Errata for
Bao–Chern–Shen
An Introduction to Riemann–Finsler Geometry

*** On page 8, lines 13 and 14 from the bottom should read: " ...... where α is positive and β is non-negative. Hence the inequality in question is valid."

*** On page 10, the second displayed statement is *wrong*. The correct inequality should be:

$$g_{ij\,(y)}\; w^i\, y^j \;\leqslant\; [\, g_{pq\,(w)}\; w^p\, w^q\,]^{1/2}\; [\, g_{rs\,(y)}\; y^r\, y^s\,]^{1/2}\ .$$

The previous version was due to faulty logic: $A \leqslant BC$ certainly does not imply $A^2 \leqslant B^2\, C^2$, because our present A can be negative.

*** On page 12, part (b) of Exercise 1.2.6 needs to be rephrased as follows: "This is an example illustrating the statement that strong convexity implies strict convexity."

*** On page 25, in Exercise 1.4.4, the lower bound for $\|A\|$ is erroneous. The correct bound is forthcoming.

*** On page 36, in Exercise 2.3.3, replace "Finslerian geodesics" by the phrase "geodesics with constant Finslerian speed". In other words, we are referring to the autoparallels of the Finsler metric. See §5.3 and Exercise 5.3.1 for further discussions.

*** On page 43, in part (b) of Exercise 2.4.6, replace $\nabla\ell$ by $(\nabla\ell)^i$.

*** On page 58, a CLARIFICATION is warranted.

- We claimed that the combination $dR_j{}^i{}_{kl} - \cdots + \cdots - \cdots - \cdots$ will show up, when in effect only the first three terms do. The last two terms, namely $-R_j{}^i{}_{tl}\,\omega_k{}^t - R_j{}^i{}_{kt}\,\omega_l{}^t$, are added in artificially because, upon contraction with $dx^k \wedge dx^l$, they do not contribute.
- Likewise, in the expression $dP_j{}^i{}_{kl} - \cdots + \cdots - \cdots - \cdots$, only the first three terms and the last term will show up. The fourth term, namely $-P_j{}^i{}_{tl}\,\omega_k{}^t$, is added in artificially because, upon contraction with $dx^k \wedge \frac{\delta y^l}{F}$, it does not contribute.

The key here is that

$$\omega_p{}^t \wedge dx^p = \Gamma^t{}_{pq}\, dx^q \wedge dx^p = 0\,,$$

since the Chern connection coefficients $\Gamma^t{}_{pq}$ in natural coordinates are symmetric in p, q.

*** On page 66, the hint in Exercise 3.8.3 is overly sketchy, to the point of being misleading. The detailed instructions in Exercise 10.2.1 (pages 265, 266) are significantly more useful.

*** On the top of page 70, we find the lines

- One can derive the **polarization identity**

$$K(\ell,V,W) \;=\; \frac{1}{4}\,K(\ell,V+W) \;-\; \frac{1}{4}\,K(\ell,V-W)\;. \tag{3.9.6}$$

The corrected version should read:

- Denote the numerator of $K(\ell,V,W)$ by $R(\ell,V,W)$. Then we have the **polarization identity**

$$R(\ell,V,W) \;=\; \frac{1}{4}\,R(\ell,V+W) \;-\; \frac{1}{4}\,R(\ell,V-W)\;. \tag{3.9.6}$$

*** This is NOT a mistake, just a CLARIFICATION. On page 76, in Proposition 3.10.1, statement (a) has an immediate implication that should perhaps be made clear.

> Since (a) is a statement in ANY coordinate induced basis, it implies that $R(v, v) = \lambda_{(x,y)}\, h(v, v)$ for every tangent vector $v \in T_x M$. And this latter statement is independent of coordinates.

Indeed, given any $v \in T_x M$, there is always a local coordinate system containing the point x, such that the v in question is equal to one of the coordinate vectors ∂_{x^i}. The said implication then follows.

