Inside-out polytopes
& a tale of seven polynomials

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Chromatic polynomials of graphs

\[ \Gamma = (V, E) \] – graph (without loops)

Proper \( k \)-coloring of \( \Gamma \): mapping \( x : V \to \{1, 2, \ldots, k\} \) such that \( x_i \neq x_j \) if there is an edge \( ij \)

Theorem (Birkhoff 1912, Whitney 1932) \( \chi_\Gamma(k) := \# \) (proper \( k \)-colorings of \( \Gamma \)) is a monic polynomial in \( k \) of degree \( |V| \).
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Theorem (Birkhoff 1912, Whitney 1932) $\chi_\Gamma(k) := \# \text{(proper $k$-colorings of $\Gamma$)}$ is a monic polynomial in $k$ of degree $|V|$.

Theorem (Stanley 1973) $(-1)^{|V|} \chi_\Gamma(-k)$ equals the number of pairs $(\alpha, x)$ consisting of an acyclic orientation $\alpha$ of $\Gamma$ and a compatible $k$-coloring. In particular, $(-1)^{|V|} \chi_\Gamma(-1)$ equals the number of acyclic orientations of $\Gamma$.

(An orientation $\alpha$ of $\Gamma$ and a $k$-coloring $x$ are compatible if $x_j \geq x_i$ whenever there is an edge oriented from $i$ to $j$. An orientation is acyclic if it has no directed cycles.)
Flow polynomials

Nowhere-zero $A$-flow on a graph $\Gamma = (V, E)$: mapping $x : E \to A \setminus \{0\}$ ($A$ an abelian group) such that for every node $v \in V$

$$\sum_{h(e) = v} x(e) = \sum_{t(e) = v} x(e)$$

$h(e) := \text{head}$ of the edge $e$ in a (fixed) orientation of $\Gamma$

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Nowhere-zero $k$-flow: $\mathbb{Z}$-flow with values in $\{1, 2, \ldots, k - 1\}$
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Nowhere-zero $k$-flow: $\mathbb{Z}$-flow with values in $\{1, 2, \ldots, k - 1\}$

Theorem

(Tutte 1954) $\varphi_\Gamma(|A|) := \#(\text{nowhere-zero } A\text{-flows})$ is a polynomial in $|A|$. (Kochol 2002) $\varphi_\Gamma(k) := \#(\text{nowhere-zero } k\text{-flows})$ is a polynomial in $k$. 
(Weak) semimagic squares

\[ H_n(t) \] – number of nonnegative integral \( n \times n \)-matrices in which every row and column sums to \( t \)

\[
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 1 & 1 \\
1 & 2 & 1 \\
\end{array}
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Theorem (Ehrhart, Stanley 1973, conjectured by Anand-Dumir-Gupta 1966) $H_n(t)$ is a polynomial in $t$ of degree $(n - 1)^2$ satisfying

\[H_n(0) = 1, \ H_n(-1) = H_n(-2) = \cdots = H_n(-n + 1) = 0,\]

and $H_n(-n - t) = (-1)^{n-1}H_n(t)$.
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What about “classical” magic squares?
Ehrhart (quasi-)polynomials

\[ \mathcal{P} \subset \mathbb{R}^d \] – convex rational polytope

For \( t \in \mathbb{Z}_{>0} \) let \( \text{Ehr}_\mathcal{P}(t) := \# (\mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d) \)
Ehrhart (quasi-)polynomials

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For $t \in \mathbb{Z}_{>0}$ let $Ehr_\mathcal{P}(t) := \# (\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d)$

Theorem
(Ehrhart 1962) $Ehr_\mathcal{P}(t)$ is a quasipolynomial in $t$ of degree $\dim \mathcal{P}$ with leading term $\text{vol} \mathcal{P}$ (normalized to $\text{aff} \mathcal{P} \cap \mathbb{Z}^d$) and constant term $Ehr_\mathcal{P}(0) = \chi(\mathcal{P}) = 1$.
(Macdonald 1971) $(-1)^{\dim \mathcal{P}} Ehr_\mathcal{P}(-t)$ enumerates the interior lattice points in $t\mathcal{P}$.

(A quasipolynomial is an expression $c_d(t) t^d + \cdots + c_1(t) t + c_0(t)$ where $c_0, \ldots, c_d$ are periodic functions in $t$.)

