Asymptotics of Ehrhart Series of Lattice Polytopes

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Let’s say we add two random 100-digit integers. How often should we expect to carry a digit?
Warm-Up Trivia

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The Eulerian polynomial $A_d(t)$ is defined through

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\sum_{m \geq 0} m^d t^m = \frac{A_d(t)}{(1 - t)^{d+1}}
$$

Persi Diaconis will tell you that the coefficients of $A_d(t)$ (the Eulerian numbers) play a role here. . .
Ehrhart Polynomials

\( \mathcal{P} \subset \mathbb{R}^d \) – lattice polytope of dimension \( d \) (vertices in \( \mathbb{Z}^d \))
Ehrhart Polynomials

$\mathcal{P} \subset \mathbb{R}^d$ – lattice polytope of dimension $d$ (vertices in $\mathbb{Z}^d$)

$L_{\mathcal{P}}(m) := \# (m\mathcal{P} \cap \mathbb{Z}^d)$ (discrete volume of $\mathcal{P}$)
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**Theorem** (Ehrhart 1962) \( L_\mathcal{P}(m) \) is a polynomial in \( m \) of degree \( d \). Equivalently,

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\text{Ehr}_\mathcal{P}(t) = \frac{h(t)}{(1 - t)^{d+1}}
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where \( h(t) \) is a polynomial of degree at most \( d \).
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Write the Ehrhart h-vector of \( \mathcal{P} \) as \( h(t) = h_d t^d + h_{d-1} t^{d-1} + \cdots + h_0 \), then

\[
L_\mathcal{P}(m) = \sum_{j=0}^{d} h_j \binom{m + d - j}{d}.
\]
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(Serious) Open Problem Classify Ehrhart h-vectors.
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(Serious) Open Problem Classify Ehrhart h-vectors.

Easier Problem Study \( \text{Ehr}_{n\mathcal{P}}(t) = 1 + \sum_{m \geq 1} L_\mathcal{P}(nm) t^m \) as \( n \) increases.
Why Should We Care?

- Linear systems are everywhere, and so polytopes are everywhere.
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▶ Many discrete problems in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.

▶ Polytopes are cool.
General Properties of Ehrhart h-Vectors

\[
\text{Ehr}_P(t) = 1 + \sum_{m \geq 1} \# (mP \cap \mathbb{Z}^d) \ t^m = \frac{h_d t^d + h_{d-1} t^{d-1} + \cdots + h_0}{(1 - t)^{d+1}}
\]

▶ (Ehrhart) \( h_0 = 1 \), \( h_1 = \# (P \cap \mathbb{Z}^d) - d - 1 \), \( h_d = \# (P^\circ \cap \mathbb{Z}^d) \)
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  \item (Stanley 1980) \( h_j \in \mathbb{Z}_{\geq 0} \)
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- (Stanley 1991) Whenever $h_s > 0$ but $h_{s+1} = \cdots = h_d = 0$, then $h_0 + h_1 + \cdots + h_j \leq h_s + h_{s-1} + \cdots + h_{s-j}$ for all $0 \leq j \leq s$. 
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- (Hibi 1994) $h_0 + \cdots + h_{j+1} \geq h_d + \cdots + h_{d-j}$ for $0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1$. 
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- (Hibi 1994) If $h_d > 0$ then $h_1 \leq h_j$ for $2 \leq j < d$. 

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General Properties of Ehrhart $h$-Vectors

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▶ (Stapledon 2009) Many more inequalities for the $h_j$'s arising from Kneser's Theorem (arXiv:0904.3035)
A triangulation $\tau$ of $P$ is unimodular if for any simplex of $\tau$ with vertices $v_0, v_1, \ldots, v_d$, the vectors $v_1 - v_0, \ldots, v_d - v_0$ form a basis of $\mathbb{Z}^d$. 
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The $h$-polynomial (h-vector) of a triangulation $\tau$ encodes the faces numbers of the simplices in $\tau$ of different dimensions.
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(Stanley 1980) If $P$ admits a unimodular triangulation then $h(t) = (1 - t)^{d+1} \text{Ehr}_P(t)$ is the $h$-polynomial of the triangulation.
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- Recent papers of Reiner–Welker and Athanasiadis use this as a starting point to give conditions under which the Ehrhart h-vector is unimodal, i.e., $h_d \leq h_{d-1} \leq \cdots \leq h_k \geq h_{k-1} \geq \cdots \geq h_0$ for some $k$. 

