Dedekind Sums: A Geometric Viewpoint

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"Ubi materia, ibi geometria."

Johannes Kepler (1571-1630)

"Ubi number theory, ibi geometria."

Variation on Johannes Kepler (1571-1630)

Ehrhart Theory

Integral (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right) = \#\left(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d\right)$

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Theorem (Ehrhart 1962) If $\mathcal{P}$ is an integral polytope, then...

- $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are polynomials in t of degree $\dim \mathcal{P}$
- Leading term: $\mathrm{vol}(P)$ (suitably normalized)
- (Macdonald 1970) $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

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- (Macdonald 1970) $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

Alternative description of a polytope:

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\} \rightleftarrows \left\{ \mathbf{x} \in \mathbb{R}^d_{\ge 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$$

Ehrhart Theory

Rational (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Q}^d$

For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right) = \#\left(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d\right)$

Theorem (Ehrhart 1962) If $\mathcal{P}$ is an rational polytope, then...

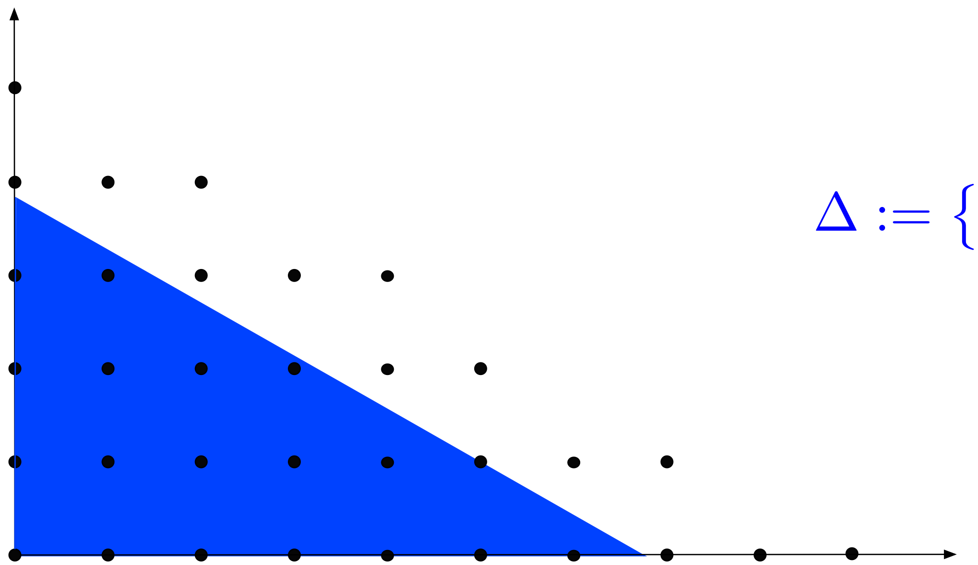
- $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are quasi-polynomials in t of degree $\dim \mathcal{P}$
- Leading term: $\mathrm{vol}(P)$ (suitably normalized)
- (Macdonald 1970) $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

Alternative description of a polytope:

$$\mathcal{P} = \left\{\mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{x} \le \mathbf{b}\right\} \rightleftarrows \left\{\mathbf{x} \in \mathbb{R}^d_{\ge 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b}\right\}$$

Quasi-polynomial – $c_d(t)\,t^d + c_{d-1}(t)\,t^{d-1} + \cdots + c_0(t)$ where $c_k(t)$ are periodic

An Example in Dimension 2



$(a = 7, b = 4, t = 23)$

$$\Delta := \left\{ (x, y) \in \mathbb{R}^2_{\geq 0} : \ ax + by \leq 1 \right\}$$