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Characteristic polynomials of hyperplane arrangements

$\mathcal{H} \subset \mathbb{R}^d$ – arrangement of affine hyperplanes

$\mathcal{L}(\mathcal{H}) := \{\bigcap S : S \subseteq \mathcal{H} \text{ and } \bigcap S \neq \emptyset\}$, ordered by reverse inclusion
Characteristic polynomials of hyperplane arrangements

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Möbius function \( \mu(r, s) := \begin{cases} 0 & \text{if } r \nsubseteq s, \\ 1 & \text{if } r = s, \\ - \sum_{r \leq u < s} \mu(r, u) & \text{if } r < s. \end{cases} \)

Characteristic polynomial

\[ p_{\mathcal{H}}(\lambda) := \sum_{s \in \mathcal{L}(\mathcal{H})} \mu(\mathbb{R}^d, s) \lambda^{\dim s} \]
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Möbius function $\mu(r, s) := \begin{cases} 
0 & \text{if } r \not\leq s, \\
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\end{cases}$

Characteristic polynomial

$p_\mathcal{H}(\lambda) := \sum_{s \in \mathcal{L}(\mathcal{H})} \mu(\mathbb{R}^d, s) \lambda^{\dim s}$

**Theorem** (Zaslavsky 1975) If $\mathbb{R}^d \not\in \mathcal{H}$ then the number of regions into which a hyperplane arrangement $\mathcal{H}$ divides $\mathbb{R}^d$ is $(-1)^d p_\mathcal{H}(-1)$. 
Graph coloring a la Ehrhart

\[ \chi_{K_2}(k) = k(k - 1) \ldots \]

\[ x_1 = x_2 \]

\[ k + 1 \]

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Inside-Out Polytopes  
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\[ \chi_{\Gamma}(k) = \# \left( \left( (0, 1)^V \setminus \bigcup \mathcal{H}(\Gamma) \right) \cap \frac{1}{k+1}\mathbb{Z}^V \right) \]
Stanley’s Theorem a la Ehrhart

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Write \((0,1)^V \setminus \bigcup \mathcal{H}(\Gamma) = \bigcup_j \mathcal{P}_j^o\), then by Ehrhart-Macdonald reciprocity

\[ (-1)^{|V|} \chi_\Gamma(-k) = \sum_j \text{Ehr}_{P_j}(k - 1) \]
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Greene’s observation

region of \(\mathcal{H}(\Gamma) \iff \text{acyclic orientation of } \Gamma \)

\(x_i < x_j \iff i \rightarrow j\)
Chromatic polynomials of signed graphs

$\Sigma$ – signed graph (without loops): each edge is labelled $+$ or $-$

Proper $k$-coloring of $\Sigma$: mapping $x : V \rightarrow \{-k, -k + 1, \ldots, k\}$ such that, if edge $ij$ has sign $\epsilon$ then $x_i \neq \epsilon x_j$
Chromatic polynomials of signed graphs

Σ – signed graph (without loops): each edge is labelled + or −

Proper $k$-coloring of $Σ$: mapping $x : V \rightarrow \{-k, -k + 1, \ldots , k\}$ such that, if edge $ij$ has sign $\epsilon$ then $x_i \neq \epsilon x_j$

Theorem (Zaslavsky 1982) $\chi_Σ(2k + 1) := \#$(proper $k$-colorings of $Σ$) and $\chi_Σ^*(2k) := \#$(proper zero-free $k$-colorings of $Σ$) are monic polynomials of degree $|V|$. The number of compatible pairs $(\alpha, x)$ consisting of an acyclic orientation $\alpha$ and a $k$-coloring $x$ of $Σ$ is equal to $(-1)^{|V|}\chi_Σ(-(2k + 1))$. The number in which $x$ is zero-free equals $(-1)^{|V|}\chi_Σ^*(-2k)$. In particular, $(-1)^{|V|}\chi_Σ(-1)$ equals the number of acyclic orientations of $Σ$. 
Theorem $\chi_{\Sigma}(2k + 1)$ and $\chi_{\Sigma}^*(2k)$ are two halves of one inside-out quasipolynomial.
Signed-graph coloring a la Ehrhart

\begin{align*}
\chi_{\Sigma}(2k + 1) \quad &\text{and} \quad \chi^*_{\Sigma}(2k) \quad \text{are two halves of one inside-out quasipolynomial.} \\
\text{Open problem} \quad &\text{Is there a combinatorial interpretation of } \chi^*_{\Sigma}(-1)？
\end{align*}
Flow polynomials revisited

\[ \varphi_\Gamma(k) := \# \text{(nowhere-zero } k\text{-flows)} \]
\[ \overline{\varphi}_\Gamma(|A|) := \# \text{(nowhere-zero } A\text{-flows)} \]

