nonnegative integers.

**Theorem** (Betke–McMullen 1985, Stapledon 2009) If $h^*_d > 0$ then

$$h^*(z) = a(z) + z b(z)$$

where $a(z) = z^d a\left(\frac{1}{z}\right)$ and $b(z) = z^{d-1} b\left(\frac{1}{z}\right)$ with nonnegative coefficients.
**Ehrhart $h^*$ Positivity Refined**

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\text{Ehr}_\mathcal{P}(z) := 1 + \sum_{t \geq 1} |t\mathcal{P} \cap \mathbb{Z}^d| z^t = \frac{h^*(z)}{(1 - z)^{d+1}}
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**Open Problem** Try to prove the analogous theorem for your favorite combinatorial polynomial with nonnegative coefficients.
More Binomial Transforms

Chromatic polynomial $\chi_G(k) = \chi_0^* \binom{k + d}{d} + \chi_1^* \binom{k + d - 1}{d} + \cdots + \chi_d^* \binom{k}{d}$

$\rightarrow$ binomial transform $\chi^*_G(z) := \chi^*_d z^d + \chi^*_{d-1} z^{d-1} + \cdots + \chi^*_0$

**Theorem** (MB–León 2019+) Let $G$ be a graph on $d$ vertices. Then there exist symmetric polynomials $a_G(z) = z^d a_G(\frac{1}{z})$ and $b_G(z) = z^{d-1} b_G(\frac{1}{z})$ with positive integer coefficients such that

$$\chi_G^*(z) = a_G(z) - b_G(z).$$

Moreover, $a_0 \leq a_1 \leq a_j$ where $1 \leq j \leq d - 1$, and $b_0 \leq b_1 \leq b_j$ where $1 \leq j \leq d - 2$. 
More Binomial Transforms

Chromatic polynomial $\chi_G(k) = \chi_0^*(\binom{k+d}{d}) + \chi_1^*(\binom{k+d-1}{d}) + \cdots + \chi_d^*(\binom{k}{d})$

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Moreover, $a_0 \leq a_1 \leq a_j$ where $1 \leq j \leq d - 1$, and $b_0 \leq b_1 \leq b_j$ where $1 \leq j \leq d - 2$.

**Theorem (Hersh–Swartz 2008)** $\chi_{d-j}^* \geq \chi_j^*$ for $2 \leq j \leq \frac{d-1}{2}$

Similar results hold for flow polynomials of graphs (Breuer–Dall 2011).
The polynomial \( h^*(z) = \sum_{j=0}^{d} h^*_j z^j \) is **unimodal** if for some \( k \in \{0, 1, \ldots, d\} \)

\[
h^*_0 \leq h^*_1 \leq \cdots \leq h^*_k \geq \cdots \geq h^*_d
\]

**Crucial Example** \( h^*(z) \) has only real roots
The polynomial $h^*(z) = \sum_{j=0}^{d} h_j^* z^j$ is unimodal if for some $k \in \{0, 1, \ldots, d\}$

$$h_0^* \leq h_1^* \leq \cdots \leq h_k^* \geq \cdots \geq h_d^*$$

**Crucial Example** $h^*(z)$ has only real roots

**Classic Example** $\mathcal{P} = [0, 1]^d$ comes with the Eulerian polynomial $h^*(z)$

**Theorem** (Schepers–Van Langenhoven 2013) $h^*(z)$ is unimodal for lattice parallelepipeds.

**Theorem** (MB–Jochemko–McCullough 2019) $h^*(z)$ is real rooted for lattice zonotopes.
Unimodal & Real-rooted Polynomials

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\]

**Crucial Example** \( h^*(z) \) has only real roots

**Conjectures** \( h^*(z) \) is unimodal/real-rooted for

- hypersimplices
- alcoved polytopes
- lattice polytopes with unimodular triangulations
- IDP polytopes (integer decomposition property)
A Polynomial Ansatz to Antimagic Graph Labelings

An antimagic labeling of $G = (V, E)$ is an assignment $E \rightarrow \mathbb{Z}_{>0}$ such that

- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sum of the labels on all edges incident with a given node is unique.
A Polynomial Ansatz to Antimagic Graph Labelings

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Conjecture [Hartsfield & Ringel 1990] Every connected graph except \( K_2 \) has an antimagic labeling.

- [Alon et al 2004] connected graphs with minimum degree \( \geq c \log |V| \)
- [Bérczi et al 2017] connected regular graphs
- open for trees
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- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sum of the labels on all edges incident with a given node is unique.

Idea Introduce a counting function: let $A_G^*(k)$ be the number of assignments of positive integers to the edges of $G$ such that
- each edge label is in $\{1, 2, \ldots, k\}$ and is distinct;
- the sum of the labels on all edges incident with a given node is unique.

Then $G$ has an antimagic labeling if and only if $A_G^*(|E|) > 0$. 
A Polynomial Ansatz to Antimagic Graph Labelings

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Then $G$ has an antimagic labeling if and only if $A^*_G(|E|) > 0$.

Bad News The counting function $A^*_G(k)$ is in general not a polynomial:

$$A^*_C(k) = k^4 - \frac{22}{3} k^3 + 17k^2 - \frac{38}{3} k + \begin{cases} 0 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd.} \end{cases}$$
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New Idea Introduce another counting function: let $A_G(k)$ be the number of assignments of positive integers to the edges of $G$ such that

- each edge label is in $\{1, 2, \ldots, k\}$;
- the sum of the labels on all edges incident with a given node is unique.

Theorem (MB–Farahmand 2017) $A_G(k)$ is a quasipolynomial in $k$ of period at most 2. If $G$ minus its loops is bipartite then $A_G(k)$ is a polynomial.

Corollary For bipartite graphs, $A_G^*(|E|) > 0$. 
One Last Picture: Birkhoff–von Neumann Roots

For more about roots of (Ehrhart) polynomials, see Braun (2008) and Pfeifle (2010).