Math 310 – Generating Functions

(1) Compute the generating function for the sequence
\[ a_k = \begin{cases} 1 & \text{if } k \text{ is a multiple of 7}, \\ 0 & \text{otherwise}. \end{cases} \]

(2) Compute the sequence \((a_k)\) that gives rise to the generating function \(\sum_{k \geq 0} a_k x^k = \left(\frac{1}{1-x}\right)^2\), by looking at the product \((1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots)\). (If you look at the result, can you think of a different way to compute \((a_k)\)?)

(3) Define a recursive sequence by setting \(a_0 = 0\) and \(a_{n+1} = 2a_n + 1\) for \(n \geq 0\).

(a) Conjecture a formula for \(a_k\) by experimenting.

(b) Now put the sequence \((a_k)\) into a generating function \(g(x)\) and find a formula for \(g(x)\) by utilizing the recursive definition of \(a_k\).

(c) Expand your formula for \(g(x)\) into partial fractions, and use the result to prove your conjectured formula for \(a_k\).

(4) We define a second recursive sequence by setting \(a_0 = 1\) and \(a_{n+1} = 2a_n + n\) for \(n \geq 0\). Find a formula for \(a_k\).

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