Theorem \((-1)^{|E| - |V| + c(\Gamma)} \varphi_\Gamma(-k)\) equals the number of pairs \((\tau, x)\) consisting of a totally cyclic orientation \(\tau\) and a compatible \((k + 1)\) - flow \(x\). In particular, the constant term \(\varphi_\Gamma(0)\) equals the number of totally cyclic orientations of \(\Gamma\).

(An orientation of \(\Gamma\) is totally cyclic if every edge lies in a coherent circle, that is, where the edges are oriented in a consistent direction around the circle. A totally cyclic orientation \(\tau\) and a flow \(x\) are compatible if \(x \geq 0\) when it is expressed in terms of \(\tau\).)
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Corollary \(\varphi_\Gamma(0) = (-1)^{|E|-|V|+c(\Gamma)} \overline{\varphi}_\Gamma(-1)\)

\(\exists\) analogous theorems for signed graphs
Open problems

Find a formula for, or a combinatorial interpretation of, the leading coefficient of $\varphi_{\Gamma}$. 
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Is there a combinatorial interpretation of $\varphi_\Gamma(-k)$ for $k \geq 2$?
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Is there a combinatorial interpretation of \( \varphi_\Gamma(-k) \) for \( k \geq 2 \)?

For some graphs, both \( \varphi_\Gamma \) and \( \varphi_\Gamma \) have integral coefficients and \( \varphi_\Gamma \) is a multiple of \( \varphi_\Gamma \). Is there a general reason for these facts?
Inside-out counting functions

Inside-out polytope: \((P, H)\)

Multiplicity of \(x \in \mathbb{R}^d\):

\[
m_{P, H}(x) := \begin{cases} 
\# \text{ closed regions of } H \text{ in } P \text{ that contain } x & \text{if } x \in P, \\
0 & \text{if } x \notin P
\end{cases}
\]

Closed Ehrhart quasipolynomial

\[
E_{P, H}(t) := \sum_{x \in \frac{1}{t} \mathbb{Z}^d} m_{P, H}(x)
\]

Open Ehrhart quasipolynomial

\[
E^o_{P, H}(t) := \# \left( \frac{1}{t} \mathbb{Z}^d \cap [P \setminus \bigcup H] \right)
\]
Basic inside-out results

Theorem If \((\mathcal{P}, \mathcal{H})\) is a closed, full-dimensional, rational inside-out polytope, then \(E_{\mathcal{P}, \mathcal{H}}(t)\) and \(E^{\circ}_{\mathcal{P}, \mathcal{H}}(t)\) are quasipolynomials in \(t\) of degree \(\dim \mathcal{P}\) with leading term \(\text{vol} \mathcal{P}\), and with constant term \(E_{\mathcal{P}, \mathcal{H}}(0)\) equal to the number of regions of \((\mathcal{P}, \mathcal{H})\). Furthermore,

\[
E^{\circ}_{\mathcal{P}, \mathcal{H}}(t) = (-1)^d E_{\mathcal{P}, \mathcal{H}}(-t).
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Basic inside-out results

**Theorem** If \((P, \mathcal{H})\) is a closed, full-dimensional, rational inside-out polytope, then \(E_{P, \mathcal{H}}(t)\) and \(E_{P^{\circ}, \mathcal{H}}(t)\) are quasipolynomials in \(t\) of degree \(\dim P\) with leading term \(\text{vol } P\), and with constant term \(E_{P, \mathcal{H}}(0)\) equal to the number of regions of \((P, \mathcal{H})\). Furthermore,

\[
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**Theorem** \((P, \mathcal{H})\) is a closed, full-dimensional, rational inside-out polytope, then

\[
E_{P, \mathcal{H}}(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\mathbb{R}^d, u) \text{Ehr}_{P \cap u}(t),
\]

and if \(\mathcal{H}\) is transverse to \(P\)

\[
E_{P, \mathcal{H}}(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\mathbb{R}^d, u)| \text{Ehr}_{P \cap u}(t).
\]