When proving that (a) implies (b), one needs to invoke the above implication in order to apply the polarization identity (3.9.6).

*** On page 80, Exercise 3.10.9 's part (c) has technical problems, and part (d) is at best clumsy. There is a better way.

- The diplayed equation in part (b) should have been simplified to read
$$\lambda \; (\, \lambda_{;i} \, \ell_j - \lambda_{;j} \, \ell_i \,) \;=\; 0 \; .$$
- Replace part (c) by the following: "Show that contraction of the above by ℓ^j, together with Euler's theorem, leads to $(\lambda^2)_{;i} = 0$. Hence λ^2 lives on M."
- Replace part (d) by the following: "Explain why λ^2 has no dependence on x either. Therefore λ^2 is constant, and so is λ."

*** On page 83, in Figure 4.1, the right half of the convex limaçon is not "fat" enough. This is because, in the range $-\pi/2 < \phi < +\pi/2$, the quantity $\cos\phi$ is always positive, and hence ρ should exceed 3 there. The displayed picture is a drawing and not an actual mathematical plot.

*** On page 83, replace the period at the end of equation (4.1.6) by a comma, and add the phrase "whenever S is parametrized clockwise." This is to ensure that the I defined in (4.1.6) is consistent with the one introduced in (4.4.1).

Here is why. Let $\bar{e}_1$ denote $\frac{1}{\sqrt{g}}(F_{y^2}\partial_{y^1} - F_{y^1}\partial_{y^2})$, and $\bar{e}_2$ denote $\frac{1}{F}(y^1\partial_{y^1} + y^2\partial_{y^2})$. These two vector fields on $\mathbb{R}^2 \setminus 0$ form a $\hat{g}$ orthonormal basis at each point. A determinant calculation shows that this ordered basis has the same orientation as $\{\partial_{y^1}, \partial_{y^2}\}$. Let us agree to draw the y^1 and y^2 axes as in Figure 4.1. Since $\bar{e}_2$ points radially outward from the origin, we must have $\bar{e}_1$ tangent to the indicatrix in a clockwise manner.

In (4.4.1), the vector e_1 from the Berwald frame is contracted three times into the Cartan tensor A to form the Cartan scalar I. The components of e_1 are identical to those of $\bar{e}_1$, even though the coordinate bases involved are different. On the other hand, (4.1.6) defines I by contracting the unit velocity $\frac{dy}{dt}$ three times into A. Now, $y(t)$ is a unit speed parametrization of the indicatrix, so its velocity is either $+\bar{e}_1$ or $-\bar{e}_1$. In order that the two definitions of I agree, the said velocity must equal $+\bar{e}_1$. Hence our unit speed parametrization of the indicatrix needs to be clockwise.

*** On page 84, in the statement of Proposition 4.1.1, add the adjective "clockwise" after "unit speed".

*** On page 86, the third displayed equation has a sign error in the exponent. The corrected equation should read:

$$\chi(t) \;=\; \chi(0)\; e^{[-\int_0^t I(\tau)d\tau]} \;.$$

*** On page 86, line 8 from the bottom should read: "Also, it does *not* have to be clockwise or counterclockwise."

*** On page 99, in part (e) of Exercise 4.4.7, replace the phrase "apply exterior differentiation" by "use (4.4.2a) for ${\omega_1}^1$".

*** On page 104, in Exercise 4.5.4, the last sentence reads "This again explains why (4.5.3) is measuring ...".

The corrected version should read: “This again explains why (4.5.2) is measuring ...”.

*** On page 105, line 5 from the bottom, change c to σ, twice.

*** On page 116, line 7 from the bottom: replace $\delta_{\alpha\beta}$ by $\delta^{\alpha\beta}$.

*** Page 116, line 10 from the bottom. After “$U^n(t) = 0$”, add the phrase “with $U = 0$ at $t_o, \ldots, t_k$.” Without this additional phrase, the condition $U^n = 0$ (namely, g_T orthogonality) does not make sense at the *a priori* kinks, because the tangent vector T is ill-defined there.

