Cyclotomic Polytopes and Growth Series of Cyclotomic Lattices

Matthias Beck & Serkan Hoşten
San Francisco State University

math.sfsu.edu/beck  Math Research Letters
Growth Series of Lattices

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\( M \) – subset that generates \( \mathcal{L} \) as a monoid
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\( S(n) \) – number of elements in \( \mathcal{L} \) with word length \( n \) (with respect to \( M \))

\( G(x) := \sum_{n \geq 0} S(n) x^n \) – growth series of \((\mathcal{L}, M)\)

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**Theorem** (Kløve–Parker) The coordinator polynomial of \( \mathbb{Z}[e^{2\pi i/p}] \), where \( p \) is prime, equals \( h_p(x) = x^{p-1} + x^{p-2} + \cdots + 1 \).
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**Conjectures** (Parker)

(1) \( h_m(x) = g(x)\frac{m}{\sqrt{m}} \) for a palindromic polynomial \( g \) of degree \( \varphi(\sqrt{m}) \).

(2) \( h_{2p}(x) = \sum_{k=0}^{p-3} \left( x^k + x^{p-1-k} \right) \sum_{j=0}^{k} \binom{p}{j} + 2^{p-1}x^{p-1} \).

(3) \( h_{15}(x) = (1 + x^8) + 7(x + x^7) + 28(x^2 + x^6) + 79(x^3 + x^5) + 130x^4 \).
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Theorem (MB–Hoşten) Parker’s Conjectures (2) & (3) are true.
Cyclotomic Polytopes

We choose a specific basis for $\mathbb{Z}[e^{2\pi i/m}]$ consisting of certain powers of $e^{2\pi i/m}$ which we then identify with the unit vectors in $\mathbb{R}^{\varphi(m)}$. The other powers of $e^{2\pi i/m}$ are integer linear combinations of this basis; hence they are lattice vectors in $\mathbb{Z}[e^{2\pi i/m}] \subset \mathbb{R}^{\varphi(m)}$. The $m^{\text{th}}$ cyclotomic polytope $C_m$ is the convex hull of all of these $m$ lattice points in $\mathbb{R}^{\varphi(m)}$, which correspond to the $m^{\text{th}}$ roots of unity.
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1. $m$ is prime
2. $m$ is a prime power
3. $m$ is the product of two coprime integers.
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When $m = p$ is a prime number, let $\zeta = e^{2\pi i/p}$ and fix the $\mathbb{Z}$-basis $1, \zeta, \zeta^2, \ldots, \zeta^{p-2}$ of the lattice $\mathbb{Z}[\zeta]$. Together with $\zeta^{p-1} = -\sum_{j=0}^{p-2} \zeta^j$, these $p$ elements form a monoid basis for $\mathbb{Z}[\zeta]$. We identify them with $e_0, e_1, \ldots, e_{p-2}, -\sum_{j=0}^{p-2} e_j$ in $\mathbb{R}^{p-1}$ and define the cyclotomic polytope $C_p \subset \mathbb{R}^{p-1}$ as the simplex

$$C_p = \text{conv} \left( e_0, e_1, \ldots, e_{p-2}, -\sum_{i=0}^{p-2} e_i \right).$$
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Cyclotomic Polytopes

For two polytopes $P \subset \mathbb{R}^{d_1}$ and $Q \subset \mathbb{R}^{d_2}$, each containing the origin in its interior, we define the direct sum $P \circ Q := \text{conv}(P \times 0_{d_2}, 0_{d_1} \times Q)$. For a prime $p$, we define the cyclotomic polytope

$$C_p^{\alpha} = C_p \circ C_p \circ \cdots \circ C_p.$$ 

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For two polytopes $P = \text{conv}(v_1, v_2, \ldots, v_s)$ and $Q = \text{conv}(w_1, w_2, \ldots, w_t)$ we define their tensor product

$$P \otimes Q := \text{conv}(v_i \otimes w_j : 1 \leq i \leq s, 1 \leq j \leq t).$$
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Our construction implies for $m = m_1 m_2$, where $m_1, m_2 > 1$ are relatively prime, that the cyclotomic polytope $C_m$ is equal to $C_{m_1} \otimes C_{m_2}$. 
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For general $m$,

$$C_m = \underbrace{C_{\sqrt{m}} \circ C_{\sqrt{m}} \circ \cdots \circ C_{\sqrt{m}}}_{\frac{m}{\sqrt{m}} \text{ times}},$$

a $0/ \pm 1$ polytope with the origin as the sole interior lattice point.
Hilbert Series

$\mathcal{L} \cong \mathbb{Z}^d$ a lattice, $M$ a minimal set of monoid generators, $K$ a field

The vectors in $M' = \{(u, 1) : u \in M \cup \{0\}\}$ encoded as monomials give rise to the monoid algebra $K[M']$, in which each monomial corresponds to 

$(v, k)$ where $v = \sum_{u_i \in M \cup \{0\}} n_i u_i$ with nonnegative integer coefficients $n_i$ such that $\sum n_i = k$. 

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$$H(K[M']; x) := \sum_{k \geq 0} \dim_K (K[M']_k) x^k,$$

where $K[M']_k$ denotes the vector space of elements of degree $k$ in this graded algebra.
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In the conditions of our theorem, the Hilbert series of \( C_m \circ C_m \) equals \((1 - x)^m\) times the square of the Hilbert series of \( C_m \), whence \( h_m(x) = h(\sqrt{m}(x))^{\frac{m}{\sqrt{m}}} \).
A polytope $P$ is **totally unimodular** if every submatrix of the matrix consisting of the vertices of $P$ has determinant $0, \pm 1$.

**Theorem** (MB–Hoşten) If $m$ is divisible by at most two odd primes then the cyclotomic polytope $C_m$ is totally unimodular.
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**Corollary** If $m$ is divisible by at most two odd primes then the cyclotomic polytope $C_m$ is **normal**, i.e., the monoid generated by $M'$ and the monoid of the lattice points in the cone generated by $M'$ are equal.
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**Remark** Total unimodularity breaks down already for $C_{3pq}$ for distinct primes $p, q > 3$. This is an indication that Parker’s Conjecture (1) might not be true in general.
Theorem (MB–Hoşten) Suppose \( m \) is divisible by at most two odd primes. 

(2) \( h_{\sqrt{m}}(x) \) is the h-polynomial of a simplicial polytope.

... follows now because \( C_{\sqrt{m}} \) has a unimodular triangulation, which induces a unimodular triangulation of the boundary of \( C_{\sqrt{m}} \). This boundary equals the boundary of a simplicial polytope \( Q \) (Stanley), and \( h_{\sqrt{m}} \) is the h-polynomial of \( Q \) (which is palindromic, unimodal, and nonnegative).
**Total Unimodularity and Normality**

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**Remark** If $C_m$ is a simplicial polytope then the coordinator polynomial $h_m$ equals the h-polynomial of $C_m$. The polytope $C_m$ is simplicial, e.g., for $m$ a prime power or the product of two primes (the latter was proved by R. Chapman and follows from the fact that $C_{pq}$ is dual to a transportation polytope with margins $p$ and $q$).
Open Problems

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► S. Sullivant computed that the dual of $C_{105}$ is not a lattice polytope, i.e., $C_{105}$ is not reflexive. If we knew that $C_{105}$ is normal, a theorem of Hibi would imply that the coordinator polynomial $h_{105}$ is not palindromic, and hence that Parker’s Conjecture (1) is not true in general.