(\(\mathcal{H}\) is transverse to \(P\) if every flat \(u \in \mathcal{L}(\mathcal{H})\) that intersects \(P\) also intersects \(P^{\circ}\), and \(P\) does not lie in any of the hyperplanes of \(\mathcal{H}\).)
(Strong) magic squares

Mag\(_n(t)\) – number of nonnegative integral \(n \times n\)-matrices with distinct entries in which every row and column sums to \(t\)

\[
\begin{array}{ccc}
4 & 3 & 8 \\
9 & 5 & 1 \\
2 & 7 & 6
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Corollary \( \text{Mag}_n(t) \) is a quasipolynomial in \( t \) of degree \( n - 2n - 1 \).

Open problem Can anything be said about the period of \( \text{Mag}_n \)? Even in the weak case, do we ever get a polynomial?
Enumeration of integer points with distinct entries

\( \mathcal{P} \subset \mathbb{R}^d \) – rational convex polytope, transverse to

\( \mathcal{H} := \mathcal{H}[K_d]^{\text{aff } \mathcal{P}} \) – arrangement corresponding to \( K_d \), induced on \( \text{aff } \mathcal{P} \)
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**Theorem** The number \( E_{\mathcal{P}^\circ, \mathcal{H}}(t) \) of integer points in \( t\mathcal{P}^\circ \) with distinct entries is a quasipolynomial with constant term equal to the number of permutations of \( [d] \) that are realizable in \( \mathcal{P} \). Furthermore, \( (-1)^{d \text{dim } s} E_{\mathcal{P}^\circ, \mathcal{H}}(-t) = E_{\mathcal{P}, \mathcal{H}}(t) := \) the number of pairs \((x, \sigma)\) consisting of an integer point \( x \in t\mathcal{P} \) and a compatible \( \mathcal{P} \)-realizable permutation \( \sigma \) of \([d]\).

(The point \( x \in \mathbb{R}^d \) and the permutation \( \tau \) are compatible if \( x_{\tau_1} < x_{\tau_2} < \cdots < x_{\tau_d} \). \( \tau \) is realizable in \( X \) if there exists a compatible \( x \in X \).)
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**Applications** (strong) magic squares, rectangles, cubes, graphs, ...
Open problems

When does $\mathcal{H}[K_d]$ change the denominator of $\mathcal{P}$?
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If $\mathcal{P}$ has integral vertices then $\text{Ehr}_\mathcal{P}$ is a polynomial. What conditions on $\mathcal{P}$ ensure that $E_{\mathcal{P},\mathcal{H}[K_d]}$ is also a polynomial? (It need not be: Consider the line segment $\mathcal{P}$ from $(0,1)$ to $(1,0)$ and let $\mathcal{H} = \{x_1 = x_2\}$.)
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The inside-out Ehrhart quasipolynomials for some magic and latin squares have striking symmetries (coefficients alternate in sign, the polynomials factor nicely, etc.). Explain.
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If $\mathcal{P}$ has integral vertices then $E_{\mathcal{P}}$ is a polynomial. What conditions on $\mathcal{P}$ ensure that $E_{\mathcal{P}, \mathcal{H}[K_d]}$ is also a polynomial? (It need not be: Consider the line segment $\mathcal{P}$ from $(0,1)$ to $(1,0)$ and let $\mathcal{H} = \{x_1 = x_2\}$.)

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The inside-out Ehrhart quasipolynomials for some magic and latin squares have much lower periods than predicted by their denominators. Explain.
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If $\mathcal{P}$ has integral vertices then $\text{Ehr}_\mathcal{P}$ is a polynomial. What conditions on $\mathcal{P}$ ensure that $E_{\mathcal{H},\mathcal{P},K_d}$ is also a polynomial? (It need not be: Consider the line segment $\mathcal{P}$ from $(0, 1)$ to $(1, 0)$ and let $\mathcal{H} = \{x_1 = x_2\}$.)