Define \( h_0(n), h_1(n), \ldots, h_d(n) \) through

\[
Ehr_{n\mathcal{P}}(t) = \frac{h_d(n) t^d + h_{d-1}(n) t^{d-1} + \cdots + h_0(n)}{(1 - t)^{d+1}}.
\]

What does the Ehrhart h-vector \((h_0(n), h_1(n), \ldots, h_d(n))\) look like as \( n \) increases?
The Main Question

Define \( h_0(n), h_1(n), \ldots, h_d(n) \) through

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What does the Ehrhart h-vector \((h_0(n), h_1(n), \ldots, h_d(n))\) look like as \( n \) increases?

Let \( h(t) = (1 - t)^{d+1} \text{Ehr}_{\mathcal{P}}(t) \). The operator \( U_n \) defined through

\[
\text{Ehr}_{n \mathcal{P}}(t) = 1 + \sum_{m \geq 1} \mathcal{L}_{\mathcal{P}}(nm) t^m = \frac{U_n h(t)}{(1 - t)^{d+1}}
\]

appears in Number Theory as a Hecke operator and in Commutative Algebra in Veronese subring constructions.
Motivation I: Unimodular Triangulations

**Theorem** (Kempf–Knudsen–Mumford–Saint-Donat–Waterman 1970's)
For every lattice polytope $P$ there exists an integer $m$ such that $mP$ admits a regular unimodular triangulation.
Motivation I: Unimodular Triangulations

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**Conjectures**

(a) For every lattice polytope $\mathcal{P}$ there exists an integer $m$ such that $k\mathcal{P}$ admits a regular unimodular triangulation for $k \geq m$. 
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(b) For every $d$ there exists an integer $m_d$ such that, if $\mathcal{P}$ is a $d$-dimensional lattice polytope, then $m_d\mathcal{P}$ admits a regular unimodular triangulation.
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(c) For every $d$ there exists an integer $m_d$ such that, if $\mathcal{P}$ is a $d$-dimensional lattice polytope, then $k\mathcal{P}$ admits a regular unimodular triangulation for $k \geq m_d$. 
Motivation II: Unimodal Ehrhart h-Vectors

Theorem (Athanasiadis–Hibi–Stanley 2004) If the $d$-dimensional lattice polytope $\mathcal{P}$ admits a regular unimodular triangulation, then the Ehrhart $h$-vector of $\mathcal{P}$ satisfies

\begin{align*}
(a) \quad & h_{j+1} \geq h_{d-j} \text{ for } 0 \leq j \leq \lfloor \frac{d}{2} \rfloor - 1 , \\
(b) \quad & h_{\lfloor \frac{d+1}{2} \rfloor} \geq h_{\lfloor \frac{d+1}{2} \rfloor + 1} \geq \cdots \geq h_{d-1} \geq h_d , \\
(c) \quad & h_j \leq \binom{h_1 + j - 1}{j} \text{ for } 0 \leq j \leq d .
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(b) $h_{\left\lfloor \frac{d+1}{2} \right\rfloor} \geq h_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1} \geq \cdots \geq h_{d-1} \geq h_d$,

(c) $h_j \leq \binom{h_1 + j - 1}{j}$ for $0 \leq j \leq d$.

In particular, if the Ehrhart $h$-vector of $\mathcal{P}$ is symmetric and $\mathcal{P}$ admits a regular unimodular triangulation, then the Ehrhart $h$-vector is unimodal.
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In particular, if the Ehrhart h-vector of \(\mathcal{P}\) is symmetric and \(\mathcal{P}\) admits a regular unimodular triangulation, then the Ehrhart h-vector is unimodal.