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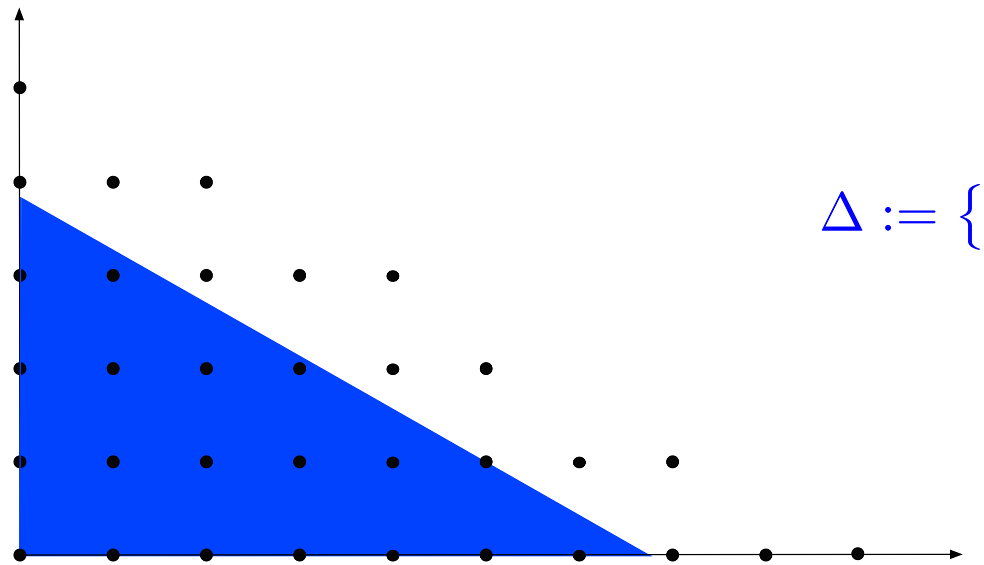

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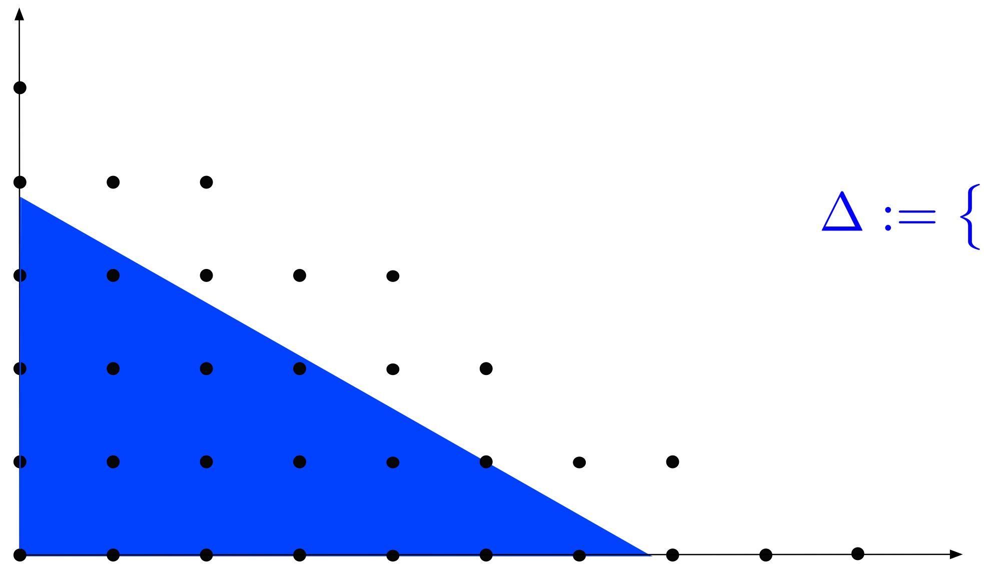

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$$\begin{aligned} L_\Delta(t) &= \# \left\{ (m,n) \in \mathbb{Z}^2_{\geq 0} : \ am + bn \leq t \right\} \\ &= \# \left\{ (m,n,s) \in \mathbb{Z}^3_{\geq 0} : \ am + bn + s = t \right\} \end{aligned}$$

