Math 310 – Partitions

(1) Prove that \( \sum_{n=0}^{\infty} \binom{n+m}{n} x^n = \frac{1}{(1-x)^{m+1}} \).

(Hint: one approach is to differentiate the identity for the geometric series.)

(2) Deduce that \( \sum_{k=0}^{\infty} \binom{m}{k} x^k = (1+x)^m \) for all \( m \in \mathbb{Z} \).

(3) Recall that \( \pi_3(n) \) counts the number of partitions of \( n \) into parts \( \leq 3 \).

(a) Find a formula for \( \pi_3(n) \) by expanding the generating function for \( \pi_3(n) \) into partial fractions. (Hint: use a similar partial-fractions decomposition as we used in class for \( \pi_2(n) \).)

(b) Deduce that \( \pi_3(n) \) equals the nearest integer to \( \frac{(n+3)^3}{12} \).

(4) Let’s define a northeast lattice path as a path through integer lattice points that uses only the steps \((1,0)\) and \((0,1)\). Let \( L_n \) be the line defined by \( x + 2y = n \). Prove that the number of northeast lattice paths from the origin to an integer lattice point on \( L_n \) is the \((n+1)\)th Fibonacci number \( f_{n+1} \).

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