*** On page 117, Exercise 5.1.1, the displayed inequality in part (b) is *wrong*. The corrected statement should be

$$g_T(U,U) \;=\; [\,(F_{y^i})_{(\sigma,T)}\, U^i\,]^2 \;+\; (F\; F_{y^i y^j})_{(\sigma,T)}\; U^i\, U^j \;,$$

which is not interesting. The erroneous previous estimate was due to faulty logic: $A \leqslant B$ certainly does not imply $A^2 \leqslant B^2$, because A can be negative. In other words, the fundamental inequality (1.2.3) can *not* be used to estimate the squared term.

*** On pages 126, 127, a clarification is needed. When we say that exp is C^1 at the 0-section, we mean that “for each fixed x, $\exp(x,y)$ is C^1 with respect to $y \in T_x M$.”

*** On page 128, part (a) of Exercise 5.3.5 needs to be corrected. The instructions should say “Differentiate (5.3.5) with respect to λ twice.” In the displayed equation that immediately follows, change all v to y.

*** On page 145, line 5 from the bottom, change c to σ, twice.

*** On page 155, in Exercise 6.2.10, change $\mathcal{B}_p^-$ to $\mathcal{B}_p^-(r)$.

*** On page 157, replace lines 4 and 5 from the top by the following: "where $\exp_{p*}$ is evaluated at the point $tv(u)$, and homogeneity implies that $\frac{dv(u)}{du}$ may be regarded as tangent to $S_p(t)$. By the Gauss lemma, we see that".

*** On page 158, line 7 from the top should read: "By (1.4.3) and (6.3.5) at time $t = t(u)$, we have".

*** On page 158, in lines 13 and 15 from the top, replace x by c.

*** On page 160, in Exercise 6.3.3, change $\mathcal{B}_p^-$ to $\mathcal{B}_p^-(r)$.

*** On page 162, in the second displayed equation, we should have $o(\,|v|^2\,)$ (little oh of $|v|$ squared) instead of $O(\,|v|^3\,)$ (Big Oh of $|v|$ cubed). This is so because our F^2 is only presumed to be of class C^2 in the discussion.

Replace "so all the higher-order terms must vanish, ..." by the following: "so in the above equation we replace v by tv, divide throughout by t^2, and then let $t \to 0$. The result is $F^2(v) = \frac{1}{2}(F^2)_{y^i y^j}(0)\, v^i v^j$, as claimed."

There is in fact a better argument: "Suppose F^2 is of class C^2 at $y = 0$. Since F^2 and its derivative both vanish at $y = 0$, the second Mean Value theorem reads

$$F^2(tv) \;=\; \frac{1}{2}\,\frac{\partial^2 F^2}{\partial y^i \partial y^j}(c_t)\,(tv)^i\,(tv)^j \;,$$

where c_t is some point along the line segment joining 0 to tv, with $t > 0$. By positive homogeneity, there is a factor of t^2 on each side, which can then be canceled off. Now let $t \to 0$. Note that c_t must approach 0 and, because

F^2 is C^2, those second order partial derivatives eventually get evaluated at $y = 0$, thereby delivering the conclusion we seek."

*** On page 163, in the first displayed equation, that $O(\,|x|^3\,)$ (Big Oh of $|x|$ cubed) should have been $o(\,|x|^2\,)$ (little oh of $|x|$ squared). This is so because d_p^2 is only presumed to be of class C^2 in the discussion.

In the third displayed equation, change that $O(\,|ty|^3\,)$ to $o(\,|ty|^2\,)$.

As remarked in the correction for page 162, there is a better argument using the second Mean Value theorem. The latter enables us to bypass the Big Oh versus little oh issue completely, and makes explicit use of the C^2 assumption.