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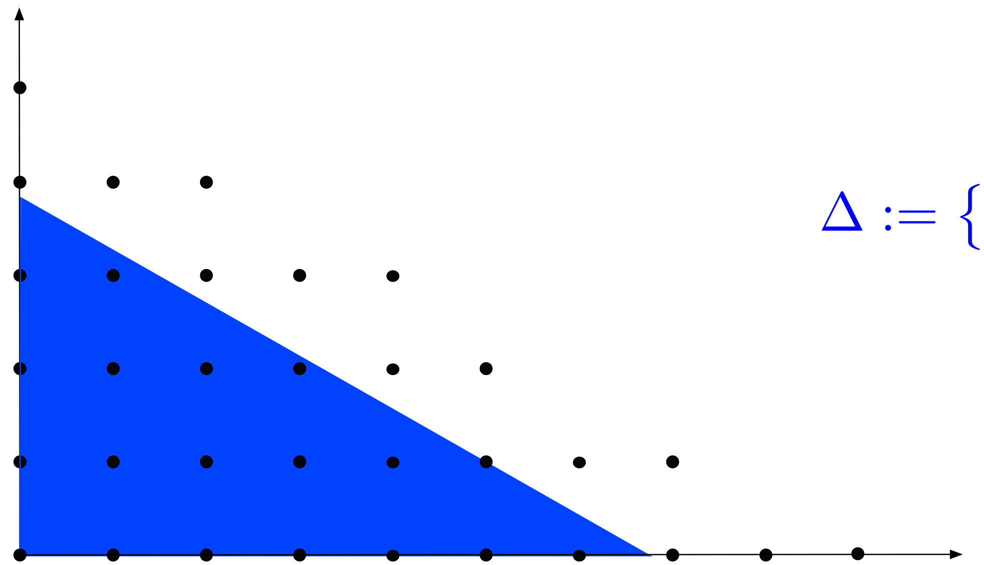

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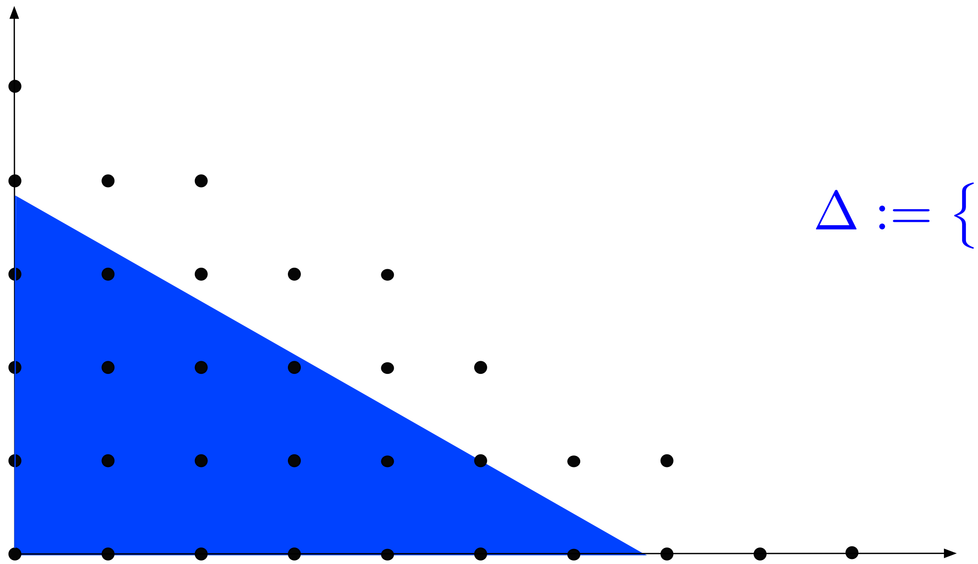

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An Example in Dimension 2

$$\Delta := \left\{ (x,y) \in \mathbb{R}^2_{\geq 0} : \; ax + by \leq 1 \right\}$$

$$f(x) := \frac{1}{\left(1 - x^a\right)\left(1 - x^b\right)\left(1 - x\right) x^{t+1}}$$

$$L_\Delta(t) \quad = \quad \frac{1}{2\pi i} \int_{|x|=\epsilon} f \, dx$$

An Example in Dimension 2

$$\Delta := \left\{ (x,y) \in \mathbb{R}^2_{\geq 0} : \ ax + by \leq 1 \right\} \qquad \gcd(a,b) = 1$$

$$f(x) := \frac{1}{(1-x^a)(1-x^b)(1-x)\,x^{t+1}} \qquad \xi_a := e^{2\pi i/a}$$

$$\begin{aligned} L_\Delta(t) &= \frac{1}{2\pi i} \int_{|x|=\epsilon} f \, dx \\ &= \operatorname{Res}_{x=1}(f) + \sum_{k=1}^{a-1} \operatorname{Res}_{x=\xi_a^k}(f) + \sum_{j=1}^{b-1} \operatorname{Res}_{x=\xi_b^j}(f) \end{aligned}$$