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Compute $\text{Mag}_4, \text{Mag}_5, \ldots$ (possibly using LattE and the Möbius function of the intersection lattice of $\mathcal{H}[K_d]$).
Latin squares and beyond

Covering cluster \((X, \mathcal{L})\) – a finite set \(X\) of points together with a family \(\mathcal{L} \subseteq P(X)\) of lines

Latin labelling of \((X, \mathcal{L})\) – assignment of integers to \(X\) such that all entries in a line are distinct.
Latin squares and beyond

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To make counting fun, we restrict the entries to the set \((0, t)\). This corresponds to the inside-out polytope \(([0, 1]^X, \mathcal{H}[\Gamma_\mathcal{L}])\), where \(\Gamma_\mathcal{L} = \bigcup_{L \in \mathcal{L}} K_L\). (Every graph is isomorphic to one of those.)
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Example: latin rectangle – lines are rows & columns, \(\Gamma_\mathcal{L} = K_m \times K_n\). Slightly more general are (partial) latin orthotopes with \(\Gamma_\mathcal{L} = K_{m_1} \times \cdots \times K_{m_j}\) (a “Hamming graph”).
Stanley’s Theorem and latinity

Theorem The number $L^\circ(t)$ of latin labellings of $(X, \mathcal{L})$ with values in $(0, t)$ is a monic polynomial of degree $|X|$ with constant term equal to the number of acyclic orientations of $\Gamma_{\mathcal{L}}$. Furthermore, $(-1)^{|X|}L^\circ(-t)$ enumerates pairs consisting of an acyclic orientation of $\Gamma_{\mathcal{L}}$ and a compatible latin labelling with values in $[0, t]$. 

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- set all line sums equal to each other;
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**Example** : latin squares, with $t = \binom{n+1}{2}$

Note that the hyperplane arrangement gets more complicated, namely $\mathcal{H}[\Gamma_\mathcal{L}]^s$, where $s$ is the subspace of $\mathbb{R}^X$ determined by the line sum conditions.
Open problem

The magic subspace of the covering cluster \(([d], \mathcal{L})\) is defined by all line sums given by \(\mathcal{L}\) being equal.

A permutation \(\sigma\) of \([d]\) defines a reverse dominance order on the power set \(P([d])\) by \(L \preceq_{\sigma} L'\) if, when \(L\) and \(L'\) are written in decreasing order according to \(\sigma\), say \(L = \{\sigma j_1, \ldots, \sigma j_l\}\) where \(j_1 > \cdots > j_l\) and \(L' = \{\sigma j'_1, \ldots, \sigma j'_{l'}\}\) where \(j'_1 > \cdots > j'_{l'}\), then \(l \leq l'\) and \(j_1 \leq j'_1, \ldots, j_l \leq j'_{l'}\).

**Conjecture** A permutation \(\sigma\) of \([d]\) is realizable by a positive point in the magic subspace of the covering cluster \(([d], \mathcal{L})\) if and only if \(\mathcal{L}\) is an antichain in the reverse dominance order due to \(\sigma\).
Antimagic

\[ f_1, \ldots, f_m \in (\mathbb{R}^d)^* \] – linear forms

\[ A^\circ(t) := \# \text{ integer points } x \in (0, t)^d \text{ such that } \]

\[ f_j(x) \neq f_k(x) \quad \text{if } j \neq k \]
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Inside-out interpretation: \( f(x) := (f_1, \ldots, f_m)(x) \notin \bigcup \mathcal{H}[K_m] \subseteq \mathbb{R}^m \)

Pullback \( \mathcal{H}[K_m]^\# \subseteq \mathbb{R}^d \) obtained from \( f^{-1}(h) \) for all \( h \in \mathcal{H}[K_m] \)

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Examples: antimagic graphs and relatives (bidirected antimagic graphs, node antimagic, total graphical antimagic), antimagic squares, cubes, etc.
Is there a combinatorial interpretation of the regions of $\mathcal{H}[K_m]^\sharp$?
Open problems

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What is the intersection-lattice structure of $\mathcal{H}[K_m]^\#$?
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Prove that every graph except $K_2$ is (strongly) antimagic, i.e., admits an antimagic labelling using the numbers $1, 2, \ldots, |E|$.
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Prove that every graph except $K_2$ is (strongly) antimagic, i.e., admits an antimagic labelling using the numbers $1, 2, \ldots, |E|$. If that’s too hard, try trees.