*** On page 174, fourth line from the top, change "5.4.6" to "5.4.4".

*** On page 176, in the second opening sentence of Exercise 7.1.4, change "conjugacy" to "non-conjugacy".

*** On page 178, in the displayed formula of Exercise 7.2.5, delete the colon ":" after $I(W, W)$.

*** On page 191, lines 8 and 9 from the bottom of the page contain a typesetting blemish. The quantity "$n - 1$" should not be separated into two pieces.

*** On page 192, the definition of the Ricci tensor for Finsler metrics was attributed to the wrong reference. The correct one is:

H. Akbar-Zadeh, Generalized Einstein manifolds, J. Geom. and Phys. **17** (1995), 342–380.

*** On page 204, in Corollary 8.2.2, "$\sigma_y(i_y\, y)$" should've been "$\sigma_y(i_y)$".

*** Page 261, line 18 from the top. Just something cosmetic: change "§4.1, 4.5" to "§4.1, §4.5".

*** On page 276, line 13 from the top, change (2.4.10) to (2.4.11). The reference to (2.4.10) is technically correct, but (2.4.11) is more efficient for the purpose at hand.

*** On page 277, equation (10.6.1) has a sign error in the exponent. The corrected equation should read:

$$K(t) \;=\; K(0)\; e^{[-\int_0^t I(\tau)d\tau]}\;.$$

*** On page 286, line 6 from the top, the left-hand side of the displayed equation should show $p\,\partial_u + q\,\partial_v$ instead of v.

*** On page 288, line 1 from the top. The first phrase should read

- Whenever $1 + C^s C_s \neq 0$ and Q is symmetric,

*** On page 288, lines 5 and 6 from the top. Replace that first remark by the following:

> * Curiously, the symmetry of Q is needed only in the statement about the inverse of $Q + CC^t$.

Originally, we had remarked that the symmetry hypothesis on Q was removable. That was wishful thinking, stemming from an oversight.

*** On page 300, that lament about the formidable obstacle in calculating explicitly the flag curvature of Randers metrics is now obsolete. It has become known in recent years that Berwald's formula, namely formula (3.8.7), is the crucial tool in overcoming this computational obstacle. See the following two references by D. Bao and Colleen Robles for an exposition:

- On Ricci and flag curvatures in Finsler geometry, in *A Sampler of Riemann–Finsler Geometry*, vol.**50**, MSRI Series, Cambridge University Press, 2004.
- On Randers spaces of constant flag curvature, *Reports on Mathematical Physics* **51** (2003), 9–42.

*** On page 302, line 13 from the bottom of the page contains a typesetting omission. There should be a y_i to the immediate right of that factor of $-\frac{1}{4}$. By y_i, we mean $g_{ij}\, y^j$, where g_{ij} is the fundamental tensor of the Finsler metric.

*** (Prior to September 11, 2003) On page 303, lines 14 and 16 from the top, line 1 from the bottom: change the upper index i on ψ to an upper index j.

*** (After September 11, 2003) On page 303, we are supposed to prove that if the Randers metric $F := \alpha + \beta$ is Berwald, equivalently if the Berwald connection

$$(G^i)_{y^j y^k} = \tilde{\gamma}^i{}_{jk} + \tilde{b}_{r|s}\, (\psi^{irs})_{y^j y^k}$$

is independent of y, then $\tilde{b}$ must be parallel with respect to $\tilde{a}$.

If $\tilde{b}$ vanishes identically, then so does $\tilde{b}_{r|s}$ and there is nothing to prove. So, let us assume that $\tilde{b} \not\equiv 0$. The argument in the book proves (by contradiction) that in this case, some component(s) of the tensor $(\psi^{irs})_{y^j y^k}$ must have a y-dependence. It then insists that in order for the tensor contraction $\tilde{b}_{r|s}\, (\psi^{irs})_{y^j y^k}$ to be independent of y, all the components of $\tilde{b}_{r|s}$ must vanish. This is premature because there are in principle at least two scenarios which could violate such a claim.