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&= \frac{t^2}{2ab} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{a} + \frac{1}{b}\right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + 3 + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) \\
&\quad + \frac{1}{a} \sum_{k=1}^{a-1} \frac{1}{(1-\xi_a^{kb})(1-\xi_a^k)\,\xi_a^{kt}} + \frac{1}{b} \sum_{j=1}^{b-1} \frac{1}{\left(1-\xi_b^{ja}\right)\left(1-\xi_b^j\right)\xi_b^{jt}}
\end{aligned}$$

An Example in Dimension 2

(Pick's or) Ehrhart's Theorem implies that

$$L_\Delta(t) = \frac{t^2}{2ab} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{a} + \frac{1}{b}\right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + 3 + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right)$$
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has constant term $L_\Delta(0) = 1$

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$$+ \frac{1}{a}\sum_{k=1}^{a-1} \frac{1}{\left(1-\xi_a^{kb}\right)\left(1-\xi_a^{k}\right)\xi_a^{kt}} + \frac{1}{b}\sum_{j=1}^{b-1} \frac{1}{\left(1-\xi_b^{ja}\right)\left(1-\xi_b^{j}\right)\xi_b^{jt}}$$

has constant term $L_\Delta(0) = 1$ and hence

$$\frac{1}{a}\sum_{k=1}^{a-1} \frac{1}{\left(1-\xi_a^{kb}\right)\left(1-\xi_a^{k}\right)} + \frac{1}{b}\sum_{j=1}^{b-1} \frac{1}{\left(1-\xi_b^{ja}\right)\left(1-\xi_b^{j}\right)}$$
$$= 1 - \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + 3 + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right)$$

An Example in Dimension 2

(Recall that $\xi_a := e^{2\pi i/a}$)

$$\frac{1}{a}\sum_{k=1}^{a-1}\frac{1}{\left(1-\xi_a^{kb}\right)\left(1-\xi_a^{k}\right)}+\frac{1}{b}\sum_{j=1}^{b-1}\frac{1}{\left(1-\xi_b^{ja}\right)\left(1-\xi_b^{j}\right)}$$

$$=1-\frac{1}{12}\left(\frac{3}{a}+\frac{3}{b}+3+\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right)$$

However...

$$\frac{1}{a}\sum_{k=1}^{a-1}\frac{1}{\left(1-\xi_a^{kb}\right)\left(1-\xi_a^{k}\right)}=-\frac{1}{4a}\sum_{k=1}^{a-1}\cot\left(\frac{\pi kb}{a}\right)\cot\left(\frac{\pi k}{a}\right)+\frac{a-1}{4a}$$

is essentially a Dedekind sum.

Dedekind Sums

Let $((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases}$ and define the Dedekind sum as

$$s(a,b) := \sum_{k=1}^{b-1} \left(\left(\frac{ka}{b} \right)\right) \left(\left(\frac{k}{b} \right)\right) = \frac{1}{4b} \sum_{j=1}^{b-1} \cot\left(\frac{\pi j a}{b} \right) \cot\left(\frac{\pi j}{b} \right).$$

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Since their introduction by Dedekind in the 1880's, these sums and their generalizations have appeared in various areas such as analytic (transformation law of η-function) and algebraic number theory (class numbers), topology (group action on manifolds), combinatorial geometry (lattice point problems), and algorithmic complexity (random number generators).

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The identity $L_{\Delta}(0) = 1$ implies...

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right)$$

the Reciprocity Law for Dedekind sums.

Dedekind Sum Reciprocity

$$s(a,b) = \frac{1}{4b} \sum_{j=1}^{b-1} \cot\left(\frac{\pi j a}{b}\right) \cot\left(\frac{\pi j}{b}\right).$$

the Reciprocity Law

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right)$$

together with the fact that $s(a,b) = s(a \text{ mod } b, b)$ implies that $s(a,b)$ is polynomial-time computable (Euclidean Algorithm).

Ehrhart Theory Revisited

For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right) = \# \left(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d \right)$.

Theorem (Ehrhart 1962) If $\mathcal{P}$ is an rational polytope, then...

- $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are quasi-polynomials in t of degree $\dim \mathcal{P}$.
- Leading term: $\mathrm{vol}(P)$ (suitably normalized)
- (Macdonald 1970) $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

In particular, if $t\mathcal{P}^\circ \cap \mathbb{Z}^d = \varnothing$ then $L_{\mathcal{P}}(-t) = 0.$

Rademacher Reciprocity

If $t\mathcal{P}^\circ \cap \mathbb{Z}^d = \varnothing$ then $L_{\mathcal{P}}(-t) = 0$.