There are (many) lattice polytopes for which (some of these) inequalities fail and one may hope to use this theorem to construct a counter-example to the Knudsen–Mumford–Waterman Conjectures.
Theorem (Brenti–Welker 2008) For any $d \in \mathbb{Z}_{>0}$, there exists real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{d-1} < \alpha_d = 0$, such that, if $h(t)$ is a polynomial of degree at most $d$ with nonnegative integer coefficients and constant term 1, then for $n$ sufficiently large, $U_n h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ and $\lim_{n \to \infty} \beta_j(n) = \alpha_j$. 
Veronese Polynomials Are Eventually Unimodal

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Here “sufficiently large” depends on \( h(t) \).
**Veronese Polynomials Are Eventually Unimodal**

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If the polynomial $p(t) = a_dt^d + a_{d-1}t^{d-1} + \cdots + a_0$ has negative roots, then its coefficients are (strictly) log concave ($a_j^2 > a_{j-1}a_{j+1}$).
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such that, if $h(t)$ is a polynomial of degree at most $d$ with nonnegative integer coefficients and constant term 1, then for $n$ sufficiently large, $U_n h(t)$ has negative real roots

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If the polynomial $p(t) = a_dt^d + a_{d-1}t^{d-1} + \cdots + a_0$ has negative roots, then its coefficients are (strictly) log concave ($a_j^2 > a_{j-1}a_{j+1}$) which, in turn, implies that the coefficients are (strictly) unimodal ($a_d < a_{d-1} < \cdots < a_k > a_{k-1} > \cdots > a_0$ for some $k$).
A General Theorem

The Eulerian polynomial $A_d(t)$ is defined through

$$\sum_{m \geq 0} m^d t^m = \frac{A_d(t)}{(1 - t)^{d+1}}.$$ 

Theorem (MB–Stapledon) Fix a positive integer $d$ and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of $A_d(t)$. There exist $M, N$ depending only on $d$ such that, if $h(t)$ is a polynomial of degree at most $d$ with nonnegative integer coefficients and constant term 1, then for $n \geq N$, $U_n h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with

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and the coefficients of $U_n h(t)$ satisfy $h_j(n) < M h_d(n)$. 

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Furthermore, if $h_0 + \cdots + h_{j+1} \geq h_d + \cdots + h_{d-j}$ for $0 \leq j \leq \lfloor \frac{d}{2} \rfloor - 1$, with at least one strict inequality, then we may choose $N$ such that, for $n \geq N$,

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An Ehrhartian Corollary

Corollary (MB–Stapledon) Fix a positive integer $d$ and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of the Eulerian polynomial $A_d(t)$. There exist $M, N$ depending only on $d$ such that, if $P$ is a $d$-dimensional lattice polytope with Ehrhart series numerator $h(t)$, then for $n \geq N$, $U_n h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\lim_{n \to \infty} \beta_j(n) = \rho_j$. 
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In particular, the coefficients of $U_n h(t)$ are unimodal for $n \geq N$.

Furthermore, they satisfy

$$1 = h_0(n) < h_d(n) < h_1(n) < \cdots < h_j(n) < h_{d-j}(n) < h_{j+1}(n)$$
$$< \cdots < h_{\left\lfloor \frac{d+1}{2} \right\rfloor}(n) < M h_d(n).$$
Ingredients I

Stapledon's Decomposition A polynomial $h(t) = h_{d+1} t^{d+1} + h_d t^d + \cdots + h_0$ of degree at most $d+1$ has a unique decomposition $h(t) = a(t) + b(t)$ where $a(t)$ and $b(t)$ are polynomials satisfying $a(t) = t^d a\left(\frac{1}{t}\right)$ and $b(t) = t^{d+1} b\left(\frac{1}{t}\right)$. 
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The coefficients of $a(t)$ are positive if and only if $h_0 + \cdots + h_j \geq h_{d+1} + \cdots + h_{d+1-j}$ for $0 \leq j < \left\lfloor \frac{d}{2} \right\rfloor$. 
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**Theorem** (Stapledon 2008) If $h(t)$ is the Ehrhart $h$-vector of a lattice $d$-polytope, then the coefficients of $a(t)$ satisfy $1 = a_0 \leq a_1 \leq a_j$ for $2 \leq j \leq d - 1$. 