- ∘ It is possible that $\tilde{b}_{r|s}$ is zero only at values of r, s for which the corresponding components of $(\psi^{irs})_{y^j y^k}$ depend on y. This was observed by Mike Crampin.

- More generally, though we have shown that it is necessary for some components of $(\psi)_{yy}$ to have y-dependences, the coefficients $\tilde{b}_{r|s}$ may be such that the resulting "linear combination" $\tilde{b}_{r|s}\,(\psi^{irs})_{y^j y^k}$ no longer depends on y.

In retrospect, the proof given in the book is wrong because it does no good to show that some component(s) of the tensor $(\psi^{irs})_{y^j y^k}$ must have a y-dependence.

Happily, there is an elegant and correct proof due to Mike Crampin. It is entirely self-contained and consists of three steps as follows.

(1) By hypothesis, the Berwald connection $(G^i)_{y^j y^k}$ is independent of y, so $\tilde{b}_{r|s}\,(\psi^{irs})_{y^j y^k} = 2E^i{}_{jk}$ for some E which contains no y-dependence. Since ψ is homogeneous of degree 2 in y, contracting both sides with y^k and using Euler's theorem gives $\tilde{b}_{r|s}\,(\psi^{irs})_{y^j} = 2\,E^i{}_{jk}\,y^k$. Now contract with y^j and cancel off a factor of 2. The result reads

$$\tilde{b}_{r|s}\,\psi^{irs} = E^i{}_{jk}\,y^j y^k\,.$$

(2) In the formula of ψ^{irs}, there is an $\ell^i := y^i/F$. Multiplying the result of (1) by $F = \alpha + \beta$ leads to

$$\begin{aligned}&\tilde{b}_{r|s}\{\,[\tilde{a}^{ir}y^s - \tilde{a}^{is}y^r](\alpha^2 + \alpha\beta) \;+\; y^i(y^r y^s + [y^r\tilde{b}^s - y^s\tilde{b}^r]\alpha)\,\}\\ &= E^i{}_{jk}\,y^j y^k\,(\alpha+\beta)\,.\end{aligned}$$

Note that $\alpha^2 = \tilde{a}_{pq}\,y^p y^q$ and $\beta = \tilde{b}_p y^p$ are both polynomial in y, whereas α is not. Sorting the terms in the above equation accordingly gives

$$\mathrm{Rat}^i + \alpha\,\mathrm{Irrat}^i \;=\; 0\,,$$

where

$$\begin{aligned}\mathrm{Rat}^i &:= \tilde{b}_{r|s}\{\,[\tilde{a}^{ir}y^s - \tilde{a}^{is}y^r]\alpha^2 \;+\; y^i y^r y^s\,\} \;-\; E^i{}_{jk}\,y^j y^k\,\beta\,,\\ \mathrm{Irrat}^i &:= \tilde{b}_{r|s}\{\,[\tilde{a}^{ir}y^s - \tilde{a}^{is}y^r]\beta \;+\; y^i[y^r\tilde{b}^s - y^s\tilde{b}^r]\,\} \;-\; E^i{}_{jk}\,y^j y^k\end{aligned}$$

are, respectively, cubic and quadratic polynomials in y. At nonzero y, the equation $\mathrm{Rat}^i + \alpha\,\mathrm{Irrat}^i = 0$ becomes $-\mathrm{Rat}^i + \alpha\,\mathrm{Irrat}^i = 0$ under $y \mapsto -y$. This implies that $\mathrm{Rat}^i = 0$, and consequently $\alpha\,\mathrm{Irrat}^i = 0$. Since $\tilde{a}$ is a Riemannian metric, α is positive. Hence $\mathrm{Irrat}^i = 0$, which gives a formula for $E^i{}_{jk}\,y^j y^k$. Substituting this formula into the equation $\mathrm{Rat}^i = 0$ produces