$t\Delta^\circ = \left\{ (x,y) \in \mathbb{R}^2_{>0} : \ ax + by < t \right\}$ does not contain any lattice points for $1 \le t < a+b$ which gives for these t

$$\frac{1}{a} \sum_{k=1}^{a-1} \frac{\xi_a^{kt}}{\left(1-\xi_a^{kb}\right)\left(1-\xi_a^{k}\right)} + \frac{1}{b} \sum_{j=1}^{b-1} \frac{\xi_b^{jt}}{\left(1-\xi_b^{ja}\right)\left(1-\xi_b^{j}\right)}$$
$$= -\frac{t^2}{2ab} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{a} + \frac{1}{b}\right) - \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + 3 + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right).$$

Rademacher Reciprocity

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$$= -\frac{t^2}{2ab} + \frac{t}{2}\left(\frac{1}{ab}+\frac{1}{a}+\frac{1}{b}\right) - \frac{1}{12}\left(\frac{3}{a}+\frac{3}{b}+3+\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right).$$

The sum $\displaystyle \frac{1}{a}\sum_{k=1}^{a-1} \frac{\xi_a^{kt}}{\left(1-\xi_a^{kb}\right)\left(1-\xi_a^{k}\right)}$ can be rewritten as a Dedekind–Rademacher sum

$$r_n(a,b) := \sum_{k=1}^{b-1} \left(\left(\frac{ka+n}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right).$$

Rademacher Reciprocity

The identity

$$\frac{1}{a}\sum_{k=1}^{a-1}\frac{\xi_a^{kt}}{\left(1-\xi_a^{kb}\right)\left(1-\xi_a^{k}\right)}+\frac{1}{b}\sum_{j=1}^{b-1}\frac{\xi_b^{jt}}{\left(1-\xi_b^{ja}\right)\left(1-\xi_b^{j}\right)}$$
$$=-\frac{t^2}{2ab}+\frac{t}{2}\left(\frac{1}{ab}+\frac{1}{a}+\frac{1}{b}\right)-\frac{1}{12}\left(\frac{3}{a}+\frac{3}{b}+3+\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right)$$

gives Knuth's version of Rademacher's Reciprocity Law (1964)

$$r_n(a,b)+r_n(b,a)=\text{ something simple }.$$

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As with $s(a,b)$, this reciprocity identity implies that $r_n(a,b)$ is polynomial-time computable.

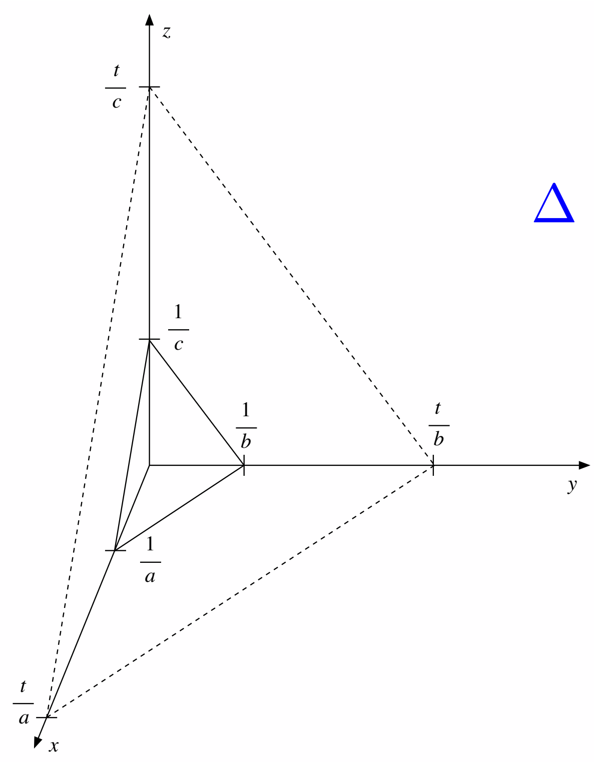
Why Bother?

- Classical connections, e.g., Dedekinds's reciprocity law implies Gauß's Theorem on quadratic reciprocity.

