(1) Fix $d \in \mathbb{Z}_{>0}$ and consider $\binom{x^d}{d}$ as a polynomial in $x$ (over your favorite field of characteristic 0).

(a) Show that, as polynomials, $(-x)^d = (-1)^d \binom{x^{d-1}}{d}$.

(b) Prove that $\sum_{n \geq 0} \binom{n+d}{d} x^n = \frac{1}{(1-x)^{d+1}}$.

(c) Show that $p(x)$ is a polynomial of degree $d$ if and only if

\[ \sum_{n \geq 0} p(n) x^n = \frac{h(x)}{(1-x)^{d+1}} \]

for some polynomial $h(x)$ of degree at most $d$ with $h(1) \neq 0$.

(2) A matrix is unipotent if it is the sum of the $d \times d$ identity matrix and a nilpotent matrix (i.e., a matrix $B$ for which there exists a positive integer $k$ such that $B^k = 0$). Fix indices $i$ and $j$, and consider the sequence $f(n) := (A^n)_{ij}$ formed by the $(i,j)$-entries of the $n$th powers of a unipotent matrix $A$. Prove that $f(n)$ agrees with a polynomial in $n$. (Hint: express $A^n$ using the binomial theorem.)

(3) Let $\Delta$ be the difference operator defined by $(\Delta f)(n) := f(n+1) - f(n)$, for a given polynomial $f(n)$. Prove that $f(n)$ is of degree $\leq d$ if and only if $(\Delta^m f)(0) = 0$ for all $m > d$. (Hint: use the shift operator $(Sf)(n) := f(n+1)$ and express $(S^n f)(0)$ using the binomial theorem.)

(4) A $d \times d$ matrix with nonnegative integer coefficients is $n$-magic if each of its rows and columns sum to $n$.

(a) Prove that every $n$-magic matrix is the sum of $n$ permutation matrices.

(b) Show that

\[ \left\{ (m_{11}, m_{12}, \ldots, m_{dd}, n) \in \mathbb{Z}_{\geq 0}^{d^2+1} : \text{m is an n-magic matrix} \right\} \]

forms a semigroup, and deduce that the number $M_d(n)$ of $n$-magic $d \times d$ matrices is a polynomial in $n$.

(5) Compute $M_3(n)$.

(6) Let $m$ be a magic labelling of a given graph $G$ with “magic sum” $n$.

(a) Define the matrix $A = (a_{ij})$ where $a_{ij}$ is the label of $m$ on the edge connecting the nodes $i$ and $j$. Show that each row and column sum of $A$ is $n$, and deduce with the Exercise (4) that

\[ 2A = \sum_{\pi} \pi + \pi^T \]

where the sum is over a certain set of permutation matrices.

(b) Prove that $2m$ is a sum of magic labellings with magic sum 2, and conclude that every completely fundamental magic labelling of $G$ has magic sum 1 or 2.

(7) Show that $\dim \text{Poly}_D(\mathbb{F}^n) = \binom{D+n}{n}$. 
(8) We proved in class that, given a finite set $S \subset \mathbb{R}^3$, there is a nonzero polynomial of degree
\[ \leq c |S|^{\frac{1}{3}} \] (for some constant $c$) that vanishes on $S$. Given $n$ lines in $\mathbb{R}^3$, prove that there is a
nonzero polynomial of degree $\leq c n^{\frac{1}{3}}$ (for some other constant $c$) that vanishes on all the lines.
Generalize.