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Corollary Hibi’s inequalities $h_0 + \cdots + h_{j+1} \geq h_d + \cdots + h_{d-j}$ for the Ehrhart h-vector are strict.
Let \( h(t) = h_{d+1}t^{d+1} + h_dt^d + \cdots + h_0 \) be a polynomial of degree at most \( d + 1 \) and expand \( \frac{h(t)}{(1-t)^{d+1}} = h_0 + \sum_{m \geq 1} g(m) t^m \), for some polynomial \( g(m) = g_d m^d + g_{d-1} m^{d-1} + \cdots + g_0 \).
Ingredients II

Let \( h(t) = h_{d+1}t^{d+1} + h_dt^d + \cdots + h_0 \) be a polynomial of degree at most \( d + 1 \) and expand \( \frac{h(t)}{(1-t)^{d+1}} = h_0 + \sum_{m \geq 1} g(m) t^m \), for some polynomial \( g(m) = g_dm^d + g_{d-1}m^{d-1} + \cdots + g_0 \).

**Theorem** (Betke–McMullen 1985) If \( h_j \geq 0 \) for \( 0 \leq j \leq d + 1 \), then for any \( 1 \leq r \leq d - 1 \),

\[
g_r \leq (-1)^{d-r} S_r(d) g_d + \frac{(-1)^{d-r-1} h_0 S_{r+1}(d)}{(d-1)!},
\]

where \( S_i(d) \) is the Stirling number of the first kind.
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**Theorem** (MB–Stapledon) If $h_0 + \cdots + h_j \geq h_{d+1} + \cdots + h_{d+1-j}$ for $0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor$, with at least one strict inequality, then

$$g_{d-1-2r} \leq S_{d-1-2r}(d-1) g_{d-1} - \frac{(h_0 - h_{d+1}) S_{d-2r}(d-1)}{2(d-2)!}.$$
Let $h(t) = h_d m^d + h_{d-1} m^{d-1} + \cdots + h_0$ be a polynomial of degree at most $d$ and expand \( \frac{h(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} g(m) t^m \), for some polynomial $g(m) = g_d m^d + g_{d-1} m^{d-1} + \cdots + g_0$. 
Ingredients III

Let \( h(t) = h_d t^d + h_{d-1} t^{d-1} + \cdots + h_0 \) be a polynomial of degree at most \( d \) and expand \( \frac{h(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} g(m) t^m \), for some polynomial \( g(m) = g_d m^d + g_{d-1} m^{d-1} + \cdots + g_0 \).

Recall our notation \( U_n h(t) = h_d(n) t^d + h_{d-1}(n) t^{d-1} + \cdots + h_0(n) \).

Lemma \( U_n h(t) = \sum_{j=0}^{d} g_j A_j(t) (1 - t)^{d-j} n^j \).
Let \( h(t) = h_d t^d + h_{d-1} t^{d-1} + \cdots + h_0 \) be a polynomial of degree at most \( d \) and expand\( \frac{h(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} g(m) t^m \), for some polynomial \( g(m) = g_d m^d + g_{d-1} m^{d-1} + \cdots + g_0 \).

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Lemma \[ U_n h(t) = \sum_{j=0}^{d} g_j A_j(t) (1-t)^{d-j} n^j. \]

In particular, for \( 1 \leq j \leq d \), \( h_j(n) \) is a polynomial in \( n \) of degree \( d \) and

\[ h_j(n) = A(d, j) g_d n^d + (A(d-1, j) - A(d-1, j-1)) g_{d-1} n^{d-1} + O(n^{d-2}). \]
Ingredients IV

Lemma \( U_n h(t) = \sum_{j=0}^{d} g_j A_j(t) (1 - t)^{d-j} n^j. \)
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Exercise The nonzero roots of the Eulerian polynomial \( A_d(t) = \sum_{j=0}^{d} A(d, j) t^j \) are negative.
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Exercise  The nonzero roots of the Eulerian polynomial \( A_d(t) = \sum_{j=0}^{d} A(d, j) t^j \) are negative.

Theorem (Cauchy) Let \( p(n) = p_d n^d + p_{d-1} n^{d-1} + \cdots + p_0 \) be a polynomial of degree \( d \) with real coefficients. The complex roots of \( p(n) \) lie in the open disc

\[
\left\{ z \in \mathbb{C} : |z| < 1 + \max_{0 \leq j \leq d} \left| \frac{p_j}{p_d} \right| \right\}.
\]
Veronese Subrings

Let $R = \oplus_{j \geq 0} R_j$ be a graded ring; we assume that $R_0 = K$ is a field and that $R$ is finitely generated over $K$. 
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By a theorem of Hilbert $H(R, m)$ is a polynomial in $m$ when $m$ is sufficiently large. Note that