- Generalized Dedekind sums measure signature effects, compute class numbers, count lattice points in polytopes, and measure randomness of random-number generators—are there intrinsic connections?

- It is not clear how to efficiently compute higher-dimensional generalizations of the Dedekind sum.

A 2-dimensional Example in Dimension 3



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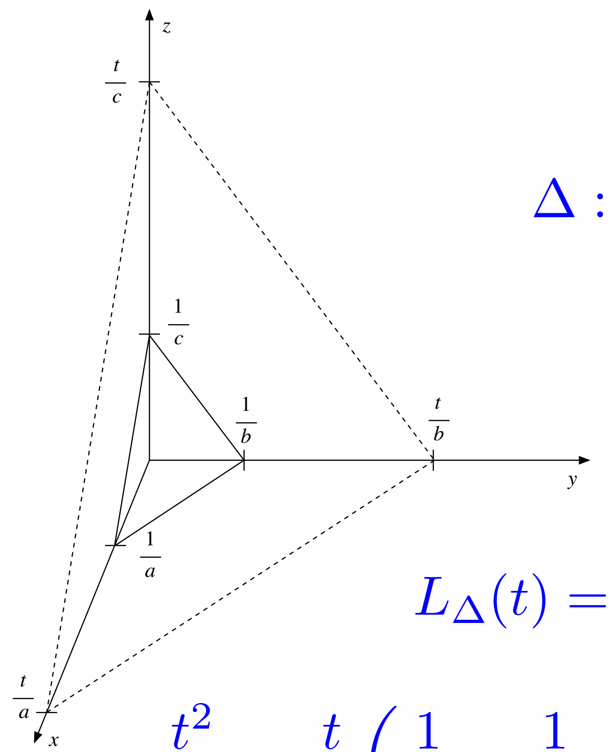

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$$\gcd(a,b) = \gcd(b,c) = \gcd(c,a) = 1$$

$$L_\Delta(t) = \frac{1}{2\pi i} \int_{|x|=\epsilon} \frac{dx}{(1-x^a)(1-x^b)(1-x^c)\, x^{t+1}}$$

$$= \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right)$$

$$+ \frac{1}{a}\sum_{k=1}^{a-1} \frac{1}{\left(1-\xi_a^{kb}\right)\left(1-\xi_a^{kc}\right)\xi_a^{kt}} + \frac{1}{b}\sum_{k=1}^{b-1} \frac{1}{\left(1-\xi_b^{kc}\right)\left(1-\xi_b^{ka}\right)\xi_b^{kt}}$$

$$+ \frac{1}{c}\sum_{k=1}^{c-1} \frac{1}{\left(1-\xi_c^{ka}\right)\left(1-\xi_c^{kb}\right)\xi_c^{kt}}$$

More Dedekind Sums

$$s(a,b;c) := \frac{1}{4c}\sum_{j=1}^{c-1} \cot\left(\frac{\pi j a}{c}\right) \cot\left(\frac{\pi j b}{c}\right)$$

The identity $L_{\Delta}(0) = 1$ implies Rademacher's Reciprocity Law (1954)

$$s(a,b;c) + s(b,c;a) + s(c,a;b) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right).$$

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Moreover,

$$t\Delta = \left\{(x,y,z) \in \mathbb{R}^3_{\geq 0} : \; ax + by + cz = t\right\}$$

has no interior lattice points for $0 < t < a+b+c$, so that Ehrhart-Macdonald Reciprocity implies that $L_\Delta(t) = 0$ for $-(a+b+c) < t < 0$, which gives Gessel's generalization of the Reciprocity Law for Dedekind–Rademacher sums (1997).

"If you had done something twice, you are likely to do it again."

Brian Kernighan & Bob Pike (*The Unix Programming Environment*)

Higher-dimensional Dedekind Sums

The Ehrhart quasi-polynomial $L_\Delta(t)$ of the simplex

$$\Delta := \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \ a_1 x_1 + \cdots + a_d x_d = 1 \right\}$$

gives rise to the Fourier–Dedekind sum (MB–Diaz–Robins 2003)

$$s_n(a_2, \ldots, a_d; a_1) := \frac{1}{a_1} \sum_{k=1}^{a_1 - 1} \frac{\xi_{a_1}^{kn}}{\left(1 - \xi_{a_1}^{k a_2}\right) \cdots \left(1 - \xi_{a_1}^{k a_d}\right)} .$$

(Here $\xi_{a_1} := e^{2\pi i / a_1}$.)