$$\begin{aligned}&\tilde{b}_{r|s}\{\,[\tilde{a}^{ir}y^s - \tilde{a}^{is}y^r]\alpha^2 \;+\; y^i y^r y^s\,\}\\ &= \tilde{b}_{r|s}\{\,[\tilde{a}^{ir}y^s - \tilde{a}^{is}y^r]\beta \;+\; y^i[y^r\tilde{b}^s - y^s\tilde{b}^r]\,\}\beta\,,\end{aligned}$$

which can be rearranged to read

$$\tilde{b}_{r|s}\{\,[\tilde{a}^{ir}y^s - \tilde{a}^{is}y^r]\,(\alpha^2-\beta^2) \;-\; y^i\,[y^r y^s - (y^r\tilde{b}^s - y^s\tilde{b}^r)\,\beta]\,\} = 0\,.$$

(3) The result of (2) has the structure $(\alpha^2-\beta^2)\,V^i - \varphi\, y^i = 0$, where $V^i := \tilde{b}_{r|s}\,[\tilde{a}^{ir}y^s - \tilde{a}^{is}y^r]$ and $\varphi := \tilde{b}_{r|s}\,[y^r y^s - (y^r\tilde{b}^s - y^s\tilde{b}^r)\,\beta]$. Almost by inspection, we see that the vectors V and y are $\tilde{a}$-orthogonal. Thus we must have $(\alpha^2-\beta^2)\,V = 0$ and $\varphi\, y = 0$. At nonzero y, the fact that $\|\tilde{b}\| < 1$ effects $\alpha^2 > \beta^2$, hence $V = 0$; also, $\varphi = 0$. These conclusions are manifestly valid at $y = 0$. In other words,

$$\tilde{b}_{r|s}\,[\tilde{a}^{ir}y^s - \tilde{a}^{is}y^r] = 0 \quad \text{and} \quad \tilde{b}_{r|s}\,[y^r y^s - (y^r\tilde{b}^s - y^s\tilde{b}^r)\,\beta] = 0\,.$$

Apply ∂_{y^j} to the first statement and then contract with $\tilde{a}_{ki}$. We get $\tilde{b}_{k|j} = \tilde{b}_{j|k}$. This symmetry reduces the second statement to $\tilde{b}_{r|s}\,y^r y^s = 0$. Differentiating off the y gives $\tilde{b}_{j|k} = 0$, which is the desired conclusion. $\square$

*** On pages 303 and 304, Exercises 11.5.1 and 11.5.2 shall be replaced by forthcoming versions, based on some material supplied by M. Crampin. The former version of these two exercises pertain to the proof on page 303, which was wrong. As a result, these exercises became irrelevant.

*** On page 305, lines 1–4 from the top: delete the sentences "We must work only with those 1-forms $\tilde{b}$ whose Riemannian norm $\|\,\tilde{b}\,\|$ is uniformly bounded. That way they can be normalized (if necessary) to satisfy $\|\,\tilde{b}\,\| < 1$, as per §11.1". Replace them by the following:

> "Since these $\tilde{b}$ are parallel, their Riemannian norms $\|\,\tilde{b}\,\|$ must be constant. See Exercise 11.6.1. Hence we can always rescale $\tilde{b}$ (if necessary) by a constant multiple to effect the condition $\|\,\tilde{b}\,\| < 1$ discussed in §11.1."