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**Example** Denote the cone over $\mathcal{P} \times \{1\}$ by cone $\mathcal{P}$. Then the semigroup algebra $K[\text{cone } \mathcal{P} \cap \mathbb{Z}^{d+1}]$ (graded by the projection to the last coordinate) gives rise to the Hilbert function $H(K[\text{cone } \mathcal{P} \cap \mathbb{Z}^{d+1}], m) = L_{\mathcal{P}}(m)$. 
Corollary (MB–Stapledon) Fix a positive integer $d$ and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of the Eulerian polynomial $A_d(t)$. There exist $M, N$ depending only on $d$ such that, if $R = \bigoplus_{j \geq 0} R_j$ is a finitely generated graded ring over a field $R_0 = K$ that is Cohen–Macauley and module finite over the $K$-subalgebra generated by $R_1$, and if the Hilbert function $H(R, m)$ is a polynomial in $m$, then for $n \geq N$, $U_n h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\lim_{n \to \infty} \beta_j(n) = \rho_j$. 

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In particular, the coefficients of $U_n h(t)$ are unimodal for $n \geq N$. 
A Veronese Corollary

Corollary (MB–Stapledon) Fix a positive integer $d$ and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of the Eulerian polynomial $A_d(t)$. There exist $M, N$ depending only on $d$ such that, if $R = \bigoplus_{j \geq 0} R_j$ is a finitely generated graded ring over a field $R_0 = K$ that is Cohen–Macauley and module finite over the $K$-subalgebra generated by $R_1$, and if the Hilbert function $H(R, m)$ is a polynomial in $m$, then for $n \geq N$, $U_n h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\lim_{n \to \infty} \beta_j(n) = \rho_j$.

In particular, the coefficients of $U_n h(t)$ are unimodal for $n \geq N$.

Furthermore, they satisfy $h_j(n) < M h_d(n)$ for $0 \leq j \leq n$ and $n \geq N$. 

Asymptotics of Ehrhart Series of Lattice Polytopes  

Matthias Beck
Open Problems

Find optimal choices for $M$ and $N$ in any of our theorems.
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**Conjecture** For Ehrhart series of $d$-dimensional polytopes, $N = d$.

(Open for $d \geq 3$)
Recall our inequalities $h_{j+1}(n) > h_{d-j}(n)$ in the main theorem.

**Theorem (MB–Stapledon)** Fix a positive integer $d$ and set $N = d$ if $d$ is even and $N = \frac{d+1}{2}$ if $d$ is odd. If $h(t)$ is a polynomial of degree at most $d$ satisfying $h_0 + \cdots + h_{j+1} > h_d + \cdots + h_{d-j}$ for $0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1$, then the coefficients of $U_n h(t)$ satisfy $h_{j+1}(n) > h_{d-j}(n)$ for $0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1$ and $n \geq N$. 
One Result about Explicit Bounds

Recall our inequalities \( h_{j+1}(n) > h_{d-j}(n) \) in the main theorem.

**Theorem** (MB–Stapledon) Fix a positive integer \( d \) and set \( N = d \) if \( d \) is even and \( N = \frac{d+1}{2} \) if \( d \) is odd. If \( h(t) \) is a polynomial of degree at most \( d \), satisfying \( h_0 + \cdots + h_{j+1} > h_d + \cdots + h_{d-j} \) for \( 0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1 \), then the coefficients of \( U_n h(t) \) satisfy \( h_{j+1}(n) > h_{d-j}(n) \) for \( 0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1 \) and \( n \geq N \).

**Corollary** Fix a positive integer \( d \) and set \( N = d \) if \( d \) is even and \( N = \frac{d+1}{2} \) if \( d \) is odd. If \( P \) is a \( d \)-dimensional lattice polytope with Ehrhart h-vector \( h(t) \), then the coefficients of \( U_n h(t) \) satisfy \( h_{j+1}(n) > h_{d-j}(n) \) for \( 0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1 \) and \( n \geq N \).
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The Message

The Ehrhart series of $nP$ becomes friendlier as $n$ increases.

In fixed dimension, you don’t have to wait forever to make all Ehrhart series look friendly.

**Homework** Figure out what all of this has to do with carrying digits when summing 100-digit numbers.