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gives rise to the Fourier–Dedekind sum (MB–Diaz–Robins 2003)

$$s_n\left(a_2, \ldots, a_d; a_1\right) := \frac{1}{a_1} \sum_{k=1}^{a_1 - 1} \frac{\xi_{a_1}^{kn}}{\left(1 - \xi_{a_1}^{k a_2}\right) \cdots \left(1 - \xi_{a_1}^{k a_d}\right)} .$$

(Here $\xi_{a_1} := e^{2\pi i / a_1}$.) These sums include as a special case (essentially $n = 0$) Zagier's higher-dimensional Dedekind sums (1973)

$$c\left(a_2, \ldots, a_d; a_1\right) := \frac{1}{a_1} \sum_{k=1}^{a_1 - 1} \cot\left(\frac{k a_2}{a_1}\right) \cdots \cot\left(\frac{k a_d}{a_1}\right) .$$

Reciprocity for Higher-dimensional Dedekind Sums

$$\Delta := \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \; a_1 x_1 + \cdots + a_d x_d = 1 \right\}$$

The identity $L_\Delta(0) = 1$ implies the reciprocity law

$$\begin{aligned} & c\,(a_2, \ldots, a_d; a_1) + c\,(a_1, a_3, \ldots, a_d; a_2) + \cdots + c\,(a_1, \ldots, a_{d-1}; a_d) \\ & \quad = \text{ something simple} \end{aligned}$$

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The right-hand side of the reciprocity law can be expressed in terms of Hirzebruch L-functions. Note that this reciprocity relation does not imply any computability properties of $c\,(a_2, \ldots, a_d; a_1)$.

Reciprocity for Fourier–Dedekind Sums

$t\Delta^\circ = \left\{ \mathbf{x} \in \mathbb{R}^d_{>0} : \ a_1 x_1 + \cdots + a_d x_d = t \right\}$ does not contain any lattice points for $t < a_1 + \cdots + a_d$ and the Ehrhart–Macdonald Theorem gives

$$L_\Delta(t) = 0 \qquad \text{for} \qquad -(a_1 + \cdots + a_d) < t < 0$$

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and hence the reciprocity relation, for $0 < n < a_1 + \cdots + a_d$,

$$\begin{aligned} s_n(a_2, \dots, a_d; a_1) + s_n(a_1, a_3, \dots, a_d; a_2) + \cdots + s_n(a_1, \dots, a_{d-1}; a_d) \\ = \text{ some simple polynomial in } n \end{aligned}$$

for the Fourier–Dedekind sums

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This reciprocity relation is a higher-dimensional analog of Rademacher Reciprocity.

Complexity of Fourier–Dedekind Sums

Barvinok's Algorithm (1993) proves polynomial-time complexity of the rational generating function

$$\sum_{(m_1,\ldots,m_d)\in\mathcal{P}\cup\mathbb{Z}^d} x_1^{m_1}\cdots x_d^{m_d}$$

for any rational polyhedra $\mathcal{P}$ in fixed dimension. Barvinok's Algorithm generalizes Lenstra's Theorem on the complexity of integral programs (1983).

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Theorem (MB–Robins 2004) For fixed d, the Fourier–Dedekind sums

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are polynomial-time computable.

Complexity of Fourier–Dedekind Sums

Open Problem Give an intrinsic reason (not dependent on Barvinok's Algorithm) why the Fourier–Dedekind sums

$$s_n\left(a_2, \dots, a_d; a_1\right) := \frac{1}{a_1} \sum_{k=1}^{a_1-1} \frac{\xi_{a_1}^{kn}}{\left(1-\xi_{a_1}^{k a_2}\right) \cdots \left(1-\xi_{a_1}^{k a_d}\right)}$$

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Partition Functions and the Frobenius Problem

The Ehrhart quasi-polynomial

$$L_\Delta(t) = \# \left\{ (m_1, \dots, m_d) \in \mathbb{Z}_{\geq 0}^d : \ m_1 a_1 + \cdots + m_d a_d = t \right\}$$

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- New approach on the Frobenius problem via Gröbner bases

Shameless Plug

M. Beck & S. Robins

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