*** On page 306, lines 13–17 from the top: delete the sentences "Let us restrict to compact oriented Riemannian surfaces without boundary. The only such surface that admits globally defined nonvanishing parallel 1-forms is the flat torus. (See Exercise 11.6.3.) Furthermore, although the resulting Randers metrics are indeed of Berwald type, they are at the same time all locally Minkowskian. (Exercise 11.6.3 again.)" Replace them by the following:

> "This is the subject matter of Exercise 11.6.3. There, we show that if a Riemannian surface $(M, \tilde{a})$ admits a nonzero parallel 1-form $\tilde{b}$, then there exists a local coordinate system in which the components of $\tilde{a}$ are δ_{ij}, and those of $\tilde{b}$ are constants. Hence the resulting Randers metric, though of Berwald type, is at the same time locally Minkowskian."

*** On page 307, replace lines 10 to 19 from the top by the following version:

* This example generalizes to all higher dimensions. Specifically, we set M to be the Cartesian product $S^{n-1} \times S^1$, where $n \geqslant 3$ and S^{n-1} is the standard unit sphere in Euclidean $\mathbb{R}^n$. Endow M with the product Riemannian metric $\tilde{a}$. The drift 1-form $\tilde{b}$ is again ϵdt, with $0 < \epsilon < 1$ a constant. The second part of Theorem 13.6.1, and Exercise 13.6.5, offer a perspective from harmonic theory.
* In fact, we can replace S^{n-1} by any Riemannian manifold $(\breve{M}, \breve{g})$, where $\breve{g}$ is not flat. Let M be the Cartesian product $S^1 \times \breve{M}$, $\tilde{a}$ be the product metric $dt \otimes dt + \breve{g}$, and $\tilde{b} := \epsilon dt$. The resulting y-global Randers metric is Berwald, and is neither Riemannian nor locally Minkowskian. See Exercise 11.6.6. If we want M to be compact and boundaryless, just pick $\breve{M}$ to be so.

*** On page 308, Exercise 11.6.3 needs to be completely rewritten. The version in print does not delineate the local computations from the topological issues. The new version reads as follows:

Let us ponder the implications of Theorem 11.5.1 for Finsler *surfaces*. Suppose a Riemannian surface $(M, \tilde{a})$ admits a globally defined nonvanishing parallel 1-form $\tilde{b}$. By part (a) of Exercise 11.6.1, there is no loss in generality if we assume $\| \tilde{b} \| < 1$.

(a) Show that there is an orthonormal frame field $\{\tilde{e}_1, \tilde{e}_2\}$ of $\tilde{a}$, where both $\tilde{e}_1$ and $\tilde{e}_2$ are parallel. Here are some hints (from Exercise 6 on p. 333 of [On]): divide $\tilde{b}$ by the *constant* $\| \tilde{b} \|$, raise its index, and use that as $\tilde{e}_1$. The covariant derivative of $\tilde{e}_1$ is zero in every direction. What does that tell you about the connection forms?

(b) Using that special frame field and torsion-freeness (suggestion: see §10.4), deduce that there exists local coordinates in which the components of $\tilde{a}$ are δ_{ij}, and those of $\tilde{b}$ are constants. Hence the Randers metric built from $\tilde{a}$ and $\tilde{b}$ is locally Minkowskian.

(c) Alternatively, deduce from part (a) that the Gaussian curvature $\tilde{K}$ of $\tilde{a}$ is zero. Then explain carefully why the hh- Chern curvature $R_j{}^i{}_{kl}$ of the resulting Randers metric F must also be zero. This, together with the vanishing of $P_j{}^i{}_{kl}$, will again imply that F is locally Minkowskian.

(d) Let us now require M to be compact, oriented, and boundaryless. Show that if $\tilde{a}$ admits a nonvanishing parallel global 1-form $\tilde{b}$, then $(M, \tilde{a})$ must be the "flat torus."

*** On page 309, rewrite Exercise 11.6.6 as follows.

This exercise demonstrates that the example studied in §11.6B is actually part of an enormous family. Let us prescribe the following data:

- Define M to be the Cartesian product $S^1 \times \breve{M}$, where $\breve{M}$ is any $(n-1)$-manifold with a non-flat Riemannian metric $\breve{g}$.
- The Riemannian metric $\tilde{a}$ on M is the product metric $dt \otimes dt + \breve{g}$, where t is the coordinate on S^1.
- The drift 1-form is $\tilde{b} := \epsilon dt$, where $0 < \epsilon < 1$ is any constant. Note: dt is globally defined on M, even though the coordinate t is not.

Write local coordinates on M as $(x^i) = (t, x^\alpha)$, where (x^α), $\alpha = 2, \ldots, n$ are local coordinates on $\breve{M}$.

(a) Check that $\| \tilde{b} \|^2 = \epsilon^2 < 1$.

(b) Let $\tilde{\omega}_j{}^i := \tilde{\gamma}^i{}_{jk}\, dx^k$ be the Levi-Civita (Christoffel) connection of $\tilde{a}$. Likewise, let $\breve{\omega}_\beta{}^\alpha := \breve{\gamma}^\alpha{}_{\beta\sigma}\, dx^\sigma$ be those of $\breve{g}$. Show that $\tilde{\omega}_1{}^1$, $\tilde{\omega}_1{}^\alpha$, $\tilde{\omega}_\beta{}^1$ are all zero, and $\tilde{\omega}_\beta{}^\alpha = \breve{\omega}_\beta{}^\alpha$. Verify that $\tilde{b}$ is a parallel 1-form of $\tilde{a}$.

(c) Let $\tilde{\Omega}_j{}^i$ and $\breve{\Omega}_\beta{}^\alpha$ be the curvature 2-forms of $\tilde{a}$ and $\breve{g}$, respectively. Show that $\tilde{\Omega}_1{}^1$, $\tilde{\Omega}_1{}^\alpha$, $\tilde{\Omega}_\beta{}^1$ are all zero, and $\tilde{\Omega}_\beta{}^\alpha = \breve{\Omega}_\beta{}^\alpha$. Explain why $\tilde{a}$ is non-flat.

(d) Does the data $(M, \tilde{a}, \tilde{b})$ produce a Berwald metric that is neither Riemannian nor locally Minkowskian?

Hints for parts (b) and (c): have a look at the formalism of warped products in §13.3A.

*** On page 334, lines 9, 6, 3 and 2 from the bottom should be modified as follows.

- Line 9 from the bottom: Insert the word "(corrected)" in front of "**Yasuda–Shimada theorem**".
- Line 6 from the bottom should read: "and $\tilde{b}^j \, (\tilde{b}_{i,x^j} - \tilde{b}_{j,x^i}) = 0$ if and only if ".
- Lines 3 and 2 from the bottom should read: "The drift 1-form $\tilde{b}$ is closed (hence locally has the exact form df) and satisfies the system of PDEs".

The correction in the Yasuda–Shimada theorem consists of amending the condition $\tilde{b}^j \, (\tilde{b}_{i,x^j} - \tilde{b}_{j,x^i}) = 0$.

*** This is NOT a mistake, just a CLARIFICATION. On page 357, we say that the metric described by the equation $Ric_{ij} = \mathfrak{c} \, g_{ij}$ is an Einstein metric with constant $\mathfrak{c}$. This *constant* is not to be confused with the *Einstein constant*, which refers to the λ in the rewrite $\mathfrak{c} = (n-1)\lambda$.

*** On page 360, Exercise 13.2.8, replace all $3r$ by r.

Part (b) should read: "Show that g is an Einstein metric satisfying the equation $Ric_{ij} = \frac{1}{r^2} \, g_{ij}$." (Incidentally, this metric does *not* have constant sectional curvature.)

The constant multiple in front of g_{ij} is customarily expressed as $(n-1)\lambda$, with the λ referred to as the *Einstein constant*. For instance, since the example here is of dimension $n = 4$, the Einstein constant in question is $\frac{1}{3\,r^2}$.

*** On page 401, there is a reference to (14.5.6). It should have referenced both (14.5.6) and (